EXERCISE 29

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Exercise 29: Let $R = k[t]$ for a field k considered as a graded ring with usual grading. Let $M = \bigoplus_{n \in \mathbb{Z}} M_n$ be a graded R module. If M is finitely generated as an R module, then
(a) $M_n = 0$ for all $n << 0$.
(b) For all $n \in \mathbb{Z}$, $\dim_k(M_n) < \infty$.

Proof: (a) By Exercise 28 it follows that if M is finitely generated, then it can be generated as an R module by a finite set of homogeneous elements. Let $\{m_1, ..., m_t\}$ be a finite set of homogeneous generators for M. Let d be the least degree of the generators of this set. Since multiplication by an element of R just raises the degree, it follows that for all $n < d$, $M_n = 0$.

(b) Let $n \in \mathbb{Z}$. Let $\Gamma = \{m_{i_1}, ..., m_{i_s}\} \subset \{m_1, ..., m_t\}$ be the subset of the generating set, consisting of elements of degree less than or equal to n. Note that Γ could be the empty set, in which case $M_n = 0$, and hence trivially finite dimensional over k. Suppose m_{i_j} has degree d_j for all $j \in \{1, ..., s\}$. Then it is easy to see that M_n is generated as a k-vector space by $\{t^{n-d_1} m_{i_1}, t^{n-d_2} m_{i_2}, ..., t^{n-d_s} m_{i_s}\}$. Thus, $\dim_k(M_n) < \infty$. This method can be generalized to the case where $R = k[t_1, ..., t_m]$.