EXERCISE 37

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Let $R = \mathbb{F}_2[S, T]$. Let X be the Fermat hypersurface. Let $G_0, ..., G_5$ be homogeneous polynomials of degree d such that $gcd(G_0, ..., G_5) = 1$, and $G_0^5 + ... + G_5^5 = 0$. Let $\varphi = (G_0, ..., G_5) : \mathbb{P}^1 \longrightarrow \mathbb{P}^5$.

We define,

$\Omega_X(\varphi) := ker(R^{\oplus 6}(-d) \xrightarrow{G_0,...,G_5} R)$. As a module, $ker(R^{\oplus 6}(-d) \xrightarrow{G_0,...,G_5} R) = ker(R^{\oplus 6} \xrightarrow{G_0,...,G_5} R(6d))$.We will work with this latter module.

$E_X(\varphi) := ker(R^{\oplus 6}(d) \xrightarrow{G_0^4,...,G_5^4} R(5d)$. As a module, $ker(R^{\oplus 6}(d) \xrightarrow{G_0^4,...,G_5^4} R(5d) = ker(R^{\oplus 6} \xrightarrow{G_0^4,...,G_5^4} R(-d))$. We will work with this latter module.

We know that both $\Omega_X(\varphi)$ and $E_X(\varphi)$ are free R modules of rank 5.

Because we are working over $\mathbb{F}_2$, if $(A_0, ..., A_5) \in \Omega_X(\varphi)$, then $A_0G_0 + ... + A_5G_5 = 0$. so $A_0^4G_0^4 + ... + A_5^4G_5^4 = 0$. Hence, $(A_0, ..., A_5) \in \Omega_X(\varphi) \Rightarrow (A_0^4, ..., A_5^4) \in E_X(\varphi)$. Let $\mathfrak{I} = \{(A_0^4, ..., A_5^4) : (A_0, ..., A_5) \in ker(R^{\oplus 6} \xrightarrow{G_0,...,G_5} R(6d))\}$.

Claim: If $(B_0, ..., B_5) \in E_X(\varphi)$ is homogeneous, then $(B_0, ..., B_5)$ can be written as a finite sum of elements of $\mathfrak{I}$ with coefficients in R.

Proof of Claim: First note that if $G \in R$ is homogeneous, then the power of S, T in each term of G^4 is a multiple of 4. This is one of the perks of working over $\mathbb{F}_2$. Suppose $deg((B_0, ..., B_5)) = D($ this means that $deg(B_i) = D$, for $i = \{0, ..., 5\})$. Then $D \equiv 0, 1, 2, 3 (mod 4)$.

Case 1: $d \equiv 0 (mod 4)$ In this case $D = 4k$. We have,
$S^{4k} = (S^k)^4$
$S^{4k-1}T = (S^{k-1})^4S^3T$
$S^{4k-2}T^2 = (S^{k-1})^4S^2T^2$
$S^{4k-3}T^3 = (S^{k-1})^4ST^3$
$S^{4(k-1)}T^4 = (S^{k-1}T)^4$
$S^{4(k-1)-1}T^5 = (S^{k-2}T)^4S^3T$
....
....

It is then easy to see that every degree D element of R is of the form $a^4 + b^4S^3T + c^4S^2T^2 + d^4ST^3$, for homogeneous $a, b, c, d \in R$.

So, for all $i \in \{0, ...5\}$, $B_i = a_i^4 + b_i^4S^3T + c_i^4S^2T^2 + d_i^4ST^3$. Since, $B_0G_0^4 + ... + B_5G_5^4 = 0$, we have $\sum_{0\leq i\leq 5}(a_iG_i)^4 + (b_iG_i)^4S^3T + (c_iG_i)^4S^2T^2 + (d_iG_i)^4ST^3 = (\sum_i a_iG_i)^4 + (\sum_i b_iG_i)^4S^3T + (\sum_i c_iG_i)^4S^2T^2 + (\sum_i d_iG_i)^4ST^3 = 0$.

It suffices to show that $\sum_i a_iG_i = \sum_i b_iG_i = \sum_i c_iG_i = \sum_i d_iG_i = 0$. Note that a_iG_i are all homogeneous of the same degree, so that $\sum_i a_iG_i$ is homogenous. Similarly, $\sum_i b_iG_i$, $\sum_i c_iG_i$, $\sum_i d_iG_i$ are homogeneous. So, to prove that these sums are 0, it suffices to prove that if for homogeneous $\alpha, \beta, \gamma, \delta \in R$, we have $\alpha^4 + \beta^4S^3T + \gamma^4S^2T^2 + \delta^4ST^3 = 0$, then $\alpha = \beta = \gamma = \delta = 0$. But, by an earlier remark, the degree of T in every term of α^4 is a multiple of 4, the degree of T in every term of β^4S^3T is congruent to 1 mod 4, the degree of T in every term of $\gamma^4S^2T^2$ is congruent to 2 mod 4, the degree of T in every term of δ^4ST^3 is congruent to 3 mod 4. Hence, no polynomial of the form $\alpha^4 + \beta^4S^3T + \gamma^4S^2T^2 + \delta^4ST^3$ can possibly equal 0, unless $\alpha, \beta, \gamma, \delta$ are 0.

