§1 - Introduction

Definition Let $L := \mathrm{Sym}^n(\mathbb{Z}^2) \cong \mathbb{Z}^{n+1}$, $SL(2,\mathbb{Z})$-repr'n. Given Λ = comm. ring with 1_Λ, let

$$L_\Lambda := L \otimes_{\mathbb{Z}} \Lambda$$

Let $X = Y_1(N) = \Gamma \backslash \mathcal{H}$, where $\Gamma = \Gamma_1(N)$ $(N \geq 3)$. Also, define

$$\mathcal{F}_\Lambda := \Gamma \backslash (L_\Lambda \times \mathcal{H})$$

Since Γ acts freely on $\mathcal{H}$, we can see this as a sheaf on the complex analytic site of $X(\mathbb{C})$. We will mostly consider

- $H^q_p(X(\mathbb{C}), \mathcal{F}_\Lambda) := \mathrm{Im}\left(H^q_c(X, \mathcal{F}_\Lambda) \to H^q(X, \mathcal{F}_\Lambda)\right)$
- $H^q_P(\Gamma, L_\Lambda) := Z_P(\Gamma, L_\Lambda) / B(\Gamma, L_\Lambda)$

It is known that

$$H^q(X(\mathbb{C}), \mathcal{F}_\Lambda) \simeq H^q(\Gamma, L_\Lambda) \quad \text{(canonically)}$$

and in fact, this induces

$$H^q_p(X(\mathbb{C}), \mathcal{F}_\Lambda) \simeq H^q_p(\Gamma, L_\Lambda) \quad \text{(canonically)}$$

We also need $\hat{L}_\Lambda$ = dual lattice as well as $\hat{\mathcal{F}}_\Lambda \cong \Gamma \backslash (\hat{L}_\Lambda \times \mathcal{H})$ = dual sheaf.

§2 - Pairings & $\Lambda = \mathbb{Z}_\ell$ or $\mathbb{R}$

Fix a prime ℓ s.t. $\ell > n$ and $\ell \nmid N$. Also, let $\Lambda_\nu = \mathbb{Z}/\ell^\nu\mathbb{Z}$. Then, by Poincaré duality

$$A_\Lambda : H^1_p(X, \xi_\Lambda) \times H^1_p(X, \tilde{\xi}_\Lambda) \to H^2_c(X, \Lambda) \xrightarrow{\sim} \Lambda$$

"is" perfect for $\Lambda = \Lambda_\nu$. Actually

$$H^1_p(X, \xi_{\Lambda_\nu}) \cong H^1_p(X, \xi_{\mathbb{Z}}) \otimes \Lambda_\nu \quad (\& \text{ for } \tilde{\xi}_{\Lambda_\nu})$$

$\Rightarrow H^1_p(X, \xi_{\mathbb{Z}_\ell}) \times H^1_p(X, \tilde{\xi}_{\mathbb{Z}_\ell}) \longrightarrow \mathbb{Z}_\ell$ is perfect.

Now: get rid of $\tilde{\xi}$.

Trick: There exists $\Theta_n \in M_{n+1}(\mathbb{Z})$ s.t.

$$\Theta_n : L_{\mathbb{Z}} \longrightarrow \tilde{L}_{\mathbb{Z}} \quad (\text{as } \Gamma\text{-modules})$$
$$x \longmapsto \Theta_n x$$

$\Rightarrow \Theta_n : \xi_\Lambda \longrightarrow \tilde{\xi}_\Lambda$. (surjective for $\Lambda = \mathbb{Z}_\ell$ & $\mathbb{Q}$).

<u>Definition</u> Let $\langle , \rangle_\Lambda$ be the pairing on $H^1_p(X, \xi_\Lambda)$

$$\langle x, y \rangle_\Lambda = A_\Lambda(x, \Theta_n y)$$

This is perfect for $\Lambda = \mathbb{Z}_\ell$.

Now, $\Lambda = \mathbb{R}$. For $k = n+2$, we have an isom.

$$\delta : S_k(\Gamma) \xrightarrow[\text{(over } \mathbb{R})]{\sim} H^1_p(X, \mathcal{F}_{\mathbb{R}}) \quad \leftarrow \text{de Rham coh.}$$

$$f \longmapsto \mathrm{Re}\left(f(z) \cdot \begin{bmatrix} z \\ 1 \end{bmatrix}^n dz \right),$$

where $[\ ]^n : \mathbb{Z}^2 \longrightarrow \mathrm{Sym}^n(\mathbb{Z}^2) \simeq \mathbb{Z}^{n+1}$.

