BOUNDS ON COMPLEXES

Contents

1. Introduction

We try to understand Neeman's paper "The categories $\mathcal{T}^c$ and $\mathcal{T}^b_c$ determine each other" in the case of derived category of quasi-coherent modules on quasi-compact and quasi-separated schemes and on Noetherian schemes.

2. Preliminaries on triangulated categories

Let $\mathcal{D}$ be an additive category. A strictly full subcategory $\mathcal{A} \subset \mathcal{D}$ is *closed under taking summands* if given an isomorphism $A \cong B \oplus C$ with A in $\mathcal{A}$, then B and C are in $\mathcal{A}$.

Let $\mathcal{D}$ be a triangulated category. A strictly full subcategory $\mathcal{A} \subset \mathcal{D}$ is *closed under extensions* if given a distinguished triangle $A \to B \to C \to A[1]$ in $\mathcal{D}$ with A and C in $\mathcal{A}$, then B is in $\mathcal{A}$.

Let $\mathcal{D}$ be a triangulated category. Let E be an object of $\mathcal{D}$. Then for $-\infty \leq a \leq b \leq \infty$ we denote

$$\langle E \rangle^{[a,b]}$$

the smallest strictly full additive subcategory of $\mathcal{D}$ which is closed under taking summands, closed under extensions, and contains $E[-i]$ for $a \leq i \leq b$. Please note the minus sign.

Let $\mathcal{D}$ be an additive category with arbitrary direct sums. A strictly full subcategory $\mathcal{A} \subset \mathcal{D}$ is *closed under arbitrary direct sums* if given a set I and objects A_i, $i \in I$ of $\mathcal{A}$ the direct sum $\bigoplus_{i \in I} A_i$ is in $\mathcal{A}$.

Let $\mathcal{D}$ be a triangulated category with arbitrary direct sums. Let E be an object of $\mathcal{D}$ Then for $-\infty \leq a \leq b \leq \infty$ we denote

$$\langle E \rangle^{[a,b],big}$$

the smallest additive subcategory of $\mathcal{D}$ which is closed under arbitrary direct sums, closed under taking summands, closed under extensions, and contains $E[-i]$ for $a \leq i \leq b$.

Lemma 2.1. *Let $\mathcal{D}$ be a triangulated category. Let E and M be objects of $\mathcal{D}$. For $-\infty \leq a \leq b \leq \infty$ if $\mathrm{Hom}(E[-i], M) = 0$ for $i \in [a,b]$, then $\mathrm{Hom}(P, M) = 0$ for all $P \in \langle E \rangle^{[a,b]}$.*

Proof. Assume $\mathrm{Hom}(E[-i], M) = 0$ for $i \in [a,b]$. The full subcategory $\mathcal{D}'$ consisting of objects X of $\mathcal{D}$ such that $\mathrm{Hom}(X, M) = 0$ is a strictly full additive subcategory of $\mathcal{D}$ which is closed under taking summands, closed under extensions, and contains $E[-i]$ for $a \leq i \leq b$. Hence $\mathcal{D}'$ contains $\langle E \rangle^{[a,b]}$ as desired. □

Lemma 2.2. *Let $\mathcal{D}$ be a triangulated category with arbitrary direct sums. Let E and M be objects of $\mathcal{D}$. For $-\infty \leq a \leq b \leq \infty$ if $\mathrm{Hom}(E[-i], M) = 0$ for $i \in [a,b]$, then $\mathrm{Hom}(P, M) = 0$ for all $P \in \langle E \rangle^{[a,b],big}$.*

Proof. Assume $\mathrm{Hom}(E[-i], M) = 0$ for $i \in [a,b]$. The full subcategory $\mathcal{D}'$ consisting of objects X of $\mathcal{D}$ such that $\mathrm{Hom}(X, M) = 0$ is a strictly full additive subcategory of $\mathcal{D}$ which is closed under arbitrary direct sums, closed under taking summands, closed under extensions, and contains $E[-i]$ for $a \leq i \leq b$. Hence $\mathcal{D}'$ contains $\langle E \rangle^{[a,b],big}$ as desired. □

3. Bounding quasi-coherent complexes

Let U be an affine scheme. Let $Z \subset U$ be a closed subset. The phrase "a Koszul complex $K(U, f_1, \ldots, f_r)$ for Z" means that $f_1, \ldots, f_r \in \Gamma(U, \mathcal{O}_U)$, that $Z = V(f_1, \ldots, f_r)$ (set theoretically), and that $K(U, f_1, \ldots, f_r)$ is the object of $D(\mathcal{O}_U)$ corresponding to the complex

$$\wedge^r(\mathcal{O}_U^{\oplus r}) \to \wedge^{r-1}(\mathcal{O}_U^{\oplus r}) \to \wedge^{r-2}(\mathcal{O}_U^{\oplus r}) \to \ldots \to \mathcal{O}_U^{\oplus r} \xrightarrow{f_1, \ldots, f_r} \mathcal{O}_U$$

with the last term sitting in degree 0. See Tag 062J. In particular, we have

$$H^0(K(U, f_1, \ldots, f_r)) = \mathcal{O}_U/(f_1, \ldots, f_r) \quad \text{and} \quad H^i(K(U, f_1, \ldots, f_r)) = 0, \ i > 0$$

Lemma 3.1. *Let X be a quasi-compact and quasi-separated scheme and $X = U \cup V$ where U is affine open and V is quasi-compact open. Denote $Z = X \setminus V$. Then*

1. *there exists a perfect object K of $D(\mathcal{O}_X)$ which restricts to a Koszul complex $K(U, f_1, \ldots, f_r)$ for Z on U and vanishes outside of U,*
2. *the category $\langle K \rangle^{[0,0]}$ contains objects K_n which restrict to the Koszul complexes $K(U, f_1^n, \ldots, f_r^n)$ on U, and*
3. *any M in $D_{QCoh}^{\leq 0}(\mathcal{O}_X)$ supported on Z is contained in $\langle K \rangle^{[-\infty,0],big}$.*

Proof. Proof of (1). Observe that Z is contained in U. Since V is quasi-compact open and X is quasi-separated, we see that $U \cap V$ is quasi-compact. Hence we can find $f_1, \ldots, f_r \in \Gamma(U, \mathcal{O}_U)$ such that that $Z = V(f_1, \ldots, f_r)$ set theoretically, see Tag 00F6. Consider the corresponding Koszul complex $K(U, f_1, \ldots, f_r)$. Exactly as in the proof of Tag 08EP we find a perfect object K of X whose restriction to U is $K(U, f_1, \ldots, f_r)$.

Proof of (2). Observe that for objects F, G of $D(\mathcal{O}_X)$ whose cohomology sheaves are supported on Z we have $\mathrm{Hom}(F,G) = \mathrm{Hom}(F|_U, G|_U)$. Hence using the distinguished triangles

$$K(U, f_1^{e_1}, \ldots, f_r^{e_r}) \to K(U, f_1^{e_1}, \ldots, f_i^{e_i+e'_i}, \ldots, f_r^{e_r}) \to K(U, f_1^{e_1}, \ldots, f_i^{e'_i}, \ldots, f_r^{e_r}) \to$$

(see proof of Tag 09IR) and an obvious induction we conclude that for every $e_1, \ldots, e_r \geq 1$ there is a perfect object of $D(\mathcal{O}_X)$ contained in $\langle K \rangle^{[0,0]}$ and restricting to $K(U, f_1^{e_1}, \ldots, f_r^{e_r})$.

Proof of (3). As above observe that $H^0(X, M) = H^0(U, M|_U)$ and that $H^0(M)$ is generated by global sections (small detail omitted). For any element $s \in H^0(M)$ there is an $n > 0$ and a map $K_n \to M$ sending the canonical element in $H^0(X, K_n)$ to s, see Tag 08E3. Thus we can find a set J, integers $n_j \geq 1$, and a map

$$L = \bigoplus\nolimits_{j \in J} K_{n_j} \longrightarrow M$$

in $D_{QCoh}(\mathcal{O}_X)$ which gives a surjection on cohomology sheaves in degree 0. Take the cone M_1 of this map. Observe that M_1 is in $D^{\leq -1}_{QCoh}(\mathcal{O}_X)$ and is supported on Z. Thus we can continue finding distinguished triangles

$$L_i \to M_i \to M_{i+1} \to L_i[1]$$

where L_i is a direct sum of complexes of the form $K_n[i]$. Note the shifts by non-negative integers. For notational convenience we set $M_0 = M$ and $L_0 = L$. Then working inductively (this is a completely standard argument):

(1) for the base case we fit the composition $L_1 \to M_1 \to L_0[1]$ into a distinguished triangle $L_0 \to N_1 \to L_1 \to L_0[1]$ and we choose a map of distinguished triangles

$$\begin{tikzcd} L_0 \arrow[r] \arrow[d] & N_1 \arrow[r] \arrow[d] & L_1 \arrow[r] \arrow[d] & L_0[1] \arrow[d] \\ L_0 \arrow[r] & M_0 \arrow[r] & M_1 \arrow[r] & L_0[1] \end{tikzcd}$$

showing that M_2 is the cone of $N_1 \to M$,

(2) continue inductively we fit the composition $L_n \to M_n \to N_{n-1}[1]$ into a distinguished triangle $N_{n-1} \to N_n \to L_n \to N_{n-1}[1]$ and we choose a map of distinguished triangles

$$\begin{tikzcd} N_{n-1} \arrow[r] \arrow[d] & N_n \arrow[r] \arrow[d] & L_n \arrow[r] \arrow[d] & N_{n-1}[1] \arrow[d] \\ N_{n-1} \arrow[r] & M \arrow[r] & M_n \arrow[r] & N_{n-1}[1] \end{tikzcd}$$

showing that M_{n+1} is the cone of $N_n \to M$.

