§0 - Notation

- $p, N \geq 1$, p prime & $p \nmid N$.
- Γ = finite-index subgroup of $PSL_2(\mathbb{Z})$
- $\Delta_0 = \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$
- V = Abelian group admitting <u>right</u> action by Γ.
- $\mathrm{Hom}(\Delta_0, V)$ is endowed w/ Γ-action

$$(\varphi|\gamma)(D) := \varphi(\gamma \cdot D)|\gamma$$

§1 - Modular Symbols

Definition A V-valued modular symbol is an element of

$$\mathrm{Symb}_\Gamma(V) := \mathrm{Hom}_\Gamma(\Delta_0, V)$$

We will care about examples where V has an additional right action by

$$S_0(p) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : p \mid c, \ p \nmid \det(\gamma) \right\}$$

$\Rightarrow$ Hecke action by T_ℓ via $\Gamma \begin{pmatrix} 1 & \\ & \ell \end{pmatrix} \Gamma = \coprod \Gamma \xi_i$

$$\varphi | T_\ell := \sum_i \varphi | \xi_i$$

Remark If $\Gamma = \Gamma_0(N)$, this is the usual action

$$\varphi | T_\ell := \varphi \Big| \begin{pmatrix} \ell & \\ & 1 \end{pmatrix} + \sum_{i=0}^{\ell-1} \varphi \Big| \begin{pmatrix} 1 & a \\ & \ell \end{pmatrix}$$

and if $q \mid N$, we also have

$$\varphi | U_q := \sum_{i=0}^{\ell-1} \varphi \Big| \begin{pmatrix} 1 & a \\ & \ell \end{pmatrix}$$

Also, if $\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$ normalizes Γ, this matrix acts as an involution, so

$$\mathrm{Symb}_\Gamma(V) := \mathrm{Symb}_\Gamma(V)^+ \oplus \mathrm{Symb}_\Gamma(V)^-,$$

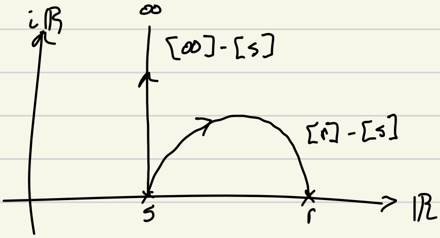
provided 2 acts invertibly on V.

Question: How do we construct modular symbols?

Answer: Need $\mathbb{Z}[\Gamma]$-generators of Δ_0 and all relations between them.

To do so, use geometry:

Given $D = [r] - [s] \in \Delta_0$, $r, s \in \mathbb{Q} \cup \{\infty\}$, identify D with geodesic in $\mathcal{H}^*$ <u>from</u> s <u>to</u> r



Given $\gamma = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in GL(2,\mathbb{Q})^+$, let $r = \frac{c}{d}$, $s = \frac{a}{b}$ and set

$$[\gamma] = [r] - [s]$$

⇒ We can span $\Delta_0\ (/\mathbb{Z})$ with $[\gamma]$, $\gamma \in GL(2,\mathbb{Q})^+$

Actually, using continued fraction, we can span Δ_0 w/ $[\gamma]$, $\gamma \in PSL(2,\mathbb{Z})$.

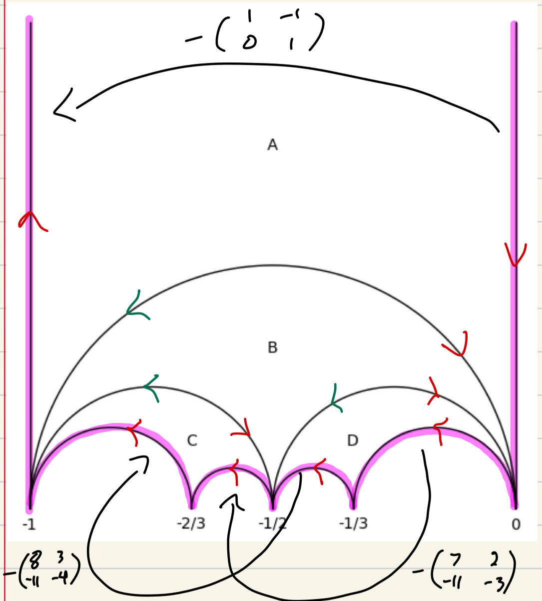
⇒ Finding generators $/\mathbb{Z}[\Gamma]$ amounts to

$$\Gamma \backslash PSL(2,\mathbb{Z}) = \{g_1, \dots, g_r\}$$

⇒ Find nice fund'l domain $\mathfrak{F}$ for Γ.

