Projective Planes

Exercise 1: Prove that the axiomatic projective plane has the same number of points as lines.

Pf: Apart from the 2 axioms of the projective plane mentioned in the notes, we assume the following additional axioms –

(3) A projective plane has atleast 3 non-collinear points

(4) Any line in the projective plane passes through at least 3 distinct points.

We will denote our projective plane by $\mathbb{P}$ and define

$$\mathcal{L} := \{\text{lines in } \mathbb{P}\}$$
$$\mathcal{T} := \{\text{points in } \mathbb{P}\}$$

We divide the proof into 2 cases:

Case 1: $\mathcal{L}$, $\mathcal{T}$ are both infinite sets

pf of Case 1: Let $\Delta_{\mathcal{L}}$, $\Delta_{\mathcal{T}}$ denote the diagonal of $\mathcal{L}\times\mathcal{L}$ and $\mathcal{T}\times\mathcal{T}$ respectively.

We need to show that $\mathcal{L}$ and $\mathcal{T}$ have the same cardinality.

It is easy to see that

$$|\mathcal{L}| = |\mathcal{L}\times\mathcal{L}| = |\mathcal{L}\times\mathcal{L} - \Delta_{\mathcal{L}}|$$
$$|\mathcal{T}| = |\mathcal{T}\times\mathcal{T}| = |\mathcal{T}\times\mathcal{T} - \Delta_{\mathcal{T}}|$$

By axioms (1), (2) of the axiomatic projective plane we have natural maps

$$\pi_1 : \mathcal{L}\times\mathcal{L} - \Delta_{\mathcal{L}} \longrightarrow \mathcal{P}$$
$$(l_1, l_2) \longmapsto l_1 \cap l_2$$

$$\pi_2 : \mathcal{P}\times\mathcal{P} - \Delta_{\mathcal{P}} \longrightarrow \mathcal{L}$$
$$(p, q) \longmapsto \overline{pq}$$

where $\overline{pq}$ denotes the unique line through p and q.

Let us show that π_1, π_2 are surjective.

If $p \in \mathcal{P}$, then by axiom (3), and the fact that $\mathcal{P}$ is infinite, $\exists$ distinct points q and r such that p, q, r are not collinear. Then clearly $\overline{pq} \neq \overline{pr}$ and $\pi_1(\overline{pq}, \overline{pr}) = p$. This shows that π_1 is surjective.

Let $l \in \mathcal{L}$. Then by axiom (4), l has atleast 2 distinct points p, q on it. Again, clearly $\pi_2(p,q) = \overline{pq} = l$. So, π_2 is surjective.

π_1 surjective $\Longrightarrow |\mathcal{P}| \leq |\mathcal{L}\times\mathcal{L} - \Delta_{\mathcal{L}}| = |\mathcal{L}|$

π_2 surjective $\Longrightarrow |\mathcal{L}| \leq |\mathcal{P}\times\mathcal{P} - \Delta_{\mathcal{P}}| = |\mathcal{P}|$

Thus, $|\mathcal{P}| \leq |\mathcal{L}| \leq |\mathcal{P}| \Longrightarrow |\mathcal{L}| = |\mathcal{P}|$

Note that if $\mathcal{L}$ is infinite then by axioms (1) and (4), $\mathcal{P}$ must be infinite and we reduce to case 1. (Here axiom (4) is used in the sense that it guarantees that every line has a point on it)

If $\mathcal{P}$ is infinite, suppose that $\mathcal{L}$ is finite. By axioms (3) and (1), every point lies on some line. So, $\exists\, l \in \mathcal{L}$ such that l has infinitely many points on it. But, by axiom (3) again,

$\exists\ p \in \mathcal{T}$ such that $p \notin l$. But then for any $q \in \mathcal{T}$ such that $q \in l$, we have a line $\overline{pq}$ which is distinct from l, and by axiom (2), if $q, q' \in \mathcal{T}$ such that $q \neq q'$ and $q, q' \in l$, then $\overline{pq} \neq \overline{pq'}$. So, this gives us infinitely many distinct lines through p intersecting l Thus, $\mathcal{L}$ is infinite, a contradiction. So, $\mathcal{L}$ must have been infinite to begin with, and we again reduce to Case 1.

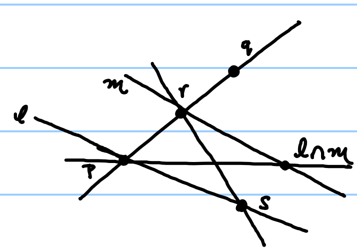
<u>Case 2</u> : $\mathcal{L}, \mathcal{T}$ are both finite sets.

We will do this proof in parts.

<u>Claim 1</u>: Let $p \in \mathcal{T}$. If $\mathcal{L}_p$ denotes the set of all lines passing through p, then $\# \mathcal{L}_p$ is independent of our choice of p.

