EXERCISE 1 FROM SECTION ON PROJECTIVE SPACES

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Exercise 1: Prove that an axiomatic projective plane has the same number of points as lines.

Proof: Apart from the two axioms of an axiomatic projective plane given in the section on projective planes in the wiki, we will assume the following additional axioms:
(3) A projective plane has at least 3 non-collinear points.
(4) Any line in the projective plane passes through at least 3 distinct points.

We will denote our projective plane by $\mathbb{P}$, and define
$\mathcal{L} := \{\text{lines in } \mathbb{P}\}$
$\Gamma := \{\text{points in } \mathbb{P}\}$

We divide the proof into 2 cases:

Case 1: $\mathcal{L}$, Γ are both infinite sets.

Proof of case 1: Let $\Delta_{\mathcal{L}}$, Δ_{Γ} denote the diagonals of $\mathcal{L} \times \mathcal{L}$, $\Gamma \times \Gamma$ respectively. We need to show that $\mathcal{L}$ and Γ have the same cardinality.
It is easy to see that
$|\mathcal{L}| = |\mathcal{L} \times \mathcal{L}| = |\mathcal{L} \times \mathcal{L} - \Delta_{\mathcal{L}}|$; $|\Gamma| = |\Gamma \times \Gamma| = |\Gamma \times \Gamma - \Delta_{\Gamma}|$.

By axioms (1), (2) of the axiomatic projective plane we have natural maps
$\pi_1 : \mathcal{L} \times \mathcal{L} - \Delta_{\mathcal{L}} \to \Gamma$ given by $\pi_1(l_1, l_2) = l_1 \cap l_2$;
$\pi_2 : \Gamma \times \Gamma - \Delta_{\Gamma} \to \mathcal{L}$ given by $\pi_2(p, q) = \overline{pq}$, where $\overline{pq}$ is the unique line passing through p and q.

We will show that π_1,π_2 are surjective.
If $p \in \Gamma$, then by axiome (3), and the fact that Γ is infinite, $\exists$ distinct points q and r such that p, q, r are not collinear. Then clearly, $\overline{pq} \neq \overline{pr}$ and $\pi_1(\overline{pq}, \overline{pr}) = p$. This shows that π_1 is surjective.
Let $l \in \mathcal{L}$. Then by axiom (4), l has at least 2 distinct points p, q on it. Again, clearly $\pi_2(p, q) = \overline{pq} = l$. So, π_2 is surjective.

Now, π_1 $surjective \Rightarrow |\Gamma| \leq |\mathcal{L} \times \mathcal{L} - \Delta_{\mathcal{L}}| = |\mathcal{L}|$; π_2 $surjective \Rightarrow |\mathcal{L}| \leq |\Gamma \times \Gamma - \Delta_{\Gamma}| = |\Gamma|$.

Thus, $|\mathcal{L}| = |\Gamma|$.

Note that if $\mathcal{L}$ is infinite then by axioms (1) and (4), Γ must be infinite and we reduce to case 1. Here axiom (4) is used in the sense that it guarantees that every line has a point on it.

If Γ is infinite, suppose $\mathcal{L}$ is finite. By axioms (3) and (1), every point lies on some line. So, $\exists\, l \in \mathcal{L}$ such that l has infinitely many points on it. By axioms (3) again, $\exists\, p \in \Gamma$ such that $p \notin l$. But then for any $q \in \Gamma$ such that $q \in l$, we have a line $\overline{pq}$ which is distinct from l, and by axiom (2), if $q, q' \in \Gamma$ such that $q \neq q'$ and $q, q' \in l$, then $\overline{pq} = \overline{pq'}$. So, this gives us infinitely many distinct lines through p intersecting l. Thus, $\mathcal{L}$ is infinite, a contradiction. So, $\mathcal{L}$ must have been infinite to begin with, and we again reduce to case 1.

Case 2: $\mathcal{L}$, Γ are both finite sets.

We will do this proof in parts.

Claim 1: Let $p \in \Gamma$. If $\mathcal{L}_p$ denotes the set of all lines passing through p, then $|\mathcal{L}_p|$ is independent of our choice of p.