We do this via an example:

Example $\Gamma = \Gamma_0(11)$ (really, its image in $PSL(2,\mathbb{Z})$)



① Start w/ A & build your way down

② Only need →

③ Relations b/w them are easy

$\Rightarrow$ Only need: $[0]-[\infty] \text{ “}D_\infty\text{”},$
$[\tfrac{-1}{3}]-[0] \text{ “}D_1\text{”},$
$[\tfrac{-1}{2}]-[\tfrac{-1}{3}] \text{ “}D_2\text{”}.$

In general, if Γ has no 2-torsion or 3-torsion:

Δ_0 is generated $/\mathbb{Z}[\Gamma]$ by $\overbrace{D_1,\dots,D_t}^{\text{1/2-bottom bdd of } \mathcal{F}}, \overset{[0]-[\infty]}{D_\infty}$

& their only relation is

$$\sum_{i=1}^{t}(\underset{\text{easily computed}}{\gamma_i^{-1}}-1)D_i = \left(\begin{pmatrix}1&-1\\&1\end{pmatrix}-1\right)D_\infty$$

Theorem Let $v_1, \dots, v_t, v_\infty \in V$ s.t.

$$\sum_{i=1}^{t} (\gamma_i - 1) v_i = \Delta^{:= \left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right) - 1} v_i$$

$\Rightarrow \exists!\, \varphi \in \mathrm{Symb}_\Gamma(V)$ s.t. $\varphi(D_i) = v_i$ & all φ's arise this way.

Example Let us stick w/ $\Gamma = \Gamma_0(11)$.

In that case, $S_2(\Gamma_0(11))$ is 1-dim'l, let

$$f \in S_2(\Gamma_0(11))$$

be the unique normalized Hecke eigenform.

$$\varphi_f([r] - [s]) := 2\pi i \int_s^r f(z)\,dz$$

is in $\mathrm{Symb}_\Gamma(\mathbb{C})$, where $\mathbb{C}$ has trivial Γ-action.

In particular, $\varphi_f([0] - [\infty]) = L(f, 1) = \frac{1}{5} \Omega_E^+$

$\Omega_E^\pm$ = Néron period of elliptic curve $E = X_0(11)$ of f.

Using involution $\left(\begin{smallmatrix} -1 & \\ & 1 \end{smallmatrix}\right)$, we can write

$$\varphi_f = \varphi_f^+ + \varphi_f^-,$$

and $\varphi_f^\pm$ takes values that are $\neq 0$, $\mathbb{Q}$-multiples of $\Omega_E^\pm$.

Then, $\varphi_f^{\pm}$ spans 2-dim'l subspace of the 3-dim'l space $\mathrm{Symb}_{\Gamma}(\mathbb{C})$.

Another line is naturally the one spanned by φ

$$v_\infty = \varphi\big(\overset{,,\,D_\infty}{[0]-[\infty]}\big) = 1, \qquad v_1 = v_2 = \varphi(D_1) = \varphi(D_2) = 0$$

and it is an "Eisenstein" contribution.

Remark $\mathrm{Symb}_{\Gamma}(\mathbb{C})$ is related to $S_2(\Gamma)$ & in general,

$$\mathrm{Symb}_{\Gamma}(\mathrm{Sym}^k(\mathbb{C}^2)) \longleftrightarrow M_{k+2}(\Gamma)$$