Observe that N_n is in $\langle K \rangle^{[-n,0],big}$ for all n. Moreover, looking at cohomology sheaves we see that $\mathrm{hocolim} N_n = M$. This means we have a distinguished triangle

$$\bigoplus N_n \to \bigoplus N_n \to M \to \bigoplus N_n[1]$$

and this finishes the proof. $\square$

Lemma 3.2. *Let X be a quasi-compact and quasi-separated scheme and $X = U \cup V$ where U is affine open and V is quasi-compact open. If Theorem 3.3 holds for U and V and $U \cap V$, then it holds for X.*

Proof. Let $T \subset X$ be a closed subset whose complement is quasi-compact. By Tag 0A9A the categories $D_{QCoh,T}(\mathcal{O}_X)$ has a perfect generator E. The result is independent of the choice of our generator, because if E and E' are generators, then $E \in \langle E' \rangle^{[a,b]}$ for some integers $a \leq b$ (because E is compact and hence is contained in $\langle E' \rangle$ by Tag 09SR) and hence

$$\langle E \rangle^{[-\infty,i],big} \subset \langle E' \rangle^{[-\infty,i-a],big}$$

Thus if we prove the result for E, then it follows for E' and vice versa by symmetry.

Set $Z = X \setminus V$. Note that the complement V' of $T' = T \cap Z$ in X is quasi-compact as well. Denote K a perfect object of $D(\mathcal{O}_X)$ which restricts Koszul complex for T' on U and vanishes outside U as in Lemma 3.1 (but with V replaced by V'). By the remarks above we may assume E contains K as a direct summand (after replacing E by $E \oplus K$ if necessary). Denote $j : V \to X$ the given open immersion. Observe that Rj_* has finite cohomological dimension, say q. Consider the canonical distinguished triangle

$$E \to Rj_*(E|_V) \to Q \to E[1]$$

Observe that $Q \in D_{QCoh,T'}(X)$ (because Rj_*E restricts to $E|_V$ on V and because Rj_*E vanishes outside T). Thus Q is in

$$\langle K \rangle^{[-\infty,b+q+1],big} \subset \langle E \rangle^{[-\infty,b+q+1],big}$$

by Lemma 3.1 where b is an integer such that $H^i(E) = 0$ for $i > b$. We conclude that $Rj_*(E|_V[i])$ is in $\langle E \rangle^{[-\infty,b+q+1-i],big}$ for all $i \in \mathbf{Z}$.

Let $i(V)$ be the integer found for $E|_V$ and $T \cap V$ in V by hypothesis. Let M be an object of $D^{\leq 0}_{QCoh,T}(\mathcal{O}_X)$. Then

$$M|_V \in \langle E|_V \rangle^{[-\infty,i(V)],big}$$

Hence we obtain

$$Rj_*(M|_V) \in \langle Rj_*E|_V \rangle^{[-\infty,i(V)],big} \subset \langle E \rangle^{[-\infty,b+q+1+i(V)],big}$$

Again since the cone of $M \to Rj_*M|_V$ is in $\langle E \rangle^{[-\infty,q+1],big}$ we conclude. $\square$

Theorem 3.3. *Let X be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset whose complement is quasi-compact. Let E be a perfect generator for $D_{QCoh,T}(\mathcal{O}_X)$. Then there exists an integer i such that*

$$D^{\leq 0}_{QCoh,T}(\mathcal{O}_X) \subset \langle E \rangle^{[-\infty,i],big}$$

Proof. By Tag 0A9A the categories $D_{QCoh,T}(\mathcal{O}_X)$ always have perfect generators, thus the statement makes sense. We use the induction principle of Tag 08DR. The case of affine scheme is Lemma 3.1. The induction step is Lemma 3.2. This finishes the proof. $\square$

4. Conclusion for compact objects

We are going to explain what Theorem 3.3 means for compact objects.

Lemma 4.1. *Let (A, d) be a differential graded algebra. The category $\langle A \rangle^{[-\infty,0],big}$ is the full subcategory of $D(A, d)$ consisting of objects which can be represented by a differential graded A-module P which has a filtration*

$$0 = F_{-1}P \subset F_0P \subset F_1P \subset \ldots \subset P$$

by differential graded submodules such that $P = \bigcup F_pP$ and the quotients $F_{i+1}P/F_iP$ are isomorphic as differential graded A-modules to a direct sum of $A[n]$ with $n \geq 0$.

Proof. If P is a differential graded A-module with a filtration as in the lemma, then we see that $F_{i+1}P/F_iP$ is contained in $\langle A \rangle^{[-\infty,0],big}$ and hence F_iP is contained in $\langle A \rangle^{[-\infty,0],big}$ and hence $P = \text{hocolim} F_iP$ is contained in $\langle A \rangle^{[-\infty,0],big}$.

To prove the converse, let $\mathcal{D}'$ be the full subcategory whose objects are represented by modules P as in the lemma. It is easy to show that $\mathcal{D}'$ contains $A[n]$ for $n \geq 0$ and is closed under arbitrary direct sums. Suppose that we have a distinguished triangle

$$X \to X' \to X'' \to X[1]$$

with X and X'' in $\mathcal{D}'$. Then choose modules P and P'' as in the statement of the lemma. The assumption means we can choose a basis $e_{i,\lambda}$, $i \geq 0$, $\lambda \in \Lambda_i$ for P and a basis $f_{j,\mu}$, $j \geq 0$, $\mu \in M_j$ for P'' as graded A-modules with the following properties

(1) $\deg(e_{i,\lambda}) \leq 0$,
(2) $\deg(f_{j,\mu}) \leq 0$,
(3) $\mathrm{d}(e_{i,\lambda}) = \sum a_{i,\lambda,i',\lambda'} e_{i',\lambda'}$ with $a_{i,\lambda,i',\lambda'} \neq 0 \Rightarrow i' < i$,
(4) $\mathrm{d}(f_{j,\mu}) = \sum b_{j,\mu,j',\mu'} f_{j',\mu'}$ with $b_{j,\mu,j',\mu'} \neq 0 \Rightarrow j' < j$

Since P'' has property (P), see Tag 09KK, we conclude that X' is reprented by a differential graded A-module P' which is an extension of P'' by P (small detail omitted). If $\tilde{f}_{j,\mu}$ is a lift of $f_{j,\mu}$ to a homogeneous element of P', then we see that $e_{i,\lambda}, \tilde{f}_{j,\mu}$ forms a basis for P'. Thus we may write

$$\mathrm{d}(\tilde{f}_{j,\mu}) = \sum b_{j,\mu,j',\mu'} \tilde{f}_{j',\mu'} + \sum c_{j,\mu,i,\lambda} e_{i,\lambda}$$

for some $c_{j,\mu,i,\lambda} \in A$. Working inductively on $j = 0, 1, 2, \ldots$, we find that there exist functions

$$n_j(-) : M_j \longrightarrow \mathbf{Z}_{\geq 0}$$

such that

$$c_{j,\mu,i,\lambda} \neq 0 \Rightarrow n_j(\mu) > i \quad \text{and} \quad b_{j,\mu,j',\mu'} \neq 0 \Rightarrow n_j(\mu) > n_{j'}(\mu')$$

(To prove existence use that the sums are always finite sums, so that the maximums we are taking are finite.) Then we can define the required filtration on P' by setting

$$F_kP' = \langle e_{i,\lambda},\ i \leq k;\ \tilde{f}_{j,\mu},\ n_j(\mu) \leq k \rangle$$

The reader easily verifies the desired properties.

