- PART I. Integration.

1. Compute the following integrals.

(a) $\int (\ln x)^2 dx$

Integrate by parts twice, the first time choose:

$$f = (\ln x)^2, \quad g = 1$$

and the second time choose:

$$f = \ln x, \quad g = 1.$$

Then

$$\int (\ln x)^2 dx = x(\ln x)^2 - 2\int x\frac{\ln x}{x}dx =$$
$$x(\ln x)^2 - 2\{x\ln x - \int x\frac{1}{x}dx\} =$$
$$x(\ln x)^2 - 2x\ln x + 2x + C.$$

(b) $\int \frac{dx}{x^2\sqrt{4x^2+1}}$

Perform the trig substitution $x = \frac{1}{2}\tan\theta, dx = \frac{1}{2}\sec^2\theta d\theta$, then

$$\int \frac{dx}{x^2\sqrt{4x^2+1}} =$$
$$= \int \frac{2\sec\theta}{\tan^2\theta} = 2\int \frac{\cos\theta}{\sin^2\theta}d\theta =$$

(use substitution $t = \sin\theta$) $= -2\frac{1}{\sin\theta} + C = -\frac{\sqrt{4x^2+1}}{x} + C$

where the last equality holds because

$$x^2 = \frac{\tan^2\theta}{4} = \frac{\sin^2\theta}{4\cos^2\theta} = \frac{\sin^2\theta}{4(1-\sin^2\theta)}.$$

Thus

$$4x^2 = 4x^2 \sin^2\theta + \sin^2\theta$$

and

$$\sin^2\theta = \frac{4x^2}{4x^2+1}.$$

This last part is not necessary. You can just substitute

$$\theta = \tan^{-1}(2x).$$

(c) Compute $\displaystyle\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}\,dx$

Apply partial fractions:

$$\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1} = \frac{x^2-2x-1}{(x-1)^2(x^2+1)}$$

and find

$$A = 1, B = C = D = -1.$$

Hence,

$$\int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}\,dx =$$

$$= \ln|x-1| - \frac{1}{x-1} - \frac{1}{2}\ln(x^2+1) - \arctan x + c.$$

2. Determine whether the following integrals are converging or diverging.

(a) $\displaystyle\int_1^{+\infty} \frac{\arctan x}{x^2+7}\,dx$

Since

$$\frac{\arctan x}{x^2+7} \le \frac{\pi}{2}\frac{1}{x^2}$$

our integral converges by the comparison test with a p-integral with $p > 1$.

(b) $\int_{-\infty}^{+\infty} x^3 e^{-x^4}\, dx$

$$\int_{-\infty}^{+\infty} x^3 e^{-x^4}\, dx =$$
$$= \int_{-\infty}^{0} x^3 e^{-x^4}\, dx + \int_{0}^{+\infty} x^3 e^{-x^4}\, dx =$$
$$= 2\int_{0}^{+\infty} x^3 e^{-x^4}\, dx = 2 \lim_{t\to+\infty} \int_{0}^{t} x^3 e^{-x^4}\, dx$$

But,

$$\int_{0}^{t} x^3 e^{-x^4}\, dx = -\frac{1}{4}e^{-t^4} + \frac{1}{4} \to \frac{1}{4}$$

as t goes to infinity. Hence our integral converges.

- Part II. Sequences and Numerical series.

1. Determine whether the sequence converges or diverges. If it converges, find the limit.

   (a) $a_n = n \sin(2/n)$

   It converges to 2, using L'Hopital ($\lim_{x \to 0} \frac{\sin 2x}{x} = 2.$)

   (b) $a_n = 2^n/n!$

   $$0 < a_n \leq 4/n \to 0,$$

   hence a_n converges to 0 by the squeeze theorem.

2. Determine whether the series is convergent or divergent.

   (a) $\sum_{n=1}^{\infty} \frac{n+4^n}{n+6^n}$.

