Intersection Theory

This is an old note on intersection theory written for a graduate student seminar in the Fall of 2007 organized by Johan de Jong. In the summer of 2009 a new group of students led by Johan de Jong and Qi You reworked this material which then because a chapter of the Stacks project. We strongly urge the reader to read this online at

https://stacks.math.columbia.edu/tag/0AZ6

instead of reading the old material below. In particular, we do not vouch for the correctness of what follows.

Cycles. Let X be a nonsingular projective variety over an algebraically closed field $\mathbf{C}$. A k-cycle on X is a finite formal sum $\sum n_i[Z_i]$ where each Z_i is a closed subvariety of dimension k.

Pushforward. Suppose that $f : X \to Y$ is a morphism of projective smooth varieties. Let $Z \subset X$ be a k-dimensional closed subvariety. We define $f_*[Z]$ to be 0 if $\dim(f(Z)) < k$ and $d \cdot [f(Z)]$ if $\dim(f(Z)) = k$ where $d = [\mathbf{C}(Z) : \mathbf{C}(f(Z))]$. Let $\alpha = \sum n_i[Z_i]$ be a k-cycle on Y. The *pushforward* of α is the sum $f_*\alpha = \sum n_i f_*[Z_i]$ where each $f_*[Z_i]$ is defined as above.

Cycle associated to closed subscheme. Suppose that X is a nonsingular projective variety and that $Z \subset X$ is a closed subscheme with $\dim(Z) \leq k$. Let Z_i be the irreducible components of Z of dimension k and let n_i be the length of the local ring of Z at the generic point of Z_i. We define the k-cycle associated to Z to be the k-cycle $[Z]_k = \sum n_i[Z_i]$.

Cycle associated to a coherent sheaf. Suppose that X is a nonsingular projective variety and that $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module on X with $\dim(Supp(\mathcal{F})) \leq k$. Let Z_i be the irreducible components of $Supp(\mathcal{F})$ of dimension k and let n_i be the length of the stalk of $\mathcal{F}$ at the generic point of Z_i. We define the k-cycle associated to $\mathcal{F}$ to be the k-cycle $[\mathcal{F}]_k = \sum n_i[Z_i]$.

Note that, if $\dim(Z) \leq k$, then $[Z]_k = [\mathcal{O}_Z]_k$.

Suppose that $f : X \to Y$ is a morphism of projective smooth varieties. Let $Z \subset X$ be a k-dimensional closed subvariety. It can be show that $f_*[Z] = [f_*\mathcal{O}_Z]_k$. See [Serre, Chapter V].

Flat pullback. Suppose that $f : X \to Y$ is a flat morphism of nonsingular projective varieties of relative dimension r, in other words all fibres have dimension r. Let $Z \subset X$ be a k-dimensional closed subvariety. We define $f^*[Z]$ to be the $k+r$-cycle associated to the scheme theoretic inverse image: $f^*[Z] = [f^{-1}(Z)]_{k+r}$. Let $\alpha = \sum n_i[Z_i]$ be a k-cycle on Y. The *pullback* of α is the sum $f_*\alpha = \sum n_i f^*[Z_i]$ where each $f^*[Z_i]$ is defined as above.

With this notation, we get that $f^*[\mathcal{F}]_k = [f^*\mathcal{F}]_{k+r}$ if $\mathcal{F}$ is a coherent sheaf on Y and the dimension of the support of $\mathcal{F}$ is at most k.

Intersection multiplicities using Tor formula. Suppose that X is a nonsingular projective variety and that $W, V \subset X$ are closed subvarieties with $\dim(W) = s$ and $\dim(V) = r$. Assume that $\dim(W \cap V) \leq r+s-\dim(X)$. We say that W and V *intersect properly* if this holds. In this case the sheaves $Tor_j^{\mathcal{O}_X}(\mathcal{O}_W, \mathcal{O}_V)$ are coherent, supported on $V \cap W$, and zero if $j < 0$ or $j > \dim(X)$. We define

$$W \cdot V = \sum_i (-1)^i [Tor_j^{\mathcal{O}_X}(\mathcal{O}_W, \mathcal{O}_V)]_{r+s-\dim(X)}.$$

With this notation, the cycle $V \cdot W$ is a formal linear combination $\sum e_i Z_i$ of the irreducible components Z_i of the intersection $V \cap W$. The integers e_i are called the *intersection multiplicities*: $e_i = e(X, V \cdot W, Z_i)$. They satisfy many good properties, see [Serre].