Finally, we have to show that $\mathcal{D}'$ is closed under taking direct summands. Let X be an object of $\mathcal{D}'$ and let $p : X \to X$ be a projector. Then the image of p is

$$Y = \text{hocolim}(X \xrightarrow{p} X \xrightarrow{p} X \to \ldots)$$

(Recall that any triangulated category with countable direct sums is Karoubian, see Tag 05QW. This formula is the proof of that lemma.) Thus we have a distinguished triangle

$$\bigoplus X \to \bigoplus X \to Y \to \bigoplus X[1]$$

Since $X[1]$ is in $\mathcal{D}'$ and since $\mathcal{D}'$ is closed under direct sums (see above) we conclude because $\mathcal{D}'$ is closed under extensions (see above). □

Remark 4.2. Let $\mathcal{D}$ be a triangulated category with direct sums. Let E be a compact object. It may very well be the case that in this generality the category $\langle E\rangle^{[-\infty,0],big}$ is the full subcategory consisting of objects which can be written as

$$X = \text{hocolim} X_n$$

where X_1 is a direct sum of shifts $E[n]$, $n \geq 0$ and each transition morphism fits into a distinguished triangle $Y_n \to X_n \to X_{n+1} \to Y_n[1]$ where Y_n is a direct sum of shifts $E[n]$, $n \geq -1$. This would be the natural analogue of the result of Lemma 4.1.

Lemma 4.3. *Let (A, d) be a differential graded algebra. Given F in $D(A, d)$ the following are equivalent*

1. *F is compact and contained in $\langle A\rangle^{[-\infty,0],big}$,*
2. *F is in $\langle A\rangle^{[-\infty,0]}$,*
3. *F is a direct summand (in the derived category) of an object represented by a differential graded A-module P which has a finite filtration*

$$0 = F_{-1}P \subset F_0P \subset F_1P \subset \ldots \subset F_dP = P$$

by differential graded submodules such that the quotients $F_{i+1}P/F_iP$ are isomorphic as differential graded A-modules to a finite direct sum of $A[n]$ with $n \geq 0$.

Proof. Observe that (3) implies (2) because P is an d-fold extension of objects which are clearly in $\langle A\rangle^{[-\infty,0]}$.

Observe that every object of $\langle A\rangle^{[-\infty,0]}$ is compact (because we start with a compact object and then take finite direct sums, direct summands, extensions, and repeat). Thus (2) implies (1).

Finally, assume (1). Then F is as in (3) by Lemma 4.1 and exactly the same arguments as those given in the proof of Tag 09R3. □

Lemma 4.4. *Let $\mathcal{D}$ be a triangulated category with direct sums. Let E, F be compact objects. If F is contained in $\langle E\rangle^{[-\infty,i],big}$ then F is contained in $\langle E\rangle^{[-\infty,i]}$.*

Proof. Consider the strictly full subcategory $\mathcal{D}'$ of $\mathcal{D}$ consisting of objects X such that for any compact object G and any morphism $G \to X$ there is a factorization $G \to G' \to X$ where G' is compact and contained in $\langle E\rangle^{[-\infty,i]}$. Clearly $\mathcal{D}'$ contains $E[n]$ for all $n \geq -i$ (this is where we use that E is compact). Equally clearly $\mathcal{D}'$ is closed under direct sums and under taking direct summands. Suppose

$$X \to X' \to X'' \to X[1]$$

is a distinguished triangle with X and X'' contained in $\mathcal{D}'$. Let $G \to X'$ be a morphism with G compact. By assumption we can choose a factorization $G \to G' \to$

X'' with G' in $\langle E\rangle^{[-\infty,i]}$. Choose a distinguished triangle $Q \to G \to G' \to Q[1]$ and a map of distinguished triangles

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & G' \arrow[r] \arrow[d] & Q[1] \arrow[d] \\ X \arrow[r] & X' \arrow[r] & X'' \arrow[r] & X[1] \end{tikzcd}$$

Choose a factorization $Q \to G'' \to X$ with G'' in $\langle E\rangle^{[-\infty,i]}$. This produces a map $G' \to G''[1]$ and hence we may choose a distinguished triangle $G'' \to G''' \to G' \to G''[1]$. Thus G''' is in $\langle E\rangle^{[-\infty,i]}$. Using the axioms of a triangulated category we may choose maps of triangles

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & G' \arrow[r] \arrow[d] & Q[1] \arrow[d] \\ G'' \arrow[r] \arrow[d] & G''' \arrow[r] \arrow[d] & G' \arrow[r] \arrow[d] & G''[1] \arrow[d] \\ X \arrow[r] & X' \arrow[r] & X'' \arrow[r] & X[1] \end{tikzcd}$$

Then the composition $G \to G''' \to X'$ may not be equal to the given map $G \to X$, but the maps agree after composining with $X' \to X''$. Hence these maps differ by a map which is a composition $G \to X \to X'$. Since we can fact the map $G \to X$ through an object of $\langle E\rangle^{[-\infty,i]}$ this finishes the proof of the fact that $\mathcal{D}'$ is closed under extensions.

We conclude that $\langle E\rangle^{[-\infty,i],big}$ is contained in $\mathcal{D}'$. Hence $\mathrm{id} : F \to F$ factors through a (compact) object of $\langle E\rangle^{[-\infty,i]}$ (here we use that F is compact). Thus F is in $\langle E\rangle^{[-\infty,i]}$ and the proof is done. □

Lemma 4.5. *Let X be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset whose complement is quasi-compact. Let E be a perfect generator for $D_{QCoh,T}(\mathcal{O}_X)$. Then there exists an integer i such that*

$$D^{\leq 0}_{QCoh,T,perfect}(\mathcal{O}_X) \subset \langle E\rangle^{[-\infty,i]}$$

First proof. Combine Theorem 3.3 and Lemma 4.4. □

Second proof. Choose a K-injective complex $\mathcal{I}^\bullet$ representing E and denote (A, d) the differential graded algebra

$$(A, \mathrm{d}) = \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}^\bullet, \mathcal{I}^\bullet))$$

Then we get an (explicit) equivalence of $D^{\leq 0}_{QCoh,T}(\mathcal{O}_X)$ with the derived category of $D(A, \mathrm{d})$, see Tag 0DJL. Under this equivalence E corresponds to A. Thus we obtain the result from Lemma 4.3. □

5. Bounding quasi-coherent complexes, II

In this section we prove dual versions of the material in Section 3.

Theorem 5.1. *Let X be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset whose complement is quasi-compact. Let E be a perfect generator for $D_{QCoh,T}(\mathcal{O}_X)$. Then there exists an integer i such that for any perfect M in $D_{QCoh,T}(\mathcal{O}_X)$ if*

$$\mathrm{Hom}(E[j], M) = 0 \text{ for all } j \leq 0$$

then $M \in D^{\leq i}_{QCoh,T}(\mathcal{O}_X)$.

Proof. Perfect generators for $D_{QCoh,T}(\mathcal{O}_X)$ exist, see Tag 0A9A, and thus the statement of the theorem makes sense. Before we start the proof we note that the result is independent of the choice of E (but the value of i does depend on the choice of E).

First assume X is affine. Choose a Koszul complex K for T on X as in Lemma 3.1. Then K is a perfect generator for $D_{QCoh,T}(\mathcal{O}_X)$, see Tag 09IR and the proof of Tag 0A9A. Moreover, the higher Koszul complexes K_n are contained in $\langle K \rangle^{[0,0]}$, see Lemma 3.1. Thus the vanishing of $\mathrm{Hom}(K[j], M)$ implies the vanishing of $\mathrm{Hom}(K_n[j], M)$ for all n, see Lemma 2.1. Then it follows from Tag 08E3. that M has vanishing cohomology in degrees ≥ 0 and this proves the result in this case.

To finish the proof we apply the induction principle of Tag 08DR. To see the induction step is true, suppose that $X = U \cup V$ where U is affine open and V is quasi-compact open. Choose a perfect object K in $D(\mathcal{O}_X)$ which restricts to a Koszul complex for the closed subset $Z = U \setminus V$ on U, see Lemma 3.1. We will use below that the higher Koszul complexes K_n are in $\langle K \rangle^{[0,0]}$, see Lemma 3.1. We also choose a perfect object K' in $D(\mathcal{O}_X)$ which restricts to a Koszul complex for the closed subset $T' = T \cap Z$ on U, see Lemma 3.1. Choose distinguished triangles

$$I_n \to \mathcal{O}_X \to K_n \to I_n[1]$$

This produces distinguished triangles

$$E \otimes^{\mathbf{L}}_{\mathcal{O}_X} I_n \to E \to E \otimes^{\mathbf{L}}_{\mathcal{O}_X} K_n \to E \otimes^{\mathbf{L}}_{\mathcal{O}_X} I_n[1]$$

and we conclude that

$$E \otimes^{\mathbf{L}}_{\mathcal{O}_X} I_n \in \langle E \oplus E \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \rangle^{[-1,0]} \subset \langle E' \rangle^{[-1,0]}$$

for all n where

$$E' = E \oplus E \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \oplus K'$$

is another perfect generator for $D_{QCoh,T}(\mathcal{O}_X)$.