   It converges by comparison with the geometric series with $r = 4/6$. Indeed,

   $$\frac{n+4^n}{n+6^n} \leq 2\frac{4^n}{6^n}.$$

   (b) $\sum_{n=1}^{\infty} (-1)^n \cos(\pi/n)$

   It diverges by the divergence test.

3. Determine whether the series is convergent or divergent.

   (a) $\sum_{n=1}^{\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}}$

   Apply the limit comparison test with

   $$a_n = \frac{n+5}{\sqrt[3]{n^7+n^2}} \quad b_n = \frac{n}{n^{7/3}} = \frac{1}{n^{4/3}}.$$

   Then,

   $$\lim_{n \to \infty} \frac{a_n}{b_n} = 1 > 0$$

and our series converges because $\sum_n b_n$ is a p-series with $p = 4/3 > 1$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

Set $f(x) = \dfrac{\ln x}{x^3}$ and check that

$$f' = -3x^{-4}\ln x + x^{-4} < 0$$

if $-3\ln x + 1 < 0$ that is for $\ln x > 1/3$ which is true if $x > e^{1/3}$. Then compute,

$$\int_1^{\infty} \frac{\ln x}{x^3} dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x^3} dx.$$

Since,

$$\int \frac{\ln x}{x^3} dx = -\frac{1}{2x^2}\ln x + \frac{1}{2}\int \frac{1}{x^3} dx = -\frac{1}{2x^2}\ln x - \frac{1}{4x^2}$$

then

$$\lim_{t\to\infty} \int_1^t \frac{\ln x}{x^3} dx = \lim_{t\to\infty} -\frac{1}{2t^2}\ln t - \frac{1}{4t^2} + \frac{1}{4} = \frac{1}{4}.$$

Thus our series converges by the integral test.

4. Determine whether the series is conditionally convergent, absolutely convergent or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2}$

By the root test, the series is diverging. Indeed,

$$\sqrt[n]{|a_n|} = \left(1 + \frac{1}{n}\right)^n \to e > 1.$$

(b) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{10^n}{(n+1)4^{2n+1}}$

By the ratio test, the series converges absolutely. Indeed,

$$\frac{|a_{n+1}|}{|a_n|} \to 10/16 < 1.$$

- Part III. Power series.

  1. Find the radius of convergence and interval of convergence of the series
  $$\sum_{n=1}^{\infty}(-1)^n \frac{x^n}{n^2 5^n}.$$

     By the ratio test,
     $$|a_{n+1}|/|a_n| \to |x|/5 < 1,$$
     hence $R = 5$. The interval of convergence is $[-5, 5]$, extrema included because
     $$x = 5 \Rightarrow \sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2}$$
     which converges by the alternating series test ($b_n = 1/n^2$ converges to 0 and it is decreasing), while
     $$x = -5 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2}$$
     which is a p-series with $p = 2 > 1$ and hence it converges.

  2. Use the definition to find the Taylor series of $f(x) = 1/\sqrt{x}$ centered at $a = 9$. Also find the associated radius of convergence.

     Compute

     $$f'(x) = -1/2x^{-3/2},\ f''(x) = 3/2^2 x^{-5/2},\ f'''(x) = -3 \cdot 5/2^3 x^{-7/2}.$$

     Hence

     $$f(9) = 1/3, f'(9) = -1/(2{\cdot}3^{-3}),\ f''(9) = 3/(2^2 3^{-5}),\ f'''(x) = -(3{\cdot}5)/(2^3 3^{-7}).$$

     Thus,
     $$f^{(n)}(9) = \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdot (2n-1)}{2^n 3^{2n+1}}.$$

The desired Taylor series is

$$\sum_{n=0}^{\infty} \frac{(-1)^n 1 \cdot 3 \cdot 5 \cdot (2n-1)}{2^n 3^{2n+1} n!}(x-9)^n.$$

Using the ratio test one finds that the associated radius of convergence is $R = 9$.