$$\text{Periods of } f \longleftrightarrow f$$

§2 - Overconvergent Modular Symbols

These are simply $\mathrm{Symb}_{\Gamma}(V)$, where

$$V = p\text{-adic space of distributions}$$

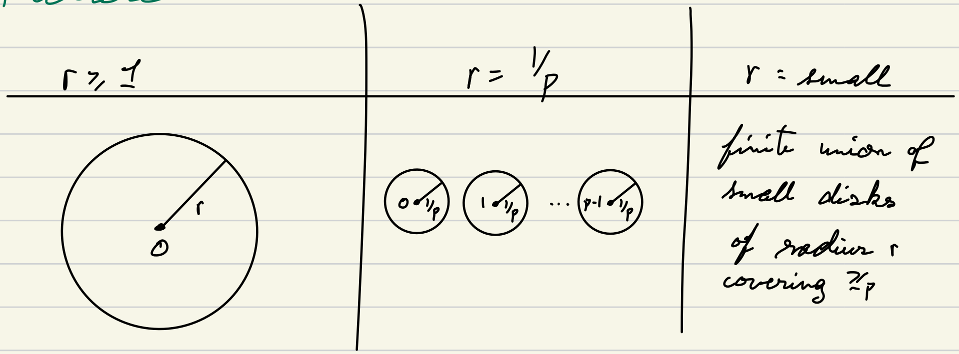
Namely, extend $\mathbb{Z}_p$ by some "radius" r & take distributions on these spaces.

Definition Let $r \in |\mathbb{C}_p^\times|$ & define

$$B[\mathbb{Z}_p, r] = \{ z \in \mathbb{C}_p \mid \exists\, a \in \mathbb{Z}_p \text{ w/ } |z-a|_p \le r \}$$

$\hookleftarrow$ $\mathbb{C}_p$-points of $\mathbb{Q}_p$-affinoid

Picture:



$$\Rightarrow A[r] = \text{functions on } B[\mathbb{Z}_p, r]$$
$$= \left\{ f(z) = \sum a_n z^n \in \mathbb{Q}_p[[z]] : |a_n|_p \cdot r^n \to 0 \right\}$$

This is a Banach space w.r.t. norm

$$\|f\| := \sup_z |f(z)|_p$$

We have natural inclusion $\mathbb{A}[r_1] \hookrightarrow \mathbb{A}[r_2]$ if $r_1 \geq r_2$

$$\Rightarrow \quad \mathcal{A}(\mathbb{Z}_p) = \varinjlim_{s>0} \mathbb{A}[s] = \{\text{fct on } \mathbb{Z}_p, \text{ loc. analytic}\}$$

$$\mathcal{A}^\dagger(\mathbb{Z}_p, r) = \varinjlim_{s>r} \mathbb{A}[s] = \{\text{fct on } \mathbb{Z}_p, \text{ analytic in "radius" } r\}$$

w/ natural Topo.

Definition $\mathbb{D}[r]$, $\mathcal{D}(\mathbb{Z}_p)$ & $\mathcal{D}^\dagger(\mathbb{Z}_p, r)$ are their natural $\mathbb{Q}_p$-linear continuous dual w/ topology from sup-norm.

$$\mathcal{A}^\dagger(\mathbb{Z}_p, r) \hookrightarrow \mathbb{A}[r] \hookrightarrow \mathcal{A}(\mathbb{Z}_p)$$

$$\mathcal{D}(\mathbb{Z}_p) \hookrightarrow \mathbb{D}[r] \hookrightarrow \mathcal{D}^\dagger(\mathbb{Z}_p, r)$$

p-adic L-functions live here

Easy to describe & work with

As we saw before, $\{z^i\}_{i=0}^\infty$ is dense in these three spaces so a distribution μ is uniquely determined by its moments

$$(\mu(z^i))_i \in \prod_{i=0}^\infty \mathbb{Q}_p$$

& we can more or less describe the "allowed" sequences of moments for some $r \in |\mathbb{C}_p^\times|$.

Let $\mathcal{D} = \mathbb{D}(\mathbb{Z}_p)$, $\mathbb{D}$ $(= \mathbb{D}(1))$ or $\mathbb{D}^+$ $(= \mathbb{D}^+(1))$.