To finish the proof, assume that M is in $D_{QCoh,T}(\mathcal{O}_X)$ such that $\mathrm{Hom}(E'[j], M) = 0$ for $j \leq 0$. Then we conclude that M satisfies

$$\mathrm{Hom}(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} I_n[j], M)$$

for all $j \leq 0$ and all $n \geq 1$, see Lemma 2.1. Observe that we have

$$\begin{aligned} \operatorname{colim} \mathrm{Hom}(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} I_n[j], M) &= \operatorname{colim} \mathrm{Hom}(I_n[j], R\mathcal{H}om(E[j], M)) \\ &= \Gamma(V, R\mathcal{H}om(E[j], M)) \\ &= \mathrm{Hom}(E[j]|_V, M|_V) \end{aligned}$$

The second equality follows from the Mayer-Vietoris sequence for $X = U \cup V$ and $R\mathcal{H}om(E[j], M)$ and the equality in Tag 08DD. We conclude by induction hypothesis that there exists an $i(V)$ such that $H^i(M)|_V = H^i(M|_V) = 0$ for $i >$

$i(V)$. In other words, the cohomology sheaves $H^i(M)$ for $i > i(V)$ are supported on T'.

To conclude we use that K' is a summand of E'. (This part of the argument is the same as the affine case.) Namely, if we denote the higher Koszul complexes K'_n then $K'_n \in \langle K' \rangle^{[0,0]}$ as before. The vanishing of $\mathrm{Hom}(K'_n, M)$ implies that that $H^i(M)$ is zero for $i > i(V) + r'$ where r' is the number of generators used in the construction of the Koszul complex K', see Tag 08E3. Some details omitted; in particular one needs to use the Mayer-Vietoris sequence and use that we already have that $H^i(M)$ is supported on T' for $i \geq i(V)$. □

Lemma 5.2. *Let X be a quasi-compact and quasi-separated scheme. Let $T \subset X$ be a closed subset whose complement is quasi-compact. Then there exists an integer i such that for any perfect M in $D_{QCoh,T}(\mathcal{O}_X)$ with*

$$\mathrm{Hom}(P, M) = 0 \textit{ for all } P \in D^{\geq 0}_{QCoh,T,perf}(\mathcal{O}_X)$$

we have $M \in D^{\leq i}_{QCoh,T}(\mathcal{O}_X)$.

Proof. This is a corollary of the stronger Theorem 5.1. □

Here is the result in the Noetherian case when you let the shifts go the opposite direction from what happens in Theorem 5.1.

Lemma 5.3. *Let X be a Noetherian scheme. Let $E \in D_{perf}(\mathcal{O}_X)$ be a classical generator. Then there exists an integer i such that for $M \in D_{Coh}(\mathcal{O}_X)$ with*

$$\mathrm{Hom}(E[j], M) = 0 \textit{ for all } j \geq 0$$

we have $M \in D^{\geq -i}_{Coh}(\mathcal{O}_X)$.

Proof. Recall that there exists an integer k such that

$$D^{\leq -k}_{perf}(\mathcal{O}_X) \subset \langle E \rangle^{[-\infty,0]}$$

see Lemma 4.5. Hence we know that $\mathrm{Hom}(Q, M) = 0$ for every Q in $D^{\geq -k}_{perf}(\mathcal{O}_X)$. Consider the map

$$\tau_{\leq -k-1} M \longrightarrow M$$

This is a map of pseudo-coherent complexes because X is Noetherian. We have to show $\tau_{\leq -k-1}M$ has vanishing cohomology. If not, then there exists an $m \leq -k-1$ such that $H^m(M) = H^m(\tau_{\leq -k-1}M)$ is nonzero. By Tag 08ES we can choose an approximation

$$Q \longrightarrow \tau_{\leq -k-1} M$$

for the triple $(X, \tau_{\leq -k-1}M, m)$. This means that Q is in $D_{perf}(X)$ and the displayed map induces isomorphisms $H^q(Q) \to H^q(\tau_{\leq -k-1}M)$ on cohomology sheaves for $q > m$ and a surjection $H^m(Q) \to H^m(\tau_{\leq -k-1}M)$ for $q = m$. This is a contradiction because it would mean that Q is in $D^{\leq -k}_{perf}(\mathcal{O}_X)$ and $H^m(Q) \to H^m(M)$ would be surjective and hence nonzero and hence $Q \to M$ would be nonzero. □

6. Neeman: sequences of subcategories

Let $\mathcal{D}$ be a triangulated category. We are going to consider sequences

$$\mathcal{P}_1 \subset \mathcal{P}_2 \subset \mathcal{P}_3 \subset \ldots$$

of strictly full additive subcategories such that for each $i \geq 1$ we have

$$\mathcal{P}_i[1] \cup \mathcal{P}_i \cup \mathcal{P}_i[-1] \subset \mathcal{P}_{i+1}$$

We will say the sequences $\{\mathcal{P}_i\}$ and $\{\mathcal{Q}_j\}$ are equivalent if and only if

(1) for each i there is a j such that $\mathcal{P}_i \subset \mathcal{Q}_j$, and
(2) for each j there is a i such that $\mathcal{Q}_j \subset \mathcal{P}_i$.

In this situation we say that a sequence

$$E_1 \to E_2 \to E_3 \to \ldots$$

of morphisms of $\mathcal{D}$ is a *Cauchy sequence* if and only if for every i there is a N such that for $n \geq N$ the maps

$$\mathrm{Hom}(P, E_n) \to \mathrm{Hom}(P, E_{n+1})$$

are isomorphisms for all P in $\mathcal{P}_i$.

Let $\mathcal{D}$ be an additive category. Denote $\mathcal{C}(\mathcal{D})$ the category of contravariant additive functors $\mathcal{D} \to Ab$. As in Neeman's paper we denote

$$Y : \mathcal{D} \longrightarrow \mathcal{C}(\mathcal{D}), \quad F \longmapsto \mathrm{Hom}(-, F)$$

the Yoneda embedding. Next, assume $\mathcal{D}$ is triangulated and $\{\mathcal{P}_i\}$ is a sequence of subcategories as above. We say A in $\mathcal{C}(\mathcal{D})$ is *compactly supported with respect to* $\{\mathcal{P}_i\}$ if there exists and $i > 0$ such that

$$\mathrm{Hom}(Y(Q), A) = 0 \text{ for all } Q \in \mathcal{P}_i^{\perp}$$

Here we recall that $\mathcal{P}_i^{\perp}$ is the full subcategory of $\mathcal{D}$ consisting of objects Q such that $\mathrm{Hom}(P, Q) = 0$ for all $P \in \mathcal{P}_i$.

Example 6.1. Let $\mathcal{D}$ be a triangulated category which has a classical generator. If we choose a classical generator G, then we can consider the sequence

$$\mathcal{P}_i = \langle G \rangle^{[-i,\infty]}$$

This sequence up to equivalence does not depend on the choice of the generator (details omitted).

7. Sequences in perf of a scheme

Let's see what we get from the definitions in Section 6 in the setting of schemes.

Lemma 7.1. *Let X be a Noetherian scheme of finite dimension. Then $D_{perf}(\mathcal{O}_X)$ has a classical generator and the equivalence class of the sequence*

$$D_{perf}^{\geq -i}(\mathcal{O}_X)$$

is the same as the canonical one constructed in Example 6.1.

Proof. We know that there exists a perfect object E of $D(\mathcal{O}_X)$ which is a generator for $D_{QCoh}(\mathcal{O}_X)$, see Tag 09IS. Next, we know E is compact and $D_{perf}(\mathcal{O}_X) = D_{compact}(\mathcal{O}_X)$ by Tag 09M1. Then finally E is a classical generator for $D_{perf}(\mathcal{O}_X)$ by Tag 09SR.

Let E be a classical generator of $D_{perf}(\mathcal{O}_X)$. Then E is bounded. Say $H^i(E) = 0$ for $i < a$. Then it is clear that

$$\langle E \rangle^{[-i,\infty]} \subset D^{\geq -i-b}_{perf}(\mathcal{O}_X)$$

This proves one inclusion.

For the other inclusion, it will suffice to show that

$$D^{\geq 0}_{perf}(\mathcal{O}_X) \subset \langle E \rangle^{[-i,\infty]}$$

for some i. By Auslander-Buchsbaum, see Tag 090V we have that objects F of $D^{\geq 0}_{perf}(\mathcal{O}_X)$ locally have bounded projective dimension, in fact bounded by $-d$ where $d = \dim(X)$. Hence the dual perfect object $F^\vee = R\mathcal{H}om(F, \mathcal{O}_X)$ will be in $D^{\leq d}_{perf}(\mathcal{O}_X)$. Since taking duals is a anti-equivalence we find that $E^\vee$ is a classical generator of $D_{perf}(\mathcal{O}_X)$ as well. By Lemma 4.5 we conclude that there exists an i, independent of F, such that

$$F^\vee \in \langle E^\vee \rangle^{[-\infty, i+d]}$$

Dualizing back we find

$$F \in \langle E \rangle^{[-i-d,\infty]}$$

as desired. $\square$

Lemma 7.2. *Let X be a quasi-compact and quasi-separated scheme. A Cauchy sequence*

$$E_1 \to E_2 \to E_3 \to \ldots$$

of $D_{perf}(\mathcal{O}_X)$ with respect to the canonical sequence of Example 6.1 is the same thing as a sequence of perfect complexes such that for all $i \geq 0$ there is a N such that for $n \geq N$ the maps

$$E_n \to E_{n+1}$$

are isomorphisms on cohomology sheaves in degrees $\geq -i$.

