EXERCISE 20

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Exercise 20: Find a pair of conics intersecting in exactly 4 points.

Proof: We will work over $\mathbb{C}$. Consider the conic C_F given by the homogeneous polynomial $F = YZ - X^2$. The affine part of this conic (i.e., when $Z = 1$) is just the parabola $Y = X^2$ (at least over $\mathbb{R}$). Consider the conic C_G given by the polynomial $G = (X+Y-Z)(-X+Y-Z) = (Y-Z)^2 - X^2$. The affine part of this conic are the two lines $X+Y-1 = 0$ and $-X+Y-1 = 0$. Note that the polynomials F and G have no common irreducible factors (F itself is irreducible over $\mathbb{C}$). So, in particular Bezout's theorem applies, and the conics C_F and C_G have at most 4 intersection points counting with multiplicities.

In fact, the affine parts of these conics intersect at 4 distinct points over $\mathbb{R}$, the points of intersection being $(\frac{-1+\sqrt{5}}{2}, (\frac{-1+\sqrt{5}}{2})^2)$, $(\frac{-1-\sqrt{5}}{2}, (\frac{-1-\sqrt{5}}{2})^2)$, $(\frac{1-\sqrt{5}}{2}, (\frac{1-\sqrt{5}}{2})^2)$, $(\frac{1+\sqrt{5}}{2}, (\frac{1+\sqrt{5}}{2})^2)$. So, the conics intersect in $\mathbb{P}^2_{\mathbb{C}}$ at the 4 distinct points $[\frac{-1+\sqrt{5}}{2} : (\frac{-1+\sqrt{5}}{2})^2 : 1]$, $[\frac{-1-\sqrt{5}}{2} : (\frac{-1-\sqrt{5}}{2})^2 : 1]$, $[\frac{1-\sqrt{5}}{2} : (\frac{1-\sqrt{5}}{2})^2 : 1]$, $[\frac{1+\sqrt{5}}{2} : (\frac{1+\sqrt{5}}{2})^2 : 1]$.