Computing intersection multiplicities. In the situation above, let $Z = Z_i$ be one of the irreducible components. Let A is be the local ring of X at the generic point of Z. Suppose that the ideal of V in A is cut out by a regular sequence $x_1, \ldots, x_c$ and suppose that the local ring of W at the generic point of Z corresponds to the quotient map $A \to B$. In this case $e(X, V \cdot W, Z)$ is equal to $c!$ times the leading coefficient in the Hilbert polynomial

$$t \mapsto \text{length}_A(B/(x_1, \ldots, x_c)^t B), \quad t >> 0.$$

Consider the case that $c = 1$, i.e., V is a(n effective) Cartier divisor. Then x_1 is a nonzero divisor on B by properness of intersection of V and W. We easily deduce

$$e(X, V \cdot W, Z) = \text{length}_A(B/x_1 B).$$

More generally, if the local ring B is Cohen-Macaulay, then we have

$$(*) \qquad e(X, V \cdot W, Z) = \text{length}_A(B/x_1 B + \ldots + x_c B).$$

Intersection product using Tor formula. Suppose that X is a nonsingular projective variety. Suppose $\alpha = \sum n_i [W_i]$ is an r-cycle, and $\beta = \sum_j m_j [V_j]$ is an s-cycle on X. We say that α and β *intersect properly* if W_i and V_j intersect properly for all i and j. In this case we define

$$\alpha \cdot \beta = \sum_{i,j} n_i m_j W_i \cdot V_j.$$

where $W_i \cdot V_j$ is as defined above using the Tor-formula.

Suppose $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves on X with $\dim(Supp(\mathcal{F})) \leq s$, $\dim(Supp(\mathcal{G})) \leq r$ and $\dim(Supp(\mathcal{F}) \cap Supp(\mathcal{G})) \leq r + s - \dim X$. In this case

$$[\mathcal{F}]_s \cdot [\mathcal{G}]_r = \sum (-1)^i [Tor_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})]_{r+s-\dim X}.$$

See [Serre, Chapter V].

Exterior product. Let X and Y be nonsingular projective vareities. Let V, resp. W be a closed subvariety of X, resp. Y. The product $V \times W$ is a closed subscheme of $X \times Y$. It is a subvariety because the ground field is algebraically closed. For a k-cycle $\alpha = \sum n_i [V_i]$ and a l-cycle $\beta = \sum m_j [V_j]$ on Y we define $\alpha \times \beta = \sum n_i m_j [V_i \times W_j]$.

Consider the subvariety $X \subset X$ with class $[X]$. Note that $pr_Y^*(\beta) = [X] \times \beta$. Note that $\alpha \times [Y]$ and $[X] \times \beta$ intersect properly on $X \times Y$. With the definitions above we have $\alpha \times \beta = (\alpha \times [Y]) \cdot ([X] \times \beta) = pr_Y^*(\alpha) \cdot pr_X^*(\beta)$.

Reduction to the diagonal. Let X be a nonsingular projective variety. Let $\Delta \subset X \times X$ denote the diagonal. We will identify Δ with X. Let α, resp. β be r-cycles, resp. s-cycles on X. Assume α and β intersect properly. In this case $\alpha \times \beta$ and $[\Delta]$ intersect properly. Note that the cycle $\Delta \cdot \alpha \times \beta$ is supported on the diagonal and hence we can think of it as a cycle on X. With this convention we have $\alpha \cdot \beta = \Delta \cdot \alpha \times \beta$. See [Serre, Chapter V].

Perhaps a less confusing formulation would be that $pr_{1,*}(\Delta \cdot \alpha \times \beta) = \alpha \cdot \beta$, where $pr_1 : X \times X \to X$ is the projection.