Then, the pairing above can be written as

$$\langle f, g \rangle_{\mathbb{R}} = \int_{\Gamma \backslash \mathcal{H}} {}^t\delta(f) \wedge \Theta_n \delta(g)$$

Using the explicit format of Θ_n, one sees

1) $\langle f, g \rangle_{\mathbb{R}} = (2i)^{n-1} \left[(f,g)_P + (-1)^{n+1} (g,f)_P \right]$

2) $\langle f, g \rangle_{\mathbb{R}} = (-1)^{n+1} \langle g, f \rangle_{\mathbb{R}}$ $\leftarrow$ (skew)-symmetric

3) $\langle f, i^{n+1} g \rangle_{\mathbb{R}} = 2^n \,\mathrm{Re}\left((f,g)_P \right)$ $\leftarrow$ perfect

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From now on, use the isom.'s

$$\begin{array}{ccc} S_k(\Gamma) & \xrightarrow[\sim]{\ \delta\ } & H^1_p(X, \mathcal{F}_{\mathbb{R}}) \\ & {\scriptstyle [S]}\searrow^{\wr} \quad \circlearrowleft \quad \nearrow^{\sim}{\scriptstyle \iota} & \\ & H^1_p(\Gamma, L_{\mathbb{R}}) & \end{array}$$

and write everything in terms of $H^1_p(\Gamma, L_{\mathbb{R}})$.

In fact, let

1) $V(N;\mathbb{R}) = H^1_p(\Gamma, L_{\mathbb{R}})$

2) $V(N;\mathbb{Z}) := \mathrm{Im}\left(H^1_p(\Gamma, L_{\mathbb{Z}}) \to H^1_p(\Gamma, L_{\mathbb{R}})\right)$

3) $\langle x, y \rangle_N := \langle \iota(x), \iota(y) \rangle^P_{\mathbb{R}}$

Proposition 1) $V(\mathbb{Z})$ is a lattice in $V(\mathbb{R})$

2) $\langle x, y \rangle_N \in \mathbb{Z}$, for $x, y \in V(\mathbb{Z})$

3) Let $V(N;\mathbb{Z}_\ell) = V(N;\mathbb{Z}) \otimes \mathbb{Z}_\ell \cong H^1_p(\Gamma, L_{\mathbb{Z}_\ell})$. The $\mathbb{Z}_\ell$-extension of $\langle\,,\,\rangle_N$ to $V(\mathbb{Z}_\ell)$ is perfect.

Let $\mathbb{T}_N$ = Hecke algebra acting faithfully on $S_k(\Gamma)$ and $\mathbb{T}_N^*$ be the "dual" Hecke algebra.

($\mathbb{T}_N \leftrightarrow \Gamma\alpha\Gamma$ & $\mathbb{T}_N^* \leftrightarrow \Gamma\,{}^{\iota}\alpha\,\Gamma$, ${}^{\iota}\alpha = \det(\alpha)\alpha^{-1}$)

Proposition $\langle x \mid T(m), y \rangle_N = \langle x, y \mid T^*(m) \rangle$

§3 - Discriminants

K = CM field or tot. real field
V = f. dim'l K - v. sp.
T = (skew-) symmetric perfect, $\mathbb{Q}$-bilinear pairing on V such that

$$\langle ax, y\rangle = \langle x, \bar{a}y\rangle, \quad \forall x, y \in V, \ a \in K$$

Let $L \subset V$ be a lattice and

$$d(T; L) = \text{Discriminant}(T|_L) \in \mathbb{Q}^\times/\mathbb{Q}^{\times, 2}$$

Let $r = [K : \mathbb{Q}]$, $s = \dim_K(V)$.

Proposition

1) If T is skew-symmetric, $d(T) = 1$
2) If s = even, K = CM-field, $d(T) = 1$
3) If s = odd, K = CM-field & T is symm, then $d(T) \equiv |d(K)| \mod \mathbb{Q}^{\times, 2}$.

Proposition Suppose $T(L, L) \subset \mathbb{Z}$ and let $L^* \subset V$ be the lattice dual to L, then

$$|d(T; L)| = [L^* : L]$$

§4 - Special Values of L-functions

Let $f(z) = \sum a_n q^n \in S_k(\Gamma)$ be a normalized Hecke newform for $\Gamma = \Gamma_1(N)$. Then,

- $f \mid T(n) = a_n f$, $n \in \mathbb{Z}_{>0}$
- $f \mid [a] = \psi(a) f$, $a \in (\mathbb{Z}/N\mathbb{Z})^\times$,

for some character ψ. Denote the primitive char. inducing ψ above by ψ again.