So, $\sum_i a_iG_i = \sum_i b_iG_i = \sum_i c_iG_i = \sum_i d_iG_i = 0 \Rightarrow (a_0, ..., a_5), (b_0, ..., b_5), (c_0, ..., c_5), (d_0, ..., d_5) \in \ker(R^{\oplus 6} \xrightarrow{G_0,...,G_5} R(6d)) \Rightarrow (a_0^4, ..., a_5^4), (b_0^4, ..., b_5^4), (c_0^4, ..., c_5^4), (d_0^4, ..., d_5^4) \in \mathfrak{I}$. Now, $(B_0, ..., B_5) = (a_0^4, ..., a_5^4 + S^3T(b_0^4, ..., b_5^4) + S^2T^2(c_0^4, ..., c_5^4) + ST^3(d_0^4, ..., d_5^4)$, so we are done.

Case 2: $D \equiv 1(\mathrm{mod} 4)$ In this case $D = 4k + 1$ for some k. By a method similar to the one above one can show that every degree D homogeneous polynomial is of the form $a^4S + b^4T + c^4S^3T^2 + d^4S^2T^3$. Again, $\alpha^4S + \beta^4T + \gamma^4S^3T^2 + \delta^4S^2T^3 = 0 \Rightarrow \alpha = \beta = \gamma = \delta = 0$. So, we get that $(B_0, ..., B_5) \in \mathfrak{I}$ by a process similar to the one above.

Case 3: $D \equiv 2(\mathrm{mod} 4)$ In this case $D = 4k + 2$ for some k. Every homogenous polynomial of degree D is of the form $a^4S^2 + b^4ST + c^4T^2 + d^4S^3T^3$, and we imitate the argument for Case 1.

Case 4: $D \equiv 3(\mathrm{mod} 4)$ In this case every homogenous polynomial of degree D is of the form $a^4S^3 + b^4S^2T + c^4ST^2 + d^4T^3$, and again we imitate argument for Case 1.

So, what we have shown is that every homogeneous element of $\ker(R^{\oplus 6} \xrightarrow{G_0^4,...,G_5^4} R(-d))$ is in the R submodule generated by $\mathfrak{I}$. Hence, $\ker(R^{\oplus 6} \xrightarrow{G_0^4,...,G_5^4} R(-d)) \subset R < \mathfrak{I} >$. But, $R < \mathfrak{I} > \subset \ker(R^{\oplus 6} \xrightarrow{G_0^4,...,G_5^4} R(-d))$. So, $\ker(R^{\oplus 6} \xrightarrow{G_0^4,...,G_5^4} R(-d)) = R < \mathfrak{I} >$. But, what does this means in terms of the generators of $\Omega_X(\varphi)$ and $E_X(\varphi)$. Well, if $Q_i = (Q_{i0}, ..., Q_{i5})$ $(i = \{1, 2, 3, 4, 5\})$ form a free basis of $\ker(R^{\oplus 6} \xrightarrow{G_0,...,G_5} R(6d))$, then $Z_i = (Q_{i0}^4, ..., Q_{i5}^4) \in \mathfrak{I}$, and moreover every element of $\mathfrak{I}$ is an R-linear combination of the element Z_i. Since, $\ker(R^{\oplus 6} \xrightarrow{G_0^4,...,G_5^4} R(-d)) = R < \mathfrak{I} >$, it follows that every element of $\ker(R^{\oplus 6} \xrightarrow{G_0^4,...,G_5^4}$

$R(-d))$ is an R linear combination of the element of $\mathfrak{I}$, and hence of the Z_i. So, $Z_1, \ldots., Z_5$ generate $\ker(R^{\oplus 6} \xrightarrow{G_0^4,\ldots,G_5^4} R(-d))$. Now, we know that $\ker(R^{\oplus 6} \xrightarrow{G_0^4,\ldots,G_5^4} R(-d)) \cong R^{\oplus 5}$. So, localizing we get $(R-0)^{-1}(\ker(R^{\oplus 6} \xrightarrow{G_0^4,\ldots,G_5^4} R(-d))) \cong (R-0)^{-1}(R^{\oplus 5})$. The latter is a vector space over $\mathsf{Frac}(R)$ of dimension 5. Hence, $(R-0)^{-1}(\ker(R^{\oplus 6} \xrightarrow{G_0^4,\ldots,G_5^4} R(-d)))$ is a vector space over $\mathsf{Frac}(R)$ of dimension 5. Since, $\frac{Z_i}{1}$ in $(R-0)^{-1}(\ker(R^{\oplus 6} \xrightarrow{G_0^4,\ldots,G_5^4} R(-d)))$ generate $(R-0)^{-1}(\ker(R^{\oplus 6} \xrightarrow{G_0^4,\ldots,G_5^4} R(-d)))$, it follows that $\frac{Z_i}{1}$ are linearly independent over $\mathsf{Frac}(R)$. As R is a domain, $R \subset \mathsf{Frac}(R)$. So, $\frac{Z_i}{1}$ are linearly independent over R. Clearly then the Z_i are linearly independent over R, completing the proof.