Flat pullback and intersection products. Suppose that $f : X \to Y$ is a flat morphism of nonsingular projective varieties. Suppose that α is a k-cycle on Y and that β is a l-cycle on Y. Assume that α and β intersect properly. Then $f^*\alpha$ and $f^*\beta$ intersect properly and $f^*(\alpha \cdot \beta) = f^*\alpha \cdot f^*\beta$. This is not hard to see from the material above.

Projection formula for flat maps. Let $f : X \to Y$ be a flat morphism of relative dimension r of nonsingular projective varieties. Let α be an k-cycle on X and let β be a l-cycle on Y. Assume that $f_*(\alpha)$ and β intersect properly, and that α and $f^*(\beta)$ intersect properly. The projection formula says that $f_*(\alpha) \cdot \beta = f_*(\alpha \cdot f^*\beta)$ in this case. See [Serre, Chapter V, Section 7, formula (10)] for a more general formula.

We explain how to prove the projection formula in the flat case. Let $W \subset X$ be a closed subvariety of dimension k. Let $V \subset Y$ be a closed subvariety of dimension l, so $f^{-1}(V)$ has pure dimension $l + r$. Assume that W and $[f^{-1}(V)]$ intersect properly. Note that $f(W \cap f^{-1}(V)) = f(W) \cap V$. Hence it follows that $f(W)$ and V intersect properly as well. Let $Z \subset f(W) \cap V$ be an irreducible component of dimension $k + l - \dim Y$. Let $Z_i \subset W \cap f^{-1}(V)$ be the irreducible components of $W \cap f^{-1}(V)$ dominating Z. Let A be the local ring of X at the generic point of Z. Let A_i be the local ring of Y at the generic point of Z_i. Let B be the local ring of $f(W)$ at the generic point of Z. Let B' be the stalk of $f_*(\mathcal{O}_W)$ at the generic point of Z. Then $B \to B'$ is finite, B' is semi-local, and the localizations B'_i of B' are the local rings of W at the generic point of the Z_i. Thus they are quotients $A_i \to B'_i$. Let C be the local ring of V at the generic point of Z. The multiplicity of Z in $f_*([W]) \cdot V$ is by definition

$$(I) = [B' : B] \sum (-1)^j \text{length}_A(Tor_j^A(B, C)).$$

Here $[B' : B]$ is the rank of the B-module B'. The multiplicity of Z in $f_*(W \cdot f^*[V])$ is by definition

$$(II) = \sum\nolimits_{i,j} (-1)^j \text{length}_{A_i}(Tor_j^{A_i}(B'_i, A_i \otimes_A C))[\kappa(A_i) : \kappa(A)]$$

Here $\kappa(-)$ indicates the residue field. The first thing is to note that $\text{length}_A(M) = [\kappa(A_i) : \kappa(A)]\text{length}_{A_i}(M)$ for a finite length A_i-module M. We can compute all the Tor groups by choosing a free resolution of C as an A-module. Doing this it is easy to see that (I) equals $\sum (-1)^j \text{length}_A(Tor_j^A(B', C))$. Finally, note that, by definition, there is an A-module map $B^{\oplus [B':B]} \to B'$ whose kernel and cokernel are supported in a proper closed subset of $Spec(B)$. From the additivity properties of the Tor-formula, see [Serre, Chapter V], it follows that $\sum (-1)^j \text{length}_A(Tor_j^A(B', C)) = [B' : B] \sum (-1)^j \text{length}_A(Tor_j^A(B, C))$ as desired.