We will have an action of weight k by

$$\Gamma_0 = \underset{\uparrow}{\Gamma} \cap \Gamma_0(p)$$

congruence subgroup of level N $(p \nmid N)$

In fact, we have an action by

$$\Sigma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) \;\middle|\; p \nmid a,\ p \mid c,\ ad-bc \neq 0 \right\}$$

$\Rightarrow$ $\forall \gamma \in \Sigma_0(p)$, $f \in A[r]$, we set

$$(f|_k \gamma)(z) = (a+cz)^k f\left(\frac{b+dz}{a+cz}\right)$$

$\rightsquigarrow$ Induces a $|_k$ action on $\mathcal{D}$ $\rightsquigarrow$ denote $\mathcal{D}_k$

Definition $\Phi \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k)$ is an overconvergent mod. symb. of wgt k.

Q: Relation with classical case?

A: Consider $V_k := \mathrm{Sym}^k(\mathbb{Q}_p^2) = \langle X^k, X^{k-1}Y, \dots, Y^k \rangle$ with

$$(P|_k \gamma)(X, Y) = P(dX - cY, -bX + aY), \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We have a map $\Sigma_0(p)$-equivariant

$$\begin{aligned} \rho_k : \mathcal{D}_k &\longrightarrow V_k \\ \mu &\longmapsto \int (Y - zX)^k \, d\mu(z) \end{aligned}$$

$$\Rightarrow \rho_k^* : \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k) \to \mathrm{Symb}_{\Gamma_0}(V_k)$$

§3- Constructing Overconvergent Mod. Symb.

This is crucial to construct p-adic L-fct.

Idea : Construct $\varphi \in \mathrm{Symb}_{\Gamma_0}(V_k)$ and lift it (uniquely?) to $\Phi \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k)$.

As we saw, constructing a V-valued mod. s. amounts to solving

$$v | \Delta = w, \quad v, w \in V, \quad \Delta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - 1$$

Proposition The map $\Delta : V_k \to V_k$ has

1) kernel $= \mathbb{Q}_p \cdot X^k$

2) Image $= \langle X^k, X^{k-1}Y, \dots, XY^{k-1} \rangle$

Proof 1) $Q | \Delta = 0 \Rightarrow Q(X,Y) = Q(X, Y-X)$.

2) $Y^k \notin \mathrm{Im}(\Delta)$ + Dimension. $\blacksquare$

Proposition The map $\Delta : \mathcal{D}_k^\dagger \to \mathcal{D}_k^\dagger$ is injective and maps bijectively onto

$$\{ \nu \in \mathcal{D}_k^\dagger : \nu(\mathbb{1}) = 0 \}$$

In fact, given ν of total measure 0, we can explicitly construct the unique μ s.t. $\mu | \Delta = \nu$.

(If $\nu \in \mathbb{D}_k$, then μ might $\notin \mathbb{D}_k$).

Theorem The map

$$\rho_k^* : \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k^\dagger) \to \mathrm{Symb}_{\Gamma_0}(V_k)$$

is surjective.

Proof For simplicity, assume Γ_0 has no 2-torsion & no 3-torsion

$\Rightarrow \varphi \in \mathrm{Symb}_{\Gamma_0}(V_k)$ is determined by

$$\underbrace{\varphi(D_1), \dots, \varphi(D_t)}_{\text{Pick } \mu_i \in \mathcal{D}_k^\dagger \text{ s.t. } \rho_k(\mu_i) = \varphi(D_i)}, \ \varphi(D_\infty) \in V_k$$

We will construct $\Phi \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k^\dagger)$ s.t.

$$\rho_k^*(\Phi) = \phi \Rightarrow \text{Set } \Phi(D_i) = \mu_i$$

We want $\mu_\infty \in \mathcal{D}_k^\dagger$ s.t.

$$\mu_\infty | \Delta = \sum_{i=1}^{t} (\gamma_i - 1)\mu_i = \nu_\infty$$

Note that $\rho_k(\nu_\infty) = \sum_i (\gamma_i - 1)\varphi(D_i) = \varphi(D_\infty)|\Delta$

$\Rightarrow \rho_k(\nu_\infty)$ has coefficient 0 in Y^k

$\Rightarrow \nu_\infty$ has total measure $0 \Rightarrow \exists \mu_\infty | \Delta = \nu_\infty$

Set $\Phi(D_\infty) = \mu_\infty$. (Some details omitted). $\square$

Corollary $\rho_k^*: \mathrm{Symb}_{\Gamma_0}(\mathbb{D}_k) \twoheadrightarrow \mathrm{Symb}_{\Gamma_0}(V_k)$.