Proof. Let E be a perfect generator for $D_{perf}(\mathcal{O}_X)$. Choose integers $a \leq b$ such that $\mathrm{Hom}(E, L) = 0$ if $L \in D_{QCoh}(\mathcal{O}_X)$ with $H^i(L) = 0$ for $i \in [a, b]$, see Tag 09M4. Suppose that $E_n \to E_{n+1}$ is an isomorphism on cohomology sheaves in degrees $\geq -i$. Choose a distinguished triangle $E_n \to E_{n+1} \to C_n \to E_n[1]$. Then $H^q(C_n) = 0$ for $q \geq -i$. Hence

$$\mathrm{Hom}(E[j], C_n) = \mathrm{Hom}(E, C_n[-j])$$

is zero if $j \leq a + i$. By Lemma 2.1 we conclude that the condition in the lemma produces a Cauchy sequence.

For the converse take cones of the maps $E_n \to E_{n+1}$ and apply Theorem 5.1. $\square$

Lemma 7.3. *Let X be a quasi-compact and quasi-separated scheme. The pseudo-coherent objects in $D(\mathcal{O}_X)$ are exactly the homotopy colimits of Cauchy sequences of $D_{perf}(\mathcal{O}_X)$ with respect to the canonical sequence of Example 6.1.*

Proof. Immediate from Lemma 7.2 and Tag 0DJN. $\square$

Lemma 7.4. *Let X be a quasi-compact and quasi-separated scheme. Given a Cauchy sequence*

$$E_1 \to E_2 \to E_3 \to \ldots$$

of $D_{perf}(\mathcal{O}_X)$ with respect to the canonical sequence of Example 6.1 with homotopy colimit E we have

$$\mathrm{Hom}(-, E) = \mathrm{colim}\, \mathrm{Hom}(-, E_n)$$

as functors on $D_{perf}(\mathcal{O}_X)$.

Proof. This is true because perfect objects of $D(\mathcal{O}_X)$ are compact and hence $\mathrm{Hom}(P, E) = \mathrm{colim}\, \mathrm{Hom}(P, E_n)$ for all P in $D_{perf}(\mathcal{O}_X)$. □

Lemma 7.5. *Let X be a Noetherian scheme. Let F be a pseudo-coherent object of $D(\mathcal{O}_X)$. Denote $A : D_{perf}(\mathcal{O}_X) \to Ab$ the functor $M \mapsto \mathrm{Hom}(M, F)$. The following are equivalent*

(1) *A is compactly supported with respect to the canonical sequence of Example 6.1 as defined in Section 6, and*
(2) *F has bounded cohomology, i.e., $F \in D^b_{Coh}(\mathcal{O}_X)$.*

Proof. Choose a perfect generator E for $D_{QCoh}(\mathcal{O}_X)$, see Tag 0A9A. Then we have $\mathcal{P}_i = \langle E \rangle^{[-i,\infty]}$ as subcategories of $D_{perf}(\mathcal{O}_X)$, Recall that $\mathcal{P}_i^\perp$ is the full subcategory of Q in $D_{perf}(\mathcal{O}_X)$ with $\mathrm{Hom}(P, Q) = 0$ for all $P \in \mathcal{P}_i$. By the Yoneda lemma we have

$$\mathrm{Hom}_{\mathcal{C}(\mathcal{D})}(Y(Q), A) = A(Q) = \mathrm{Hom}(Q, F)$$

Thus we have to show the following two are equivalent

(1) there exists an i such that $\mathrm{Hom}(Q, F) = 0$ for all Q in $\mathcal{P}_i^\perp$, and
(2) $F \in D^b_{Coh}(\mathcal{O}_X)$.

By Theorem 5.1 the sequence of subcategories $\mathcal{P}_i^\perp$ is equivalent to the sequence of subcategories $D^{\leq -i}_{perf}(\mathcal{O}_X)$. Thus we have to show the following two are equivalent

(a) there exists an i such that $\mathrm{Hom}(Q, F) = 0$ for all Q in $D^{\leq -i}_{perf}(\mathcal{O}_X)$, and
(b) $F \in D^b_{Coh}(\mathcal{O}_X)$.

The implication from (b) to (a) is immediate by taking i large enough. Suppose that (a) holds for some i. Consider the map

$$\tau_{\leq -i-1} F \longrightarrow F$$

This is a map of pseudo-coherent complexes because X is Noetherian[1]. We have to show $\tau_{\leq -i-1} F$ has vanishing cohomology. If not, then there exists an $m \leq -i-1$ such that $H^m(F) = H^m(\tau_{\leq -i-1} F)$ is nonzero. By Tag 08ES we can choose an approximation

$$Q \longrightarrow \tau_{\leq -i-1} F$$

for the triple $(X, \tau_{\leq -i-1} F, m)$. This means that Q is in $D_{perf}(X)$ and the displayed map induces isomorphisms $H^q(Q) \to H^q(\tau_{\leq -i-1} F)$ on cohomology sheaves for $q > m$ and a surjection $H^m(Q) \to H^m(\tau_{\leq -i-1} F)$ for $q = m$. This is a contradiction because it would mean that Q is in $D^{\leq -i}_{perf}(\mathcal{O}_X)$ and $H^m(Q) \to H^m(F)$ would be surjective and hence nonzero and hence $Q \to F$ would be nonzero. □

[1] This is the only step where we need to use that X is Noetherian and it might well be that the lemma holds for X quasi-compact and quasi-separated.

8. Neeman: metrics on triangulated categories

Let $\mathcal{D}$ be a triangulated category. A *metric* on $\mathcal{D}$ is given by a sequence of additive subcategories

$$\mathcal{M}_1 \supset \mathcal{M}_2 \supset \mathcal{M}_3 \supset \ldots$$

each closed under extensions. The metric $\{\mathcal{M}_i\}$ is *finer* than the metric $\{\mathcal{N}_j\}$ if for every j there is an i such that $\mathcal{M}_i \subset \mathcal{N}_j$. If you think of the subcategories $\mathcal{M}_i$ as the "ball of radius $1/i$ around 0", then this means the "topology" defined by the sequence $\{\mathcal{M}_i\}$ is finer than the topology defined by the sequence $\{\mathcal{N}_j\}$.

A *Cauchy sequence* in $\mathcal{D}$ with respect the metric $\{\mathcal{M}_i\}$ is a sequence

$$E_1 \to E_2 \to E_3 \to \ldots$$

such that for any $i > 0$ and $j \in \mathbf{Z}$ there is an N such that for $n' > n \geq N$ we have

$$C_{n,n'}[j] \in \mathcal{M}_i$$

where $C_{n,n'}$ is the cone of $E_n \to E_{n'}$, i.e., it sits in a distinguished triangle

$$E_n \to E_{n'} \to C_{n,n'} \to E_n[1]$$

of $\mathcal{D}$. A trivial remark is that it suffices to check $\forall i > 0$ $\forall j \in \mathbf{Z}$ $\exists N$ such that for $n > N$ we have $C_{n,n+1}[j] \in \mathcal{M}_i$. This is true because we have distinguished triangles

$$C_{n,n'-1} \to C_{n,n'} \to C_{n'-1,n'} \to C_{n,n'-1}[1]$$

by the octahedral axiom of triangulated categories. Hence if $C_{n,n'-1}[j]$ and $C_{n'-1,n'}[j]$ are in $\mathcal{M}_i$, then so is $C_{n,n'}$ because $\mathcal{M}_i$ is closed under extensions. Thus $C_{n,n+1}[j] \in \mathcal{M}_i$ for all $n > N$ implies $C_{n,n'}[j] \in \mathcal{M}_i$ for all $n' > n > N$.

Recall the Yoneda embedding $Y : \mathcal{D} \to \mathcal{C}(\mathcal{D})$ into the category of contravariant additve functors to Ab discussed in Section 6. We say an object A of of $\mathcal{C}(\mathcal{D})$ is *compactly supported with respect to the metric* $\{\mathcal{M}_i\}$ if for all $j \in \mathbf{Z}$ there exists an $i > 0$ such that

$$\mathrm{Hom}(Y(M[j]), A) = 0$$

for all M in $\mathcal{M}_i$.

Now we have all the notation required to formulate Neeman's theorem.

