The eigencurve – A survey

4/13/21

p - prime, $(N,p)=1$.

The eigencurve of tame level N is a rigid analytic curve/$\mathbb{Q}_p$ whose points are overconvergent eigenforms of tame level N. It encapsulates all families of o.c. eigenforms which vary p-adically over their weights.

Today Give an overview of the construction of the eigencurve when $N=1$, and discuss some properties.

§1 Some background

$N=1$. $G_{p,\infty} :=$ Gal gr of max xtn of $\mathbb{Q}$ unr away from p,∞.

Let $\bar{\rho}: G_{p,\infty} \to GL_2(\mathbb{F}_q)$ ($q=p^n$, some n) be a cts rep.

For example, f eigenform of level $\Gamma_1(p^m)$, ρ_f its p-adic Gal rep, then $\rho_f \pmod p$ is such a rep.

Reps obtained in this way are called p-modular.

Assume $\bar{\rho}$ is p-modular. Let $\lambda_{\bar{\rho}} = (\bar{a}, \bar{d}, \bar{x})$ be its assoc. pseudo rep. (We'll often write $\lambda = a+d = \mathrm{Tr}(\lambda)$.)

Fact. $\exists!$ $\mathbb{Z}_p$-alg $R_{\bar{\rho}}$ s.t. for any local artinian $\mathbb{F}_q$-alg (A, m_A), we have

$$\mathrm{Hom}_{\text{rings}}(R_{\bar{\rho}}, A) \cong \{\text{ps.rep's } \lambda : G_{p,\infty} \to A \mid \lambda \equiv \lambda_{\bar{\rho}} \pmod{m_A}\}$$

It comes with a univ ps.rep $\lambda_{\bar{\rho}}^{\text{univ}} : G_{p,\infty} \to R_{\bar{\rho}}$.

Its $\mathbb{C}_p$-pts are cts ps.rep's $G_{p,\infty} \to \mathbb{C}_p$ which reduce to $\lambda_{\bar{\rho}}$ mod $m_{\mathcal{O}_{\mathbb{C}_p}}$.

Now let $\mathcal{W}$ = rig an space / $\mathbb{Q}_p$ given on fin xtn's K of $\mathbb{Q}_p$ by

$$\mathcal{W}(K) = \mathrm{Hom}_{\text{cont}}(\mathbb{Z}_p^\times, K^\times).$$

$\mathcal{W}$ is called wt space, $\mathcal{W}(\mathbb{C}_p) = \mathrm{Hom}_{\text{cont}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$.

$k \in \mathbb{Z}$ gives $\chi_k \in \mathcal{W}(\mathbb{C}_p)$ by $z \mapsto z^k$.

$\mathcal{W} = (\mathrm{Spf}\,\Lambda)^{\text{an}}$, $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$.

There is a map $R_{\bar{\rho}} \xrightarrow{\kappa} \mathcal{W}$. On $\mathbb{C}_p$ pts, it is $\lambda \mapsto (\mathbb{Z}_p^\times \ni z \mapsto z \cdot (\det \lambda)((\mathrm{CFT}^{-1}(z)))$, where $\mathrm{CFT}: G_{p,\infty}^{ab} \cong \mathbb{Z}_p^\times$

If λ is attached to an eigenform f, then this gives the weight of f.

Let

$$R_p = \prod_{\bar{\rho}\ p\text{-modular}} R_{\bar{\rho}}, \qquad X_p = \mathrm{Sp}(R_p) = \text{rig an space assoc. w/ } R_p.$$

(↖ finite product)

Let $\lambda_p^{\text{univ}} = \prod \lambda_{\bar{\rho}}^{\text{univ}}$.

§2 Construction of the eigencurve $\mathcal{C}_p$

Let $\mathcal{H}' = \Lambda[T_\ell \mid \ell \neq p]$, and $\mathcal{H} = \mathcal{H}'[U_p]$.

Let $\iota: \mathcal{H}' \to R_p$, $T_\ell \mapsto \lambda_p^{\text{univ}}(\mathrm{Frob}_\ell)$.

A consequence of U_p having a good spectral theory is: $\forall\, \alpha \in \mathcal{H}'$ s.t. $t(\alpha) \in R_p^\times$, $\exists$ an entire power series

$$P_{t(\alpha)U_p}(T) \in \Lambda\{\{T\}\} := \left\{ \sum a_i T^i \in \Lambda[[T]] \,\middle|\, a_i \in m_\Lambda^{c_i} \text{ for some } c_i \text{ w/ } \tfrac{c_i}{n} \to \infty \right\}$$

s.t. for all $(\kappa: \Lambda \to \mathbb{C}_p) \in \mathcal{W}(\mathbb{C}_p)$, we have

$$\kappa(P_{t(\alpha)U_p}(T)) = \det(1 - (\alpha U_p)T \mid M_\kappa^\dagger(\mathbb{C}_p))$$

($\det$: Fredholm det; $M_\kappa^\dagger(\mathbb{C}_p)$: oc mod fns of wt κ.)