Proof Apply U_p. $\square$

Q: Can we do better?

Let $\varphi \in \mathrm{Symb}_{\Gamma_0}(V_k) \otimes K$ be an eigensymbol, where K is a finite ext'n of $\mathbb{Q}_p$ containing all eigenvalues of φ.

Let λ = eigenvalue of φ under U_p, so

$$h = \text{slope of } \varphi = \mathrm{val}_K(\lambda) \in \{0, \dots, k+1\}$$

Definition $X_\varphi : (\rho_k^*)^{-1}(K \cdot \varphi) \subseteq \mathrm{Symb}_{\Gamma_0}(\mathbb{D}_k) \otimes K$

Proposition There is a unique decomposition

$$X_\varphi = X_\varphi^{(=h)} \oplus X_\varphi^{(>h)}$$

where $\lambda^{-1} U_p$ is topologically nilpotent on $X_\varphi^{(>h)}$ and s.t. $\{\lambda^n U_p^{-n}\}$ forms a bdd seq. on $X_\varphi^{(=h)}$. Also,

$$\dim_K \left(X_\varphi^{(=h)} \right) < +\infty$$

Theorem φ as above, $\exists\, 0 \neq \Phi \in \mathrm{Symb}_{\Gamma_0}(\mathbb{D}_k) \otimes K$ with the same system of eigenvalues

Proof ρ_k^* kills $X_\varphi^{(>h)} \Rightarrow \rho_k^* : X_\varphi^{(=h)} \twoheadrightarrow K \cdot \varphi$. $\square$

If slope $h < k+1$

In practice: Lift φ to $\Phi \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k^+) \otimes K$ as previously.

$\Rightarrow$ Apply $\lambda^{-1} U_p$ to be in $\mathrm{Symb}_{\Gamma_0}(\mathbb{D}_k^+) \otimes K$

$\Rightarrow$ ordinary projection in $X_\varphi^{(=h)}$ is $\lim\limits_{n \to +\infty} U_p^{n!}$.

Theorem [Comparison]

$$\mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p))^{(<k+1)} \xrightarrow{\sim} \mathrm{Symb}_{\Gamma_0}(\mathbb{D}_k)^{(<k+1)} \xrightarrow{\sim} \mathrm{Symb}_{\Gamma_0}(V_k)^{(<k+1)}$$

§4 - Connection to p-adic L-functions

Let $f \in S_{k+2}(\Gamma, \overline{\mathbb{Q}}) \subset S_{k+2}(\Gamma, \overline{\mathbb{Q}}_p)$ be a normalized eigenform of slope $h < k+1$.

It's p-adic L-function is $\mu_f \in \mathcal{D}(\mathbb{Z}_p^\times)$ s.t.

$$\mu_f^{\pm}\Big(\underset{j \in \{0,\dots,k\}}{z^j} \cdot \overset{\text{f. char, cond. } p^n}{\chi}\Big) = \underset{U_p\text{-eigenvalue of } f}{\alpha^{-1}} \cdot \frac{p^{n(j+1)}}{(-2\pi i)^j} \cdot \frac{j!}{\tau(\chi^{-1})} \cdot \frac{L(f, \chi^{-1}, j+1)}{\Omega_f^{\pm}}$$

& $\mu_f = \mu_f^+ + \mu_f^-$.

We can construct it w/ overconvergent mod. symbols. Let

same system of eigenvalues as f

$$\varphi_f^{\pm}([r]-[s]) = \frac{\pi i}{\Omega_f^{\pm}}\left(\int_r^s f(z)(zX+Y)^k dz \pm (-1)^k \int_{-r}^{-s} f(z)(zX-Y)^k dz\right)$$

(only integral values & at least one unit)

$$\in \mathrm{Symb}_{\Gamma_0}(V_k)^{\pm}$$

By comparison theorem,

$$\exists!\ \Phi_f^{\pm} \in \mathrm{Symb}_{\Gamma_0}(\mathcal{D}_k(\mathbb{Z}_p)) \text{ lifting } \varphi_f^{\pm}$$

Let $\Phi_f := \Phi_f^+ + \Phi_f^-$.