Let $\alpha_p, \beta_p \in \mathbb{C}$ be the roots of

$$x^2 - a_p x + \psi(p) p^{k-1}$$

(0 if $p \mid N$)

and given any Dirichlet character ω of cond. M,

$$L(s, f, \omega) = \prod_p \left[(1 - \omega(p)\alpha_p^2 p^{-s})(1 - \omega(p)\alpha_p \beta_p p^{-s})(1 - \omega(p)\beta_p^2 p^{-s}) \right]^{-1}$$

This can be re-interpreted as L-function of the product of two cusp forms and one obtains

Theorem Let $N(\psi)$ be the conductor of ψ, then

$$L(k, f, \bar{\psi}) = \frac{2^{2k} \pi^{k+1} (k-1)!}{N \cdot N(\psi) \cdot \varphi(N/N(\psi))} \cdot (f, f)_\Gamma$$

(φ: Euler's totient function)

§5 - Discriminants associated to Newforms

Take $f \in S_k(\Gamma)$ as above. Then, set

- $K_f = \mathbb{Q}(\{a_n\}) = \mathbb{Q}(\{a_n\} : (n,N)=1)$
  $= CM$-field or tot. real field
- $I_f = \{\text{Embeddings } K_f \overset{\sigma}{\hookrightarrow} \mathbb{C}\}$
- $f^\sigma = \sum a_n^\sigma q^n \in S_k(\Gamma), \ \forall \sigma \in I_f$
- $S(f) = \sum_{\sigma \in I} \mathbb{C}. f^\sigma \in S_k(\Gamma)$
- $W_f(\mathbb{R}) =$ Image of $S(f)$ inside $V(\mathbb{R})$
- $L_f = W_f(\mathbb{R}) \cap V(\mathbb{Z}) \rightsquigarrow$ A lattice in $V(\mathbb{R})$.

We want $L_f \subset K_f$ - v. space. Take

- $W_f(\mathbb{Q}) = L_f \otimes \mathbb{Q} = W_f(\mathbb{R}) \cap V(\mathbb{Q})$
- $a_n x := x | T(n), \ \forall n \in \mathbb{Z}_{>0}$ & $x \in W_f(\mathbb{Q})$

Since f is a newform, then we indeed have

$$\langle a x, y \rangle_N = \langle x, \bar{a} y \rangle_N, \quad \forall a \in K, \ x, y \in W_f(\mathbb{Q})$$

Let T_f = restriction of $\langle , \rangle_N$ of $W_f(\mathbb{Q})$. We want to compute $d(T_f ; L_f) =: d(f)$.

Let $r = [K_f : \mathbb{Q}]$ and relabel the f^σ by $\{f_1, \dots, f_r\}$
Then, let $\varphi : S_k(\Gamma) \xrightarrow{\sim} V(\mathbb{R})$ and set

$$\omega_\nu = \begin{cases} \varphi(f_\nu), & 1 \le \nu \le r \\ \varphi(i f_{\nu - r}) & r < \nu \le 2r \end{cases}$$

$\Rightarrow$ ω_ν is a basis of $W_f(\mathbb{R})$.

Let $\{\delta_1, \dots, \delta_{2r}\}$ be any basis of L_f. Then,

$$\exists\, U \in GL_{2r}(\mathbb{R}) \text{ s.t. } (\delta_1, \dots, \delta_{2r}) U = (\omega_1, \dots, \omega_{2r})$$

Define $u(f) := |\det(U)|$

Theorem Let $r(f) = [K_f : \mathbb{Q}]$, $n = k-2 \geq 0$, then

$$d(f) = [L_f^* : L_f] = \Big[u(f)^{-1} 2^{nr(f)} \prod_{\sigma \in I_f} (f^\sigma, f^\sigma)_\Gamma \Big]^2$$

This is a straight-forward computation using the properties of $\langle , \rangle_N$ mentioned previously.

Definition Let $c(f) := u(f)^{-1} 2^{nr(f)} \prod_\sigma (f^\sigma, f^\sigma)_\Gamma$
$$= + \sqrt{d(f)}$$

Theorem $c(f) \in \mathbb{Z}$ except if

1) k is odd ($\Rightarrow T_f$ is symmetric)
2) $r(f) = [K_f : \mathbb{Q}]$ is odd ($\Rightarrow K_f$ is tot. real)
3) There exists some quad. imag. field M, and Hecke character $\lambda : \mathbb{A}_M^\times / M^\times \to \mathbb{C}$ s.t. $\lambda(\bar{x}) = \overline{\lambda(x)}$

and $f(z) = \int_0^{+\infty} L(t, \lambda) \cdot t^{z-1} dt$.