<u>Pf of claim 1</u>: Let $p, q \in \mathcal{T}$ be two distinct points. It suffices to show that $\# \mathcal{L}_p = \# \mathcal{L}_q$.

By axiom (1), $\exists !$ line $\overline{pq}$ passing through p and q.



Now by axiom (4), $\exists$ a point r on $\overline{pq}$ distinct from p and q. Let $l \in \mathcal{L}_p - \{\overline{pq}\}$, $m \in \mathcal{L}_r - \{\overline{pq}\}$. By axiom (2), l and m are distinct. By axiom (2), $l \cap m$ is a single point which is clearly not on $\overline{pq}$. Let $T_{\mathbb{P}-\overline{pq}}$ denote the set of points of $\mathbb{P}$ not on $\overline{pq}$. Then we get a map

$$\varphi : (\mathcal{L}_p - \{\overline{pq}\}) \times (\mathcal{L}_r - \{\overline{pq}\}) \longrightarrow T_{\mathbb{P}-\overline{pq}}$$
$$(l \quad , \quad m) \longmapsto l \cap m$$

Note that $T_{\mathbb{P}-\overline{pq}} \neq \phi$ by axiom (3). The map φ is a bijection with inverse

$$\tau : T_{\mathbb{P}-\overline{pq}} \longrightarrow (\mathcal{L}_p - \{\overline{pq}\}) \times (\mathcal{L}_r - \{\overline{pq}\})$$
$$s \longmapsto (\overline{ps} \quad , \quad \overline{rs})$$

Thus, $(\# \mathcal{L}_p - 1)(\# \mathcal{L}_r - 1) = \# T_{\mathbb{P}-\overline{pq}}$. One can similarly show that

$$(\# \mathcal{L}_q - 1)(\# \mathcal{L}_r - 1) = \# T_{\mathbb{P}-\overline{pq}}.$$

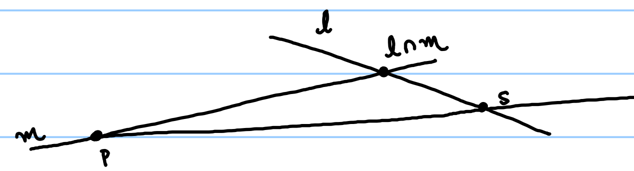
Thus, $\# \mathcal{L}_p - 1 = \# \mathcal{L}_q - 1 \implies \# \mathcal{L}_p = \# \mathcal{L}_q$

This proves claim 1.

Let us denote the number of lines through any point, which is a constant, by c.

Claim 2: Let $l \in \mathcal{L}$. Let T_l denote the set of points on $\mathbb{P}$ passing through l. Then $\# T_l$ is irrespective of l, and $\# T_l = c$ for all $l \in \mathcal{L}$.

Pf of claim 2: Let p be a point not on $\mathcal{L}$. Again such a p exists by axiom 3.



In particular $\ell \notin \mathcal{L}_p$. Define a map

$$\begin{aligned} \chi : \mathcal{L}_p &\longrightarrow \mathcal{P}_\ell \\ m &\longmapsto \ell \cap m \end{aligned}$$

Then χ has inverse

$$\begin{aligned} \chi^{-1} : \mathcal{P}_\ell &\longrightarrow \mathcal{L}_p \\ s &\longmapsto \overline{ps} \end{aligned}$$

So, $\# \mathcal{P}_\ell = \# \mathcal{L}_p = c$ by claim 1. Since ℓ was arbitrary, this proves claim 2.

We are now in a position to prove Case 2. We will basically count the number of points and the number of lines and show that these two numbers agree.

Let $p, q \in \mathcal{P}$ be 2 distinct points. Note that

$$\mathcal{P} = \mathcal{P}_{\mathbb{P}-\overline{pq}} \cup \mathcal{P}_{\overline{pq}} \quad \text{where } \mathcal{P}_{\mathbb{P}-\overline{pq}} \cap \mathcal{P}_{\overline{pq}} = \phi.$$

Now, by claim 1, $\# \mathcal{P}_{\mathbb{P}-\overline{pq}} = (c-1)(c-1)$ and by claim 2, $\# \mathcal{P}_{\overline{pq}} = c$. So,

$$\# \mathcal{P} = (c-1)(c-1) + c = c^2 - 2c + 1 + c = c(c-1) + 1.$$

On the other hand, let $\ell \in \mathcal{L}$.

Then $\mathcal{L} = \left(\coprod_{q \in \mathcal{P}_\ell} (\mathcal{L}_q - \{\ell\}) \right) \amalg \{\ell\}$ (I use the disjoint union symbol just to emphasize that the sets are mutually disjoint)

Now, by claim 2 $\# T_\ell = c$ and by claim 1, $\# \mathcal{L}_q - \{\ell\} = c-1$. Thus,

$$\# \mathcal{L} = c(c-1)+1.$$

$\boxtimes$