Theorem 8.1. *Let $\mathcal{D}$ be a triangulated category. Let $\{\mathcal{M}_i\}$ be a metric on $\mathcal{D}$. Consider the strictly full subcategory*

$$\mathcal{S} \subset \mathcal{C}(\mathcal{D})$$

consisting of objects A which have the following two properties:

1. *there is a Cauchy sequence $E_1 \to E_2 \to E_3 \to \ldots$ with respect to the metric and $A = \mathrm{colim}\, Y(E_n)$ where $Y : \mathcal{D} \to \mathcal{C}(\mathcal{D})$ is the Yoneda embedding, and*
2. *A is compactly supported with respect to the metric $\{\mathcal{M}_i\}$.*

Then $\mathcal{S}$ is a triangulated category with shift functor given by $A \mapsto A[1] = A \circ [-1]$ and distinguished triangles given by those triangles

$$A \to B \to C \to A[1]$$

such that there exist a system of distinguished triangles

$$E_n \to F_n \to G_n \to E_n[1]$$

such that $\{E_n\}$, $\{F_n\}$, $\{G_n\}$ *are Cauchy sequences and such that* $A = \operatorname{colim} Y(E_n)$, $B = \operatorname{colim} Y(B_n)$, $C = \operatorname{colim} Y(G_n)$ *compatible with the maps.*

Proof. The rather terrible proof can be found in Neeman's paper. □

9. Translation between the two notions

This section does the trivial translation between the notions introduced in Sections 6 and 8. In particular, it follows that Theorem 8.1 applies to the notions of Cauchy sequences and compactly supported objects introduced in Section 6.

Lemma 9.1. *Let* $\mathcal{D}$ *be a triangulated category. Let*

$$\mathcal{P}_1 \subset \mathcal{P}_2 \subset \mathcal{P}_3 \subset \ldots$$

be a sequence of subcategories as in Section 6. Set

$$\mathcal{M}_i = \mathcal{P}_i^\perp = \{M \in \mathcal{D} \mid \operatorname{Hom}(P, M) = 0 \text{ for all } P \in \mathcal{P}_i\}$$

Then $\{\mathcal{M}_i\}$ *is a metric on* $\mathcal{D}$.

Proof. Since $\mathcal{P}_i \subset \mathcal{P}_{i+1}$ we see that $\mathcal{M}_i \supset \mathcal{M}_{i+1}$. Moreover, we have $\mathcal{M}_i$ is closed under extensions: if

$$X \to X' \to X'' \to X[1]$$

is a distinguished triangle and X and X'' are in $\mathcal{M}_i$, then for $P \in \mathcal{P}_i$ the exact sequence

$$\operatorname{Hom}(P, X) \to \operatorname{Hom}(P, X') \to \operatorname{Hom}(P, X'')$$

shows that $\operatorname{Hom}(P, X') = 0$ and hence X' is in $\mathcal{M}_i$. □

Lemma 9.2. *Let* $\mathcal{D}$ *be a triangulated category. Let*

$$\mathcal{P}_1 \subset \mathcal{P}_2 \subset \mathcal{P}_3 \subset \ldots$$

be a sequence of subcategories as in Section 6. Set $\mathcal{M}_i = \mathcal{P}_i^\perp$ *as in Lemma 9.1. In this case the following are true*

(1) *for every* $j \in \mathbf{Z}$*, there exists an* N *such that for* $i > N$ *we have*

$$\mathcal{M}_i[j] \subset \mathcal{M}_{i-|j|}$$

(2) *a Cauchy sequence with respect to the metric* $\{\mathcal{M}_i\}$ *is the same as a Cauchy sequence with respect to the sequence* $\{\mathcal{P}_i\}$ *in the sense of Section 6,*

(3) *an object* A *of* $\mathcal{C}(\mathcal{D})$ *is compactly supported with respect to the metric* $\{\mathcal{M}_i\}$ *if and only if it is compactly supported with respect to the sequence* $\{\mathcal{P}_i\}$ *in the sense of Section 6.*

Proof. Proof of (1). By our definition in Section 6 and a simple induction argument we have for $i > a > 0$ that

$$\mathcal{P}_i \supset \bigcup\nolimits_{-a \leq j \leq a} \mathcal{P}_{i-a}[j]$$

Hence $\mathcal{P}_i[j] \supset \mathcal{P}_{i-|j|}$ by taking $a = |j|$ and shifting. The inclusion in (1) follows by taking perpendiculars and using that $\mathcal{M}_i[j] = (\mathcal{P}_i[j])^\perp$ as $[j]$ is an equivalence of categories.

Proof of (2). By part (1) we see that a sequence

$$E_1 \to E_2 \to E_3 \to \ldots$$

is a Cauchy sequence with respect to the metric if and only if for every i we can find an N such that the cones $C_{n,n+1}$ are in $\mathcal{M}_i$ for $n > N$. This is equivalent to asking for all $P \in \mathcal{P}_i$ the map $\mathrm{Hom}(P, E_n) \to \mathrm{Hom}(P, E_{n+1})$ to be surjective and the map $\mathrm{Hom}(P, E_n[1]) \to \mathrm{Hom}(P, E_{n+1}[1])$ to be injective for all $n > N$. Since $\mathcal{P}_i$ contains $\mathcal{P}_{i-1}[-1]$ we conclude that $\mathrm{Hom}(P, E_n) \to \mathrm{Hom}(P, E_{n+1})$ is bijective for P in $\mathcal{P}_{i-1}$ and $n > N$. This means that we have a Cauchy sequence with respect to the sequence $\{\mathcal{P}_i\}$. The converse argument is exactly the same and we omit it.

Proof of (3). This is trivial if you know (1). □

Example 9.3. Let (A, d) be a differential graded algebra. Denote $D(A, \mathrm{d})$ the derived category of A and denote $D_c(A, \mathrm{d})$ the full subcategory of compact objects. Then A is a generator of $D_c(A, \mathrm{d})$ and the sequence in Example 6.1 is given by

$$\mathcal{P}_i = \langle A \rangle^{[-i,\infty]}$$

for $i \geq 1$. Since $\mathrm{Hom}(A[-i], E) = H^i(E)$ we see that a sequence

$$E_1 \to E_2 \to E_3 \to \ldots$$

of morphisms of $D_c(A, \mathrm{d})$ is a Cauchy sequence if and only if for all $i \geq 1$ there exists an N such that for $n > N$ the maps $H^j(E_n) \to H^j(E_{n+1})$ are isomorphisms for all $j \geq -i$. Set

$$F(-) = \mathrm{colim}\, \mathrm{Hom}(-, E_n)$$

in $\mathcal{C}(D_c(A, \mathrm{d}))$. Next, Q in $D_c(A, \mathrm{d})$ is in $\mathcal{P}_i^{\perp}$ if and only if $H^q(Q) = 0$ for all $q \geq -i$. Thus F is compactly supported with respect to $\{\mathcal{P}_i\}$ if and only if there exists an i such that

$$\mathrm{colim}\, \mathrm{Hom}(Q, E_n)$$

is zero for every Q in $D_c(A, \mathrm{d})$ such that $H^q(Q) = 0$ for $q \geq -i$.

If $H^t(A) = 0$ for $t \gg 0$, then two things happen: first we see that every object of $D_c(A, \mathrm{d})$ has cohomology bounded above and hence for a Cauchy sequence the homotopy colimit $E = \mathrm{hocolim} E_n$ equally has cohomology bounded above, second we see that $A[q]$ will be contained in $\mathcal{P}_i^{\perp}$ for q large enough and hence we see E has cohomology bounded below. Thus the objects of the triangulated category constructed in Theorem 8.1 have "bounded cohomology" in this case.

Suppose $A^t = 0$ for $t \neq 0$. Then the homotopy colimits E of "compactly supported Cauchy sequences" are those E in $D^b(A, \mathrm{d})$ such that for every $n \gg 0$ there is an $E_n \in D_c(A, \mathrm{d})$ and a map $E_n \to E$ which induces an isomorphism on cohomology in degrees $\geq -n$. If A^0 is a (possibly noncommutative) Noetherian ring, this is just $D^b_{\mathrm{Coh}}(A^0)$. If A^0 is any commutative ring (not necessarily Noetherian) this gives the pseudo-coherent complexes with bounded cohomology.

10. Recovering DbCoh from Dperf

This is Neeman's first result (if you are only interested in derived categories of schemes). Recall that for an additive category $\mathcal{D}$ we denote $\mathcal{C}(\mathcal{D})$ the category of contravariant additive functors $\mathcal{D} \to Ab$.

Lemma 10.1. *Let X be a Noetherian scheme. Then the functor*

$$Y' : D^b_{Coh}(\mathcal{O}_X) \longrightarrow \mathcal{C}(D_{perf}(\mathcal{O}_X)), \quad F \longmapsto (M \mapsto \mathrm{Hom}(M, F))$$

is fully faithful.

