Proof. Consider a singular subic curve $C : F = 0$ in $\mathbb{P}^2$ with singular point p. Then, by a previous exercise, there exists an automorphism $\phi : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ such that $p \mapsto (0 : 0 : 1)$. Dehomogenizing the general form for a cubic by setting $Z = 1$, we obtain the general equation for a cubic curve of the form,

$$F(X, Y, 1) = AX^3 + BX^2Y + CX^2 + DXY^2 + EX + FXY + GY^3 + H + IY^2 + JY$$

Since $(0, 0, 1)$ lies on the cubic, $H = 0$. Moreover, since $(0 : 0 : 1)$ is a singular point,

$$F_X(0, 0, 1) = 3AX^3 + 2BXY + 2CX + DY^2 + E + FY = F = 0$$
$$F_Y(0, 0, 1) = 3GY^2 + 2DXY + J + BX^2 + 2IY + FX = J = 0$$

Thus, we obtain the new cubic equation,

$$F(X, Y, 1) = AX^3 + BX^2Y + CX^2 + DxY^2 + FXY + GY^3 + IY^2$$

Parametrizing by the line $Y = tX$ along the zero set of F, we obtain,

$$\begin{aligned} F(X, tX, 1) &= AX^3 + BX^3t + CX^2 + DX^3t^2 + FX^2t + GX^3t^3 + IX^2t^2 = 0 \\ &= AX + BXt + C + DXt^2 + Ft + GXt^3 + It^2 \end{aligned}$$

Thus we obtain the parametrization,

$$(X, Y) = (-\frac{It^2 + Ft + C}{Gt^3 + Dt^2 + Bt + A}, -\frac{It^3 + Ft^2 + Ct}{Gt^3 + Dt^2 + Bt + A})$$

And finally rehomogenizing using the variable s, we obtain the morphism from $\psi : \mathbb{P}^1 \longrightarrow \mathbb{P}^2$ defined by,

$$(s : t) \mapsto (It^2 + Ft + C, It^3 + Ft^2 + Ct, Gt^3 + Dt^2 + Bt + A)$$

and our parametrization given by $\phi^{-1} \circ \psi : \mathbb{P}^1 \longrightarrow \mathbb{P}^2$. Thus we have constructed a parametrization for any singular cubic curve.

$\square$