Rational Equivalence. Let X be a nonsingular projective variety. Let $\alpha = \sum n_i [W_i]$ be a $(k+1)$-cycle on $X \times \mathbf{P}^1$, and let a, b be two closed points of $\mathbf{P}^1$. Assume that $X \times a$ and α intersect properly, and that $X \times b$ and α intersect properly. This will be the case if each W_i dominates $\mathbf{P}^1$ for example. Let $pr_X : X \times \mathbf{P}^1 \to X$ be the projection morphism. A *cycle rationally equivalent to zero* is any cycle of the form

$$pr_{X,*}(\alpha \cdot X \times a - \alpha \cdot X \times b).$$

This is a k-cycle. Note that these cycles are easy to compute in practice (given α) because they are obtained by proper intersection with Cartier divisors (see formula above). It is a fact that the collection of k-cycles rationally equivalent to zero is a additive subgroup of the group of k-cycles. We say two k cycles are *rationally equivalent*, notation $\alpha \sim_{rat} \alpha'$ if $\alpha - \alpha'$ is a cycle rationally equivalent to zero. See Chapter I of [Fulton].

Pushforward and rational equivalence. Suppose that $f : X \to Y$ is a morphism of projective smooth varieties. Let $\alpha \sim_{rat} 0$ be a k-cycle on X rationally equivalent to 0. Then the pushforward of α is rationally equivalent to zero: $f_*\alpha \sim_{rat} 0$. See Chapter I of [Fulton].

Pullback and rational equivalence. Suppose that $f : X \to Y$ is a flat morphism of relative dimension r of projective smooth varieties. Let $\alpha \sim_{rat} 0$ be a k-cycle on Y rationally equivalent to 0. Then the pullback of α is rationally equivalent to zero: $f^*\alpha \sim_{rat} 0$. See Chapter I of [Fulton].

Moving Lemma. The moving lemma states that given an r-cycle α and a s cycle β there exists α', $\alpha' \sim_{rat} \alpha$ such that α and β intersect properly. See [Samuel], [Chevalley], or [Fulton, Example 11,4,1].

Intersection product and rational equivalence. With definitions as above we show that the intersection product is well defined modulo rational equivalence. Let X be a nonsingular projective algebraic variety. Let α, resp. β be a s, resp. r cycle on X. Assume that α and β intersect properly so that $\alpha \cdot \beta$ is defined. Finally, assume that $\alpha \sim_{rat} 0$. Goal: show that $\alpha \cdot \beta \sim_{rat} 0$.

After some formal arguments this amounts to showing the following statement. Let $W \subset X \times \mathbf{P}^1$ be a $(s+1)$-dimensional subvariety dominating $\mathbf{P}^1$. Let W_a, resp. W_b be the fibre of $W \to \mathbf{P}^1$ over a, resp. b. Let V be a r-dimensional subvariety of X such that V intersects both W_a and W_b properly. Then $V \cdot [W_a] \sim_{rat} V \cdot [W_b]$.

In order to see this, note first that $[W_a] = pr_{X,*}(W \cdot X \times a)$ and similar for $[W_b]$. Thus we reduce to showing

$$V \cdot pr_{X,*}(W \cdot X \times a) \sim_{rat} V \cdot pr_{X,*}(W \cdot X \times b).$$

The projection formula – which may be applied – says $V \cdot pr_{X,*}(W \cdot X \times a) = pr_{X,*}(V \times \mathbf{P}^1 \cdot (W \cdot X \times a))$, and similar for b. Thus we reduce to showing

$$pr_{X,*}(V \times \mathbf{P}^1 \cdot (W \cdot X \times a)) \sim_{rat} pr_{X,*}(V \times \mathbf{P}^1 \cdot (W \cdot X \times b))$$

Associativity for the intersection multiplicities (see [Serre, Chapter V]) implies that $V \cdot (W \cdot X \times a) = (V \times \mathbf{P}^1 \cdot W) \cdot X \times a$ and similar for b. Thus we reduce to showing

$$pr_{X,*}((V \times \mathbf{P}^1 \cdot W) \cdot X \times a) \sim_{rat} pr_{X,*}((V \times \mathbf{P}^1 \cdot W) \cdot X \times b)$$

which is true by definition of rational equivalence.