Proof. Duppose we have F and G in $D^b_{Coh}(\mathcal{O}_X)$. Let $\alpha : F \to G$ be a map with $Y'(\alpha) = 0$. We want to show that α is zero. Write $F = \text{hocolim} F_n$ as in Tag 0DJN. We obtain an exact sequence

$$R^1 \lim \text{Hom}(F_n[1], G) \to \text{Hom}(F, G) \to \lim \text{Hom}(F_n, G)$$

(details omitted). However, since the cone $C_{n,n+1}$ on $F_n \to F_{n+1}$ has nonvanishing cohomology sheaves only in degrees $< n$ (by construction) we see that $\text{Hom}(C_{n,n+1}[i], G) = 0$ for $i = -1, 0, 1, 2$ and $n \gg 0$ (because G is bounded). This implies that the inverse system $\{\text{Hom}(F_n[1], G)\}$ is eventually constant and hence has vanishing $R^1 \lim$. Thus the term on the left is zero. But if $Y'(\alpha) = 0$, then α maps to zero on the right and hence is zero.

Fullyness follows in exactly the same manner. □

Theorem 10.2. *Let X be a Noetherian scheme. Consider the canonical sequence of subcategories of $D_{perf}(X)$ as in Example 6.1. Then the functor*

$$Y' : D^b_{Coh}(\mathcal{O}_X) \longrightarrow \mathcal{C}(D_{perf}(\mathcal{O}_X)), \quad F \longmapsto (M \mapsto \text{Hom}(M, F))$$

identifies $D^b_{Coh}(\mathcal{O}_X)$ with the full subcategory of functors which are both colimits of Cauchy sequences and compactly supported with respect to the canonical sequence. Moreover, the triangulated structure given in Theorem 8.1 on the essential image of the functor is equal to the triangulated structure on the source of the functor.

Proof. The functor is fully faithful by Lemma 10.1. The description of the essential image follows immediately from Lemmas 7.2, 7.3, 7.4, and 7.5.

To show that the distinguished triangles agree we have to show two things

(1) given a system of distinguished triangles $E_n \to F_n \to G_n \to E_n[1]$ in $D_{perf}(\mathcal{O}_X)$ such that $\{E_n\}$, $\{F_n\}$, $\{G_n\}$ are Cauchy sequences with $\text{colim}\, Y(E_n)$, $\text{colim}\, Y(F_n)$, and $\text{colim}\, Y(G_n)$ compactly supported we can get a distinguished triangle in $D^b_{Coh}(\mathcal{O}_X)$ by "taking homotopy colimits", and
(2) given a distinguished triangle $E \to F \to G \to E[1]$ in $D^b_{Coh}(\mathcal{O}_X)$ we can write it as a "homotopy colimit" of a sequence of distinguished triangles as in (1).

The quotes indicate we only need to verify equality after applying the functor Y', which makes this doable.

Proof of (1). Set $E = \text{hocolim} E_n$, $F = \text{hocolim} F_n$, and $G = \text{hocolim} G_n$. The assumption means that $E, F, G \in D^b_{Coh}(\mathcal{O}_X)$ and that the maps $E_n \to E$, $F_n \to F$, and $G_n \to G$ are isomorphisms on cohomology sheaves in degrees $\geq -i$ for $n \gg 0$. Thus we can pick $n \gg m \gg 0$ such that the maps $E_n \to E$, $F_n \to F$, $G_n \to G$ factor through isomorphisms

$$\tau_{\geq -m} E_n \to E, \ \tau_{\geq -m} F_n \to F, \ \tau_{\geq -m} G_n \to G$$

Then we get the desired distinguished triangle in $D^b_{Coh}(\mathcal{O}_X)$ by using

$$\tau_{\geq -m} E_n \to \tau_{\geq -m} F_n \to \tau_{\geq -m} G_n \to \tau_{\geq -m} E_n[1]$$

this works because we have the distinghuished triangle $E_n \to F_n \to G_n \to E_n[1]$ and truncating it as shown gives a distinghuished triangle because E_n will have vanishing cohomology sheaves in degrees $-m-1, -m, -m+1$ if n, m are large enough.

Proof of (2). Choose $E = \text{hocolim} E_n$ and $F = \text{hocolim} F_n$ as in Tag 0DJN. Choose an $m_1 \geq 1$ and a map $E_1 \to F_{m_1}$ such that the diagram

$$\begin{tikzcd} E_1 \arrow[r] \arrow[d] & E \arrow[d] \\ F_{m_1} \arrow[r] & F \end{tikzcd}$$

commutes. This is possible as $F = \text{hocolim} F_n$ and E_1 is compact, see Tag 094A. We inductively choose for $n > 1$ an integer $m_n \geq m_{n-1}$ and a map $E_n \to F_{m_n}$ such that the diagram

$$\begin{tikzcd} E_n \arrow[r] \arrow[d] & E_{n-1} \arrow[r] \arrow[d] & E \arrow[d] \\ F_{m_n} \arrow[r] & F_{m_{n-1}} \arrow[r] & F \end{tikzcd}$$

commutes. (Use the same lemma two times.) Choose distinguished triangles

$$E_n \to F_{m_n} \to G_n \to E_n[1]$$

and choose maps of distinguished triangles

$$\begin{tikzcd} E_n \arrow[r] \arrow[d] & F_{m_n} \arrow[r] \arrow[d] & G_n \arrow[r] \arrow[d] & E_n[1] \arrow[d] \\ E_{n+1} \arrow[r] & F_{m_{n+1}} \arrow[r] & G_{n+1} \arrow[r] & E_{n+1}[1] \end{tikzcd}$$

Since the cohomology sheaves of the sequences $\{E_n\}$ and $\{F_n\}$ stabilize in degrees $\geq -i$ the same is true for the sequence of cohomology sheaves of G_n. Thus $\text{hocolim} G_n$ is in $D^b_{Coh}(\mathcal{O}_X)$. (Warning: we do not yet know this is G; at this point we do not even have a map from this to G or vice versa.) Thus we have produced a system of distinguished triangles as in (1). We still have to show that our original distinguished triangle agrees with the "homotopy colimit" of this system of triangles, at least after applying Y.

Now we can also choose maps of distinguished triangles

$$\begin{tikzcd} E_n \arrow[r] \arrow[d] & F_{m_n} \arrow[r] \arrow[d] & G_n \arrow[r] \arrow[d] & E_n[1] \arrow[d] \\ E \arrow[r] & F \arrow[r] & G \arrow[r] & E[1] \end{tikzcd}$$

using the axioms of triangulated categories but the problem is that we don't know we can do this compatibly in n for the maps $G_n \to G$. However, we can do the same thing as before and use that for $n \gg m \gg 0$ the vertical maps in the previous diagram factor through the truncations to give a commutative diagram

$$\begin{tikzcd} \tau_{\geq -m} E_n \arrow[r] \arrow[d] & \tau_{\geq -m} F_{m_n} \arrow[r] \arrow[d] & \tau_{\geq -m} G_n \arrow[r] \arrow[d] & \tau_{\geq -m} E_n[1] \arrow[d] \\ E \arrow[r] & F \arrow[r] & G \arrow[r] & E[1] \end{tikzcd}$$

whose vertical arrows are isomorphisms. (Some details omitted.) Then we fix one $n \gg m > 0$ like this and for $n' > n$ we use the composition

$$G_{n'} \to \tau_{\geq m} G_{n'} \leftarrow \tau_{\geq -m} G_n \to G$$

which is possible because the middle left arrow can be inverted. With these maps it is clear that $G = \text{hocolim} G_n$ (look at cohomology sheaves). Then the reader may check that we have enough commutativity to show that

$$Y'(E) \to Y'(F) \to Y'(G) \to Y'(E)[1]$$

is the colimit of the triangles

$$Y(E_n) \to Y(F_n) \to Y(G_n) \to Y(E_n)[1]$$

as desired. □

11. Neeman: dual canonical metric

Let $\mathcal{D}$ be a triangulated category. Given subcategories $\mathcal{P}$ and $\mathcal{Q}$ we say $\mathcal{P}$ is *smaller* that $\mathcal{Q}$ if $\mathcal{P}[n] \subset \mathcal{Q}$ for some $n \in \mathbf{Z}$. This defines a partial ordering on the set of subcategories.

Let G be an object of $\mathcal{D}$. Consider the subcategory

$$\mathcal{Q}(G) = (\langle G \rangle^{[-\infty,0]})^{\perp}$$

Observe that M is an object of $\mathcal{Q}(G)$ if and only if

$$\text{Hom}(G[j], M) = 0 \text{ for all } j \geq 0$$

Thus we clearly have $\mathcal{Q}(G) \supset \mathcal{Q}(G)[-1] \supset \mathcal{Q}(G)[-2] \supset \ldots$.

If there exists an object G_0 such that $\mathcal{Q}(G_0)$ is minimal with respect to the ordering defined above, then we say "$\mathcal{Q}(\mathcal{D})$ is defined" and we set $\mathcal{Q}(\mathcal{D}) = \mathcal{Q}(G_0)$. This is only well defined up to equality in the partial order above.