If these three conditions are satisfied, let $d \in \mathbb{Z}$ be pos. square-free such that $M = \mathbb{Q}(\sqrt{-d})$. Then,

$$c(f)/\sqrt{d} \in \mathbb{Z}$$

<u>Proof</u> [Sketch] Note that $s = \dim_{K_f} W_f(\mathbb{R}) = 2$

1) If k is even $\Rightarrow T_f$ is skew-symmetric $\Rightarrow$ Done

Assume k is odd now.

2) If $K = $ CM-field $\Rightarrow$ Done (b/c $s = 2 = $ even)

Assume K_f is tot. real $\Rightarrow \bar{\psi} = \psi$ is real. Also, since k is odd, $\psi(-1) = (-1)^k = -1$, so ψ is non-trivial & quadratic.

Hence, $\psi = $ quadratic residue symbol of some quadratic imaginary field M.

Let π be the automorphic representation of $GL(2, \mathbb{A}_{\mathbb{Q}})$ associated to f. Then,

f is a newform + K_f is real

$\Rightarrow \pi \cong \pi \otimes \psi \Rightarrow f$ is a Mellin transform as above.

To conclude, find an action of M on $W_f(\mathbb{Q})$. $\square$

Conclusion Let $Z(s,f) = \prod_\sigma L(s, f^\sigma, \psi^\sigma)$. Then

$$c(f) = \frac{1}{u(f)} \left(\frac{(k-1)! \cdot N \cdot N(\psi) \cdot \varphi(N/N(\psi))}{2^{k+2} \pi^{k+1}} \right)^{r(f)} \cdot Z(k, f)$$

is a positive integer, unless (*) is satisfied, in which case $c(f)/\sqrt{d}$ is a positive integer.

§ Congruences of Cusp Forms

Let $C(f) = \begin{cases} c(f)/\sqrt{d}, & \text{if } (*) \text{ holds} \\ c(f), & \text{otherwise} \end{cases}$

Theorem Let f be as above and ℓ be a prime such that $\ell > k-2$, $\ell \nmid N$ and $\ell \mid C(f)$.

Then, $\exists$ a normalized eigenform g of $S_k(\Gamma)$ and a prime $\mathfrak{q}$ of $\bar{\mathbb{Q}}$ above ℓ such that

1) g is not conjugate to f
2) $g \equiv f \mod \mathfrak{q}$

Remark Say $g = \sum b_n q^n$. We have canonical maps

$$\mathbb{Z}[\{b_n\}] = \mathcal{O}_g \xleftarrow{\phi_g} \mathbb{T}_N \xrightarrow{\phi_f} \mathcal{O}_f = \mathbb{Z}[\{a_n\}]$$

$$b_n \longleftarrow T(n) \longmapsto a_n$$

Let $\mathfrak{q}_1 \subset \mathbb{Z}[a_n]$ be a prime above ℓ and

1) $\mathfrak{q}_0 = \phi_f^{-1}(\mathfrak{q}_1)$
2) $\mathfrak{q}_2 = \phi_g(\mathfrak{q}_0) \subset \mathcal{O}_g$

$\Rightarrow \mathcal{O}_f/\mathfrak{q}_1 \cong \mathbb{T}_N/\mathfrak{q}_0 \longrightarrow \mathcal{O}_g/\mathfrak{q}_2$

$\Rightarrow \mathfrak{q}_2 = \mathcal{O}_g$ OR $\mathcal{O}_g/\mathfrak{p}_2 \cong \mathcal{O}_f/\mathfrak{p}_1$ and we say

In that case, we get congruence b/w f & g for free

$$f \mod \mathfrak{p}_1 \equiv g \mod \mathfrak{p}_2$$

Proof Let $Y = W_f(\mathbb{R})^\perp \subset V(\mathbb{R})$.

We see that $V(\mathbb{R}) = W_f(\mathbb{R}) \oplus Y$ and $L_Y = Y \cap V(\mathbb{Z})$ is a lattice in Y.

