EXERCISE 23.5

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Exercise 23.5: Show that the ideal $\mathfrak{I} = \{Q \in k[x, y] | Q(f, g) = 0\}$ is a principal ideal in $k[x, y]$.

Proof: By the well-ordering principle, $\mathfrak{I}$ has a non-zero element Q such that $deg_x(Q)$ is the least among all the other elements of $\mathfrak{I}$, and such that the coefficient of the leading term of Q when regarded as an element of $k[y][x]$ is of the least possible degree. Similarly, $\mathfrak{I}$ has a non-zero element P such that $deg_y(P)$ is least among all the elements of $\mathfrak{I}$, and such that the coefficient of the leading term of P when regarded as an element of $k[x][y]$ is of the least possible degree.

Suppose $deg_x(Q) = 0 = deg_y(P)$. Then $Q \in k[y]$ and $P \in k[x]$. We cannot have $deg(Q) = 0 = deg(P)$, for then Q, P will be non-zero constant polynomials, and so will never satisfy $Q(f, g) = 0 = P(f, g)$. Now, $f, g \in k(t)$. Since, $Q \in k[y] \subset k[t][y]$, and $Q(f, g) = 0$, it follows that g is integral over $k[t]$. Since $k[t]$ is integrally closed in $k(t)$, we must have $g \in k[t]$. Similarly, $f \in k[t]$. But if $deg(f)$ or $deg(g) > 0$ in $k[t]$, then by degree considerations it follows that f, g cannot satisfy $Q(f, g) = 0 = P(f, g)$. On the other hand, if $deg(f) = 0 = deg(g)$, then f, g are constant polynomials in $k[t]$. But, then $x - f \in \mathfrak{I}$, and $y - g \in \mathfrak{I}$. So, $(x - f, y - g) \in \mathfrak{I}$. Since, $\mathfrak{I} \neq (1)$, and $(x - f, y - g)$ is a maximal ideal in $k[x, y]$, it follows that $\mathfrak{I} = (x - f, y - g)$, and in this situation $\mathfrak{I}$ is not principle.

So, we reach the following conclusions from our considerations in the previous paragraph:

(i) If f or $g \in k(t) - k[t]$, then we cannot have $deg_x(Q) = 0 = deg_y(P)$.

(ii) If f and $g \in k[t]$, and $deg(f)$ or $deg(g) > 0$, then we cannot have $deg_x(Q) = 0 = deg_y(P)$.

(iii) If $f, g \in k[t]$, and $deg(f) = 0 = deg(g)$, then $\mathfrak{I}$ is not a principal ideal.

Suppose we are in the situation of (i) or (ii). Then assume without loss of generality that $deg_x(Q) \neq 0$. It is easy to see that Q is irreducible in $k[x, y]$.

Let $R \in \mathfrak{I}$ be a non-zero element. Then $deg_x(Q) \leq deg_x(R)$, by our choice of Q. We can divide R by Q in the ring $k(y)[x]$, giving us $T, S \in k(y)[x]$, such that $R = TQ + S$, and such that either $S \neq 0$ and $\deg(S) < \deg_x(Q)$ or $S = 0$.

Suppose $S \neq 0$. We have, $T = a_n(y)x^n + \ldots + a_0(y)$ and $S = b_m(y)x^m + \ldots + b_0(y)$, where $a_i(y), b_j(y) \in k(y)$, and $m < \deg_x(Q)$. Let d be the common denominator of the rational functions $a_i(y), b_j(y)$. Then multiplying $R = TQ + S$ by d, we get $dR = (dT)Q + (dS)$, where dT and dS are now elements of $k[x, y]$. Note that $d \in k[y]$, so $\deg_x(dS) = m < \deg_x(Q)$. Rearranging, we get $dS = dR - (dT)Q$. Hence, $dS(f, g) = 0$, and so $dS \in I$. But, this is impossible, since $\deg_x(dS) < \deg_x(Q)$. So, we must have that $S = 0$.

So, $Q|R$ in $k(y)[x]$. Since, $k[y]$ is a UFD and Q is a non-constant irreducible polynomial in $k[y][x]$ (this where the assumption that $\deg_x(Q) > 0$ is used), we must have $Q|R$ in $k[y][x]$ by Gauss's Lemma. So, $R \in (Q)$. This proves that $\mathfrak{I} = (Q)$.