If $\mathcal{Q}(\mathcal{D})$ is defined, then we consider the sequence

$$\mathcal{P}_i = \mathcal{Q}(\mathcal{D})[i], \quad i \geq 1$$

which is a sequence of subcategories as in Section 6. Moreover, it is immediate that this sequence is well defined up to equivalence. Let us call this the *dual canonical sequence of subcategories*.

Let $\mathcal{D}$ be a triangulated category. Let $\{\mathcal{P}_i\}$ be a sequence of subcategories as in Section 6 (for example the dual canonical sequence above). We say that an inverse sequence

$$E_1 \leftarrow E_2 \leftarrow E_3 \leftarrow \ldots$$

of morphisms of $\mathcal{D}$ is an *inverse Cauchy sequence* if and only if for every i there is a N such that for $n \geq N$ the maps

$$\text{Hom}(E_n, P) \to \text{Hom}(E_{n+1}, P)$$

are isomorphisms for all P in $\mathcal{P}_i$.

Let $\mathcal{D}$ be an additive category. Denote $\mathcal{C}(\mathcal{D}^{opp})$ the category of **covariant** additive functors $\mathcal{D} \to Ab$. Denote

$$U : \mathcal{D}^{opp} \longrightarrow \mathcal{C}(\mathcal{D}^{opp}), \quad F \longmapsto \text{Hom}(F, -)$$

the Yoneda embedding. Next, assume $\mathcal{D}$ is triangulated and $\{\mathcal{P}_i\}$ is a sequence of subcategories as above. We say B in $\mathcal{C}(\mathcal{D}^{opp})$ is *compactly supported with respect to* $\{\mathcal{P}_i\}$ if there exists and $i > 0$ such that

$$\mathrm{Hom}(B, U(Q)) = 0 \text{ for all } Q \in {}^{\perp}\mathcal{P}_i$$

Here we recall that ${}^{\perp}\mathcal{P}_i$ is the full subcategory of $\mathcal{D}$ consisting of objects Q such that $\mathrm{Hom}(Q, P) = 0$ for all $P \in \mathcal{P}_i$.

12. Recovering Dperf from DbCoh

This is Neeman's second result (if you are only interested in derived categories of schemes). Recall that for an additive category $\mathcal{D}$ we denote $\mathcal{C}(\mathcal{D})$ the category of contravariant additive functors $\mathcal{D} \to Ab$.

Lemma 12.1. *Let X be a Noetherian scheme. Then $\mathcal{Q}(D^b_{Coh}(\mathcal{O}_X))$ exists (see Section 11) and is equal to $D^{b,\geq 0}_{Coh}(\mathcal{O}_X)$ up to equivalence.*

Proof. First choose $E \in D(\mathcal{O}_X)$ a perfect generator for $D_{QCoh}(\mathcal{O}_X)$, see Tag 0A9A. Then Lemma 5.3 tells us that $Q(E)$ is contained in $D^{b,\geq -i}_{Coh}(\mathcal{O}_X)$ for some i. On the other hand, for G in $D^b_{Coh}(\mathcal{O}_X)$ we see that $Q(G)$ contains $D^{b,\geq -j}_{Coh}(\mathcal{O}_X)$ for some $j = j(G) \in \mathbf{Z}$ by looking at cohomology sheaves. These two assertions combined imply the lemma. ☐

Lemma 12.2. *Let X be a Noetherian scheme. Let*

$$E_1 \leftarrow E_2 \leftarrow E_3 \leftarrow \ldots$$

be an inverse sequence in $\mathcal{Q}(D^b_{Coh}(\mathcal{O}_X))$. The following are equivalent

1. *$\{E_n\}$ is an inverse Cauchy sequence with respect to the dual canonical sequence of subcategories,*
2. *for every $i \in \mathbf{Z}$ there is an N such that for $n > N$ the maps $E_{n+1} \to E_n$ induce isomorphisms on cohomology sheaves in degrees $\geq -i$.*

Proof. By Lemma 12.1 the assertion makes sense and the sequence of subcategories we are looking at are

$$D^{b,\geq -1}_{Coh}(\mathcal{O}_X) \subset D^{b,\geq -2}_{Coh}(\mathcal{O}_X) \subset D^{b,\geq -3}_{Coh}(\mathcal{O}_X) \subset \ldots$$

Then it is trivial to conclude by taking truncations for example. ☐

Lemma 12.3. *Let X be a Noetherian scheme. The pseudo-coherent objects of $D(\mathcal{O}_X)$ are exactly the derived limits of inverse Cauchy sequences of $D^b_{Coh}(\mathcal{O}_X)$ with respect to the dual canonical sequence.*

Proof. It is clear from Lemma 12.2 and Tag 0A0J that the derived limit of an inverse Cauchy sequence is pseudo-coherent. Conversely, if E is pseudo-coherent, then we can write $E = R\lim \tau_{\geq -n} E$, see Tag 08D3. ☐

Lemma 12.4. *Let X be a Noetherian scheme. Given an inverse Cauchy sequence*

$$E_1 \leftarrow E_2 \leftarrow E_3 \leftarrow \ldots$$

of $D^b_{Coh}(\mathcal{O}_X)$ with respect to the dual canonical sequence with derived limit E we have

$$\mathrm{Hom}(E, -) = \mathrm{colim}\, \mathrm{Hom}(E_n, -)$$

as functors on $D^b_{Coh}(\mathcal{O}_X)$.

Proof. Say M is in $D^b_{Coh}(\mathcal{O}_X)$. If C_n is the cone of $E \to E_n$, then C_n has vanishing cohomology in degrees $\geq -i$ for $n \gg 0$. Hence $\mathrm{Hom}(C_n[i], M) = 0$ for $n \gg 0$ and $i = -1, 0, 1$. From this the reader easily concludes that the colimit is essentially constant with the correct value. $\square$

Lemma 12.5. *Let X be a Noetherian scheme. Let F be a pseudo-coherent object of $D(\mathcal{O}_X)$. Denote $B = U(F) : D^b_{Coh}(\mathcal{O}_X) \to Ab$ the functor $M \mapsto \mathrm{Hom}(F, M)$. The following are equivalent*

1. *B is compactly supported with respect to the dual canonical sequence as defined in Section 11, and*
2. *F is perfect.*

Proof. By Lemma 12.1 we may choose our sequence to be the sequence $\mathcal{P}_i = D^{b,\geq -i}_{Coh}(\mathcal{O}_X)$. Using the canonical truncations it follows immedately that ${}^\perp\mathcal{P}_i = D^{b,\leq -i-1}_{Coh}(\mathcal{O}_X)$. Thus we have to show the following two are equivalent

1. there exists an i such that $\mathrm{Hom}(F, Q) = 0$ for all Q in $D^{b,\leq -i-1}_{Coh}(\mathcal{O}_X)$, and
2. F is perfect.

The implication (2) $\Rightarrow$ (1) follows from Tag 09M4.

Conversely assume (1) is true for some i. Let $x \in X$ be a closed point. Let κ be the residue field at x. If we apply (1) to κ viewed as a coherent sheaf on X, then we see that

$$\begin{aligned} R\,\mathrm{Hom}(F, \kappa) &= R\Gamma(X, R\,\mathcal{H}om(F, \kappa)) \\ &= R\,\mathrm{Hom}_{\mathcal{O}_{X,x}}(F_x, \kappa) \\ &= R\,\mathrm{Hom}_\kappa(F \otimes^{\mathbf{L}}_{\mathcal{O}_X} \kappa, \kappa) \end{aligned}$$

has vanishing cohomology in degrees $\geq i+1$. (The second equality holds because x is a closed point.) Thus $F \otimes^{\mathbf{L}}_{\mathcal{O}_X} \kappa$ only has vanishing cohomology in degrees $\leq -i-1$. Hence we see that F is perfect in a neighbourhood of x for example by Tag 0BCC. This finishes the proof (as any closed subset of X contains a closed point and being perfect is local on X). $\square$

Theorem 12.6. *Let X be a Noetherian scheme. Consider the dual canonical sequence of subcategories of $D^b_{Coh}(X)$. Then the functor*

$$U' : D_{perf}(\mathcal{O}_X)^{opp} \longrightarrow \mathcal{C}(D^b_{Coh}(\mathcal{O}_X)^{opp}), \quad F \longmapsto (M \mapsto \mathrm{Hom}(F, M))$$

identifies $D_{perf}(\mathcal{O}_X)$ with the full subcategory of functors which are both colimits of inverse Cauchy sequences and compactly supported with respect to the dual canonical sequence. Moreover, the triangulated structure given in Theorem 8.1 on the essential image of the functor is equal to the triangulated structure on the source of the functor.

Proof. The fully faithfulness of U' is clear from the fact that $D_{perf}(\mathcal{O}_X) \subset D^b_{Coh}(\mathcal{O}_X)$. The essential image of the functor is as given in the statement by Lemmas 12.1, 12.2, 12.3, 12.4, and 12.5. We omit the proof that the distinguished triangles agree. $\square$