Theorem $\mu_f = \Phi_f([\infty]-[0])\big|_{\mathbb{Z}_p^\times}$

§5 - Computations

Use the filtration:

$$\mathbb{D}_k^0 = \{ \mu(z^j) \in \mathbb{Z}_p, \forall j \}$$

$$\mathrm{Fil}^M(\mathbb{D}_k^0) = \{ \mu \in \mathbb{D}_k^0 \mid \mu(z^j) \in p^{M+1-j}\mathbb{Z}_p \}$$

$$\mathcal{F}_k(M) = \mathbb{D}_k^0 / \mathrm{Fil}^M(\mathbb{D}_k^0) \cong \left(\mathbb{Z}/p^{M+1}\mathbb{Z}\right) \times \dots \times \left(\mathbb{Z}/p\mathbb{Z}\right)$$

$\mathcal{F}_k(M)$: finite set

$\mu(1) \bmod p^{M+1}$ . . . $\mu(z^{M+1}) \bmod p$

Given $\varphi \in \mathrm{Symb}_{\Gamma_0}(V_k)$, Greenberg has an alg. to lift it to $\mathrm{Symb}_{\Gamma_0}(\mathcal{F}(M))$ for $M = 1, 2, \dots$

Example $\Gamma = \Gamma(1)$, $p = 11$, $\Gamma_0 = \Gamma_0(11)$.

Let $\varphi = \varphi_f$ = $\mathbb{Q}_p$-mod. symb. of weight 0 associated to $f \in S_2(\Gamma_0)$

$$\varphi(D_\infty) = \frac{1}{5}, \quad \varphi(D_1) = \frac{-3}{2}, \quad \varphi(D_2) = \frac{1}{2}$$

Using the alg., we compute its lift $\Phi = \Phi_f$, or really the approx. $\Phi_M \in \mathrm{Symb}_{\Gamma_0}(\mathcal{F}_0(M))$, $M = 1, 2, \dots$

$$\Phi_1 \longleftrightarrow [(97, 3), (59, 4), (61, 4)]$$

reduction of $\Phi(D_2)$

reduction of $\Phi(D_\infty)$ in $\mathcal{F}_0(1) \cong \mathbb{Z}/121\mathbb{Z} \times \mathbb{Z}/11\mathbb{Z}$

reduction of $\Phi(D_1)$

For $M = 2, \ldots, 10$:

$$\begin{aligned}
&\Phi_2 \longleftrightarrow [(1065, 47, 6), (664, 37, 5), (666, 70, 1)] \\
&\Phi_3 \longleftrightarrow [(11713, 894, 72, 6), (7319, 1126, 27, 5), (7321, 1159, 100, 9)] \\
&\Phi_4 \longleftrightarrow [(128841, 7549, 798, 50, 4), (80524, 1126, 1237, 60, 2), (80526, 13138, 463, 86, 6)] \\
&\quad\vdots \\
&\Phi_{10} \longleftrightarrow \\
&\quad [(228249336489, 4107437612, 751372925, 96227115, 15695904, 1666537, 14032, 10489, 929, 102, 0), \\
&\quad (142655835304, 17332646489, 213170204, 172173501, 12969871, 150949, 143485, 7580, 1257, 2, 2), \\
&\quad (142655835306, 5630575713, 733658311, 186492667, 19306282, 1166702, 17021, 8513, 1017, 57, 6)]
\end{aligned}$$

This can easily be implemented in SAGE

Remark This is crucial when computing p-adic approximations of Stark-Heegner points.

Doing the above up to 11^{100}, we find

$$\begin{aligned}
x &= 10816241366446925396670846851116849, \\
y &= -19391462977749218369160989980706200472762157755000 \\
&\quad -4503481327176251972713258756168602406570456354 93\sqrt{101}.
\end{aligned}$$

on $X_0(11)$.