EXAMPLES OF ALGEBRAIC STACKS

Contents

1. Introduction

This is a list of examples of algebraic stacks. The reason for being of this list is to serve also as a list of topics for lectures by students in the Graduate Student Algebraic Geometry Seminar held at Columbia University in the Fall of 2009.

As references on algebraic stacks we use

(1) Deligne and Mumford [DM69, Section 4],
(2) Fantechi, [Fan01],
(3) Laumon and Moret-bailly, [LMB00],
(4) book in progress by many authors, `http://www.math.unizh.ch/index.php?pr_vo_det&key1=1287&key2=580&no_cache=1`
(5) algebraic stacks project, see [spa], and
(6) various publications by Kai Behrend et al, see [Beh03], [Beh04], and [BN05].

Most of these can be found on the web.

2. Quotient stacks

Introduce $[X/G]$ and give lots of examples.

(1) $B(G)$ where G is either an abstract group or a group scheme, see [spa, Examples 0370 and 0371],
(2) spell out what happens for $B(\mathbf{G}_m)$,
(3) spell out what happens for $B(\mathrm{GL}_n)$,
(4) $[\mathbf{A}^1/\mathbf{G}_m]$ with the "usual" action, and
(5) $\mathcal{P}(a_0, \ldots, a_n)$, the weighted projective space stack – also discuss what happens if all a_i are 1.

3. Stacky curves

In this lecture we say curve when we really mean nonsingular curve over an algebraically closed field (the complex numbers if you like). Discuss the following facts:

(1) A stacky curve is always a gerb over an orbi-curve. Explain this.
(2) What is a universal covering? When do they exist.
(3) When is a stacky curve a quotient stack?
(4) What is the genus of a stacky curve?
(5) What is the canonical sheaf?
(6) Is there a dualizing sheaf?
(7) What is a morphism from a stacky (generically nongerby) curve into $B(S_n)$?

4. Actions of group(schemes) on algebraic stacks

It is a little bit confusing since the collection of algebraic stacks forms a 2-category. Discuss the following

(1) What should come out when H acts on $B(G)$. (Ask Michael Thaddeus.)
(2) Use this to come up with a definition.
(3) Do examples:
    (a) H acting on $B(G)$,
    (b) when can $\mathbf{G}_m$ act nontrivially on a stacky curve?
    (c) Define toric stacks, and examples of toric stacks.

5. Picard stacks

Here is a list of things to discuss:

(1) The Picard stack of a scheme, or more generally a scheme over another scheme. See [spa, Example 0372]
(2) The Picard stack of an algebraic stack; here you have to be a little careful.
(3) Discuss the inertia stack in general, and apply this to the Picard stack (see [spa, Example 0375] and $[X/G]$ (see [spa, Example 0374]).
(4) When is the Picard stack supposed to be algebraic? This may be a good place to discuss (briefly) Artin's axioms and a little deformation theory.

6. Vector bundles on algebraic stacks

Discuss a tiny little bit the notion of a quasi-coherent sheaf on an algebraic stack. Especially: Vector bundles on quotient stacks. Vector bundles on $B(G)$ where G is a group or an algebraic group. Classify vector bundles on $[\mathbf{P}^1/\mathbf{G}_m]$ (ask Michael Thaddeus).

7. Coarse module spaces (schemes)

Elementary remarks and introduce the theorem of Keel-Mori. Applications: Existence without GIT. Discuss how this leads to coarse moduli schemes in certain cases (for example $\overline{\mathcal{M}}_g$).

8. GIT-stacks?

Definition 8.1. Let k be a field. A *GIT-stack*[1] $\mathcal{X}$ over k is a quotient stack of the form

$$\mathcal{X} = [X^{ss}/G]$$

where G is a reductive linear algebraic group over k, and X^{ss} is the semi-stable locus for an action of G on a is a projective scheme $(X, \mathcal{O}_X(1))$ over k.

A GIT-stack $\mathcal{X}$ does not come equipped with the data of $(G, X, \mathcal{O}_X(1))$. Show GIT-stacks have coarse moduli schemes. Show GIT-stacks saitsfy the existence part of the valuative criterion of properness. Are GIT-stacks proper? Ask Jarod Alper about some of the good properties of the morphism $\mathcal{X} \to X$ from a GIT-stack to its coarse moduli scheme. Examples:

(1) Let C be a projective, smooth, geometrically connected curve over k. Let

$$\mathcal{M} = Sh(C, r, \mathcal{O}_C)$$

be the stack parametrizing locally free sheaves of rank r on C with a given trivialization of the determinant. Show that $\mathcal{M}$ contains an open dense substack which is a GIT-stack.

(2) Another similar example is to take *Curves* the algebraic stack parametrizing (all) flat, proper families of curves whose geometric fibres are 1-dimensional proper schemes having arithmetic genus 1. Then $\overline{\mathcal{M}}_g \subset$ *Curves* is an open substack which is a GIT-stack.

(3) Weighted projective space stacks.

9. Irreducible, connected, normal, etc algebraic stacks

In other words, discuss some elementary properties of algebraic stacks. Also: dimension, closed substacks, etc. If X is a connected smooth algebraic stack over $\mathbf{C}$ why is dimension the dimension at any point the same?

10. Cohomology groups of stacks

What is the problem with defining cohomology? (Different sites associated to an algebraic stack.) What should be the answer? What should be the answer for $[X/G]$? Give a definition that just works. Compute examples, such as $\mathcal{M}_{1,1,\mathbf{C}}$ for example. Lefschetz trace formula for fixed points of Frobenius for a stack over a finite field (focus on what should be true, not on what can be proved).

11. Intersection theory on stacks

Something about how on quotient stacks. Compute the following Chow rings:

(1) $A_*(B(\mathbf{G}_m))$,
(2) $A_*(B(G))$ where G is a finite cyclic group,
(3) $A_*(\mathcal{P}(a_0, \ldots, a_n))$ where

$$\mathcal{P}(a_0, \ldots, a_n) = [\mathbf{A}^{n+1} \setminus \{0\}/\mathbf{G}_m]$$

with $\mathbf{G}_m$ acting with weights $a_0, \ldots, a_n$ on $\mathbf{A}^{n+1}$,
(4) $A_*(B(\mathrm{GL}_n))$, and

[1] This is likely nonstandard notation.

(5) $A_*([\mathbf{A}^n/\mathrm{GL}_n])$ (this is actually trivial if you know the answer to the previous one).

GIT-stacks + fundamental class problem.

12. Examples of algebraic spaces

Examples of algebraic spaces which are not schemes, and which are still completely natural from the point of view of moduli theory:

(1) Example of noneffective descent from [BLR90, Section 6.7]. The descent is effective in the category of algerbaic spaces. Think of this as giving a flat family of curves over a base scheme whose total space is an algebraic space and all of whose fibres are projective curves.
(2) Similarly: Take a (suitably general) 1-parameter family of smooth surfaces of degree $d \geq 4$ in $\mathbf{P}^3$ degenerating to a surface having an ordinary double point. After possibly making a double cover of the base here exists an algebraic space all of whose fibres are smooth surfaces, which is the same as the original family except for the central fibre, and with smooth central fibre. But the total space is not a scheme.
(3) Related to previous example: Small resolutions of threefolds with ordinary double points. There are $2^{\#\text{of nodes}}$ of them. Explain "nonseparatedness" of moduli stack of surfaces.
(4) Hironaka example mod $\mathbf{Z}/2\mathbf{Z}$ if you like.

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[spa] The stacks project authors. *Stacks Project*. `http://math.columbia.edu/algebraic_geometry/stacks-git`.