**EXERCISE 13**

Suppose that $F = ax^2 + bxy + cxz + dy^2 + eyz + fz^2$ is a conic in $\mathbb{P}^2_K$ for an algebraically closed field $K$ with $char(K) \neq 2$.

Taking the partial derivatives of $F$, we get

$$F_x = 2ax + by + cz$$

$$F_y = bx + 2dy + ez$$

$$F_z = cx + ey + 2fz$$

.

Hence, supposing $(x, y, z) \in K^3 - \{0\}$ is a common zero of the partial derivatives, $(x, y, z)$ must satisfy the matrix equation

$$\begin{pmatrix} 2a & b & c \\ b & 2d & e \\ c & e & 2f \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

.

Since $(x, y, z) \neq 0$, this gives $det(A) = 0$, where A is the 3x3 matrix. Now we note that the original conic equation can be rewritten as

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 2a & b & c \\ b & 2d & e \\ c & e & 2f \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

.

Since A is symmetric and $char(K) \neq 2$, we can find a matrix B $\in GL_3(K)$ s.t. $A' = B^t AB$ is a *diagonal matrix*. We apply $(x, y, z)$ to $B^t$ to get a change of coordinates $(x', y', z') = (x, y, z)B^t$, noting that $B$ applied to $(x, y, z)^t$ is $(B^t)^t$ so $B(x, y, z)^t = ((x, y, z)B^t)^t = (x', y', z')^t$. This allows us to consider the projectively equivalent conic equation

$$\begin{pmatrix} x' & y' & z' \end{pmatrix} \begin{pmatrix} a' & 0 & 0 \\ 0 & b' & 0 \\ 0 & 0 & c' \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

Where the 3x3 matrix is $A'$. Since $det(A) = 0$, we also get $det(A') = 0$ i.e. $a'b'c' = 0$. WLOG assume $c' = 0$. This gives the conic equation $ax'^2 + by'^2$, which is not irreducible in $K[x, y, z]$ since K is algebraically closed. This is a contradiction, so our supposition that $(x, y, z)$ is a common zero of the partial derivatives is false.