Proof of Claim 1: Let $p, q \in \Gamma$ be two distinct points. It suffices to show that $|\mathcal{L}_p| = |\mathcal{L}_q|$. By axiom (1), $\exists!$ line $\overline{pq}$ passing through p, and q. Now, by axiom (4), $\exists$ a point r on $\overline{pq}$ distinct from p and q. Let $l \in \mathcal{L}_p - \{\overline{pq}\}$, $m \in \mathcal{L}_r - \{\overline{pq}\}$. By axiom (2), l and m are distinct. By axiom (2), $l \cap m$ is a single point which is clearly not on $\overline{pq}$. Let $\Gamma_{\mathbb{P}-\overline{pq}}$ denote the set of points of $\mathbb{P}$ not on $\overline{pq}$. Then, we get a map
$\varphi : (\mathcal{L}_p - \{\overline{pq}\}) \times (\mathcal{L}_r - \{\overline{pq}\}) \longrightarrow \Gamma_{\mathbb{P}-\overline{pq}}$ given by $\varphi(l, m) = l \cap m$.

Note that $\Gamma_{\mathbb{P}-\overline{pq}} \neq \emptyset$ by axiom (3). The map φ is a bijection with inverse $\varphi^{-1} : \Gamma_{\mathbb{P}-\overline{pq}} \longrightarrow (\mathcal{L}_p - \{\overline{pq}\}) \times (\mathcal{L}_r - \{\overline{pq}\})$, $\varphi^{-1}(s) = (\overline{ps}, \overline{rs})$.

Thus, $(|\mathcal{L}_p| - 1)(|\mathcal{L}_r| - 1) = |\Gamma_{\mathbb{P}-\overline{pq}}|$. One can similarly show that $(|\mathcal{L}_q| - 1)(|\mathcal{L}_r| - 1) = |\Gamma_{\mathbb{P}-\overline{pq}}|$. This proves that $|\mathcal{L}_p| = |\mathcal{L}_q|$.

Let us denote the number of lines through any point, which is a constant, by c.

Claim 2: Let $l \in \mathcal{L}$. Let Γ_l denote the set of points in $\mathbb{P}$ passing through l. Then $|\Gamma_l|$ is independent of l, and $|\Gamma_l| = c$ for all $l \in \mathcal{L}$.

Proof of claim 2: Let p be a point not on l. Again, such a p exists by axiom (3). In particular $l \notin \mathcal{L}_p$. Define a map $\chi : \mathcal{L}_p \longrightarrow \Gamma_l$, where $\chi(m) = l \cap m$.
Then χ has inverse $\chi^{-1} : \Gamma_l \longrightarrow \mathcal{L}_p$, where $\chi^{-1}(s) = \overline{ps}$. So, $|\Gamma_l| = |\mathcal{L}_p| = c$ by claim 1. Since l was arbitrary, this proves claim 2.

We are not in a position to prove Case 2. We will basically count the number of points and the number of lines and show that these two numbers agree.

Let $p, q \in \Gamma$ be two distinct points. Note that $\Gamma = \Gamma_{\mathbb{P}-\overline{pq}} \cup \Gamma_{\overline{pq}}$, where $\Gamma_{\mathbb{P}-\overline{pq}} \cap \Gamma_{\overline{pq}} = \emptyset$. Now by claim 1, $|\Gamma_{\mathbb{P}-\overline{pq}}| = (c-1)(c-1)$ and by claim 2, $|\Gamma_{\overline{pq}}| = c$. So, $|\Gamma| = (c-1)(c-1)+c = c(c-1)+1$.

On the other hand, let $l \in \mathcal{L}$. Then $\mathcal{L} = \left(\coprod_{q\in\Gamma_l}(\mathcal{L}_q - \{l\})\right)\coprod\{l\}$ (I use the disjoint union symbol just to emphasize that the sets are mutually disjoint). By claim 2, $|\Gamma_l| = c$ and by claim 1, $|\mathcal{L}_q - \{l\}| = c-1$, for all $q \in \Gamma_l$. Thus, $|\mathcal{L}| = c(c-1)+1$.