Then $P_{t(\alpha)U_p}(T)$ cuts out a curve in $\mathcal{W} \times \mathbb{A}^1 =$ rig an space assoc w/ $\Lambda\{\{T\}\}$, called Z_α. We have

$$Z_\alpha(\mathbb{C}_p) = \{ (\kappa, u) \in (\mathcal{W} \times \mathbb{A}^1)(\mathbb{C}_p) \mid \exists\, f \in M_\kappa^\dagger(\mathbb{C}_p) \text{ s.t. } \alpha U_p f = u^{-1} f \}.$$

Then we pull back to $X_p \times \mathbb{A}^1$: There is a rig an morphism $r_\alpha : X_p \times \mathbb{A}^1 \to \mathcal{W} \times \mathbb{A}^1$ given on pts by

$$r_\alpha(\lambda, t) = \left(\kappa(\lambda), \frac{t}{\lambda(t(\alpha))}\right)$$

We take the <u>eigencurve</u> to be

$$C_p := \bigcap_{\alpha \text{ s.t. } t(\alpha) \in R_p^\times} r_\alpha^{-1}(Z_\alpha)$$

<u>Fact</u> (Gouvêa–Mazur) $\forall$ (normalized) finite slope oc eigenform f, $\exists!$ a $\mathbb{C}_p$-ps. rep λ_f s.t.

$$\lambda_f(\mathrm{Frob}_\ell) = \ell\text{-th Fourier coeff of } f.$$

Thus we get

$$\{\text{oc eigenforms}\} \to (X_p \times \mathbb{A}^1)(\mathbb{C}_p), \quad f \longmapsto (\lambda_f, (U_p\text{-eigenvalue of } f)^{-1})$$

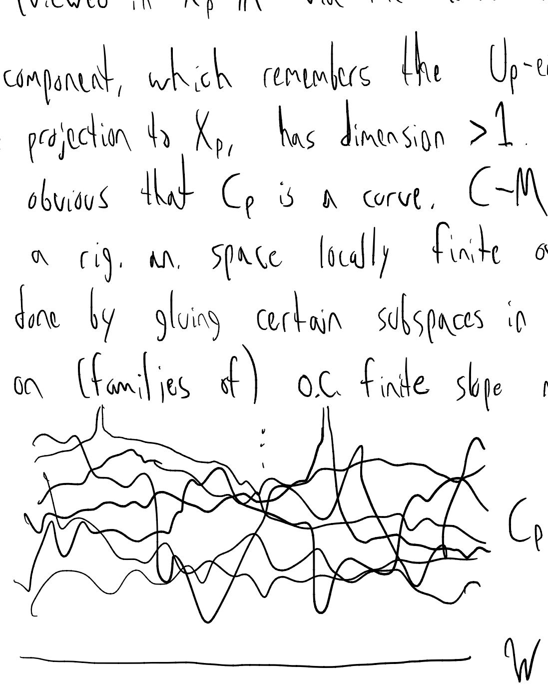
<u>Thm</u> (Coleman–Mazur) The image of this map is $C_p(\mathbb{C}_p)$

<u>Rmks</u> (i) Coleman and Mazur prove that C_p is the "Fredholm closure" of the set of all classical eigenforms [viewed in $X_p \times \mathbb{A}^1$ via the above map].

(ii) The $\mathbb{A}^1$ component, which remembers the U_p-eigenvalue, is necessary. In fact, the image of C_p under the projection to X_p, has dimension > 1.

(iii) It's not obvious that C_p is a curve. C–M prove this by constructing C_p^{red} in a different way as a rig. an. space locally finite over $\mathcal{W}$ (hence a curve).
This is done by gluing certain subspaces in spaces $\mathrm{Sp}(\mathbb{T})$, where $\mathbb{T}$'s are certain Hecke algebras acting on (families of) o.c. finite slope modular forms.

<u>Picture</u> (Take 1)



§3. Some properties.

Problem Is $\mathcal{C}_p$ "proper?" That is, are there holes? So can a family of finite slope oc modular forms converge p-adically to an infinite slope one?

Rmk This question is equivalent to whether $\mathcal{C}_p^{red}$ is proper.

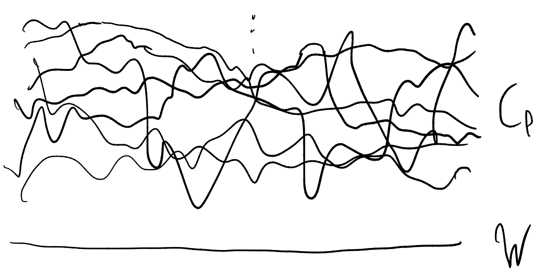
Thm (Diao-Liu '16) $\mathcal{C}_p$ is proper in this sense.

Rmk on pf They approach this from the Galois perspective. This is proved using p-adic Hodge theory, reducing it to a statement about the interpolation of certain p-adic periods.

Thm (L. Ye, '20) The eigencurve is still proper in this sense.

Rmk on pf This time Ye proves it from the Hecke perspective, using an older idea of Buzzard-Calegari. The new input is a good, geometric def. of $M_\kappa^\dagger$ when $\kappa \notin \mathbb{Z}$, building on a def due to Pilloni (Before, $M_\kappa^\dagger$ was defined for non-integral wts as the space obtained from an integral wt $M_k^\dagger$ by dividing by an appropriate Eisenstein series of wt $k-\kappa$.)

Picture (Take 2)



(The difference is that there are now no poles)

Conj (Spectral Halo conjecture) Let $0<r<1$, and let $\mathcal{W}^{>r} = \{\kappa \text{ s.t. } |\kappa(\exp(p))|>r\} \subseteq \mathcal{W}