Upshot: Chow rings. Using the above, for any nonsingular projective X we set $A_k(X)$ equal to the group of k-cycles on X modulo rational equivalence. Since it is more convenient we also use $A^c(X) = A_{\dim X - c}(X)$ to denote the group of codimension c cycles modulo rational equivalence. The intersection product defines a product

$$A^k(X) \times A^l(X) \longrightarrow A^{k+l}(X)$$

defined as follows: for $a \in A^k(X)$ and $b \in A^l(X)$ we can find a codimension k cycle α representing α, a codimension l cycle β representing b such that α and β intersect properly. We define $a \cdot b$ to be the rational equivalence class of $\alpha \cdot \beta$. End result: A commutative and associative graded ring $A^*(X)$ with unit $1 = [X]$.

Pullback for a general morphism. Let X and Y be nonsingular projective varieties, and let $f : X \to Y$ be a morphism. We define

$$f^* : A_k(Y) \to A_{k+\dim X - \dim Y}(X)$$

by the rule

$$f^*(\alpha) = pr_{X,*}(\Gamma_f \cdot pr_Y^*(\alpha))$$

where $\Gamma_f \subset X \times Y$ is the graph of f. Note that it is defined only on cycle classes and not on cylces. This pullback satisfies:
(1) $f^* : A^*(Y) \to A^*(X)$ is a ring map,
(2) $(f \circ g)^* = g^* \circ f^*$ for a composable pair f, g,
(3) the projection formula holds: $f_*(\alpha) \cdot \beta = f_*(\alpha \cdot f^*\beta)$, and
(4) if f is flat then it agrees with the previous definition.

All of these follow easily from the above. For (1) you have to show that $pr_{X,*}(\Gamma_f \cdot \alpha \cdot \beta) = pr_{X,*}(\Gamma_f \cdot \alpha) \cdot pr_{X,*}(\Gamma_f \cdot \beta)$. It is easy to see that if α intersects Γ_f properly, then $\Gamma_f \cdot \alpha = \Gamma_f \cdot pr_X^*(pr_{X,*}(\Gamma_f \cdot \alpha))$ as cycles because Γ_f is a graph. Thus we get

$$\begin{aligned} pr_{X,*}(\Gamma_f \cdot \alpha \cdot \beta) &= pr_{X,*}(\Gamma_f \cdot pr_X^*(pr_{X,*}(\Gamma_f \cdot \alpha)) \cdot \beta) \\ &= pr_{X,*}(pr_X^*(pr_{X,*}(\Gamma_f \cdot \alpha)) \cdot (\Gamma_f \cdot \beta)) \\ &= pr_{X,*}(\Gamma_f \cdot \alpha) \cdot pr_{X,*}(\Gamma_f \cdot \beta) \end{aligned}$$

the last step by the projection formula in the flat case. Properties (2) and (3) are formal [for (3) use the flat projection formula twice]. Property (4) rests on identifying the intersection product $\Gamma_f \cdot \alpha$ in the case f is flat.

Pullback of cycles. Suppose that X and Y be nonsingular projective varieties, and let $f : X \to Y$ be a morphism. Suppose that $Z \subset Y$ is a closed subvariety. Let $f^{-1}(Z)$ be the scheme theoretic inverse image:

$$\begin{array}{ccc} f^{-1}(Z) & \to & Z \\ \downarrow & & \downarrow \\ X & \to & Y \end{array}$$

is a fibre product diagram of schemes. In particular $f^{-1}(Z) \subset X$ is a closed subscheme of X. In this case we always have

$$\dim f^{-1}(Z) \geq \dim Z + \dim X - \dim Y.$$

If equality holds in the formula above, then $f^*[Z] = [f^{-1}(Z)]_{\dim Z + \dim X - \dim Y}$ provided that the scheme Z is Cohen-Macaulay at the images of the generic points of $f^{-1}(Z)$. This follows by identifying $f^{-1}(Z)$ with the scheme theoretic intersection of Γ_f and $X \times Z$ and using the computation $(*)$ of the intersection multiplicities we gave above.

References

[Serre] Algèbre Locale Multipliciés, LNM 11.
[Samuel] Rational Equivalence of Arbitrary Cycles.
[Chevalley] Less classes d'équivalence rationelle, I, II
[Fulton] Intersection Theory