Let M_f and M_Y be the projections of $V(\mathbb{Z})$ on $W_f(\mathbb{R})$ and Y respectively, both lattices and

$$M_f \supset L_f \ , \quad M_Y \supset L_Y$$

Denote all their closures in $V(\mathbb{Q}_\ell) = V(\mathbb{Z}) \otimes \mathbb{Q}_\ell$ with a $\bar{\cdot}$. Using the fact that $V(\mathbb{Z}_\ell)$ is self-dual inside $V(\mathbb{Q}_\ell)$, one can show that

$$M_f = L_f^* \ , \quad M_Y = L_Y^*$$

$\Rightarrow \ell \mid [M_f : L_f]$

The two projections $M_f \leftarrow V(\mathbb{Z}) \rightarrow M_Y$ induce

$$M_f/L_f \overset{pr_f}{\underset{\sim}{\longleftarrow}} V(\mathbb{Z})/(L_f \oplus L_Y) \overset{pr_Y}{\underset{\sim}{\longrightarrow}} M_Y/L_Y$$

One can show that L_f and M_f are stable under the action of $\mathbb{T}_N'^*$, so L_Y and M_Y are stable under the action of $\mathbb{T}_N'$. The above are isom.'s of $\mathbb{T}_N'$-modules.

Let $\mathbb{T}_f$ and $\mathbb{T}_Y$ be the Hecke algebras acting faithfully on $W_f(\mathbb{R})$ and Y respectfully.

Then restricting the action of $\mathbb{T}'_N$ on $V(\mathbb{R})$ to either $W_f(\mathbb{R})$ or Y induces surjections

$$\mathbb{T}'_f \xleftarrow{\varphi_f} \mathbb{T}'_N \xrightarrow{\varphi_Y} \mathbb{T}'_Y$$

Then, M_f/L_f is a $\mathbb{T}'_f$-module and $p \mid \#(M_f/L_f)$, so

$\exists$ max'l ideal $\mathfrak{p}_f \subset \mathbb{T}'_f$ containing $\mathrm{Ann}(M_f/L_f)$ such that $\mathbb{T}'_f/\mathfrak{p} \cong \mathbb{Z}/p\mathbb{Z}$

Take

$$\begin{array}{ccccc} \mathbb{T}'_Y & \xleftarrow{\varphi_Y} & \mathbb{T}'_N & \xrightarrow{\varphi_f} & \mathbb{T}'_f \\ \mathfrak{p}_Y := \varphi_Y(\mathfrak{p}) & & \mathfrak{p} := \varphi_f^{-1}(\mathfrak{p}_f) & & \mathfrak{p}_f \end{array}$$

$\Rightarrow$ 1) $\mathrm{Ann}(V(\mathbb{Z})/(L_f \oplus L_Y)) \subseteq \mathfrak{p} \subsetneq \mathbb{T}'_N$

2) $\mathrm{Ann}(M_Y/L_Y) \subseteq \mathfrak{p}_Y \subsetneq \mathbb{T}'_Y$

3) $$\begin{array}{ccc} (M_Y/L_Y) \otimes_{\mathbb{T}'_Y} \mathbb{T}_Y/\mathfrak{p}_Y & & (M_f/L_f) \otimes_{\mathbb{T}'_f} \mathbb{T}'_f/\mathfrak{p}_f \\ \wr\wr & & \wr\wr \\ \mathbb{T}'_Y/\mathfrak{p}_Y & \cong & \mathbb{T}'_f/\mathfrak{p}_f \cong \mathbb{Z}/p\mathbb{Z} \end{array}$$

4) $\mathbb{T}'_f \cong \mathcal{O}_f = \mathbb{Z}[\{a_n\}] \Rightarrow \mathfrak{p}_f$ gives rise to some prime $\mathfrak{P}$ of $\overline{\mathbb{Q}}$.

Since T_n acts via multiplication by a_n mod $\mathfrak{p}_f$ on RHS, it acts via multiplication by a_n mod $\mathfrak{p}_Y$ on LHS.

$\Rightarrow$ This can be lifted to a subrepresentation ρ of $\mathbb{T}'_Y$ inside $M_Y \otimes \overline{\mathbb{Q}}$ such that $\rho(T(n)) \equiv a_n$ modulo $\mathfrak{P}$.

Note that $M_Y \otimes_{\mathbb{Z}} \mathbb{C} \cong Y^2$ as a $\mathbb{T}_Y \otimes_{\mathbb{Z}} \mathbb{C}$-module, so $\exists g \in Y$ such that $g \equiv f \mod q$.

The fact that g is not conjugate to f follows from $g \notin W_f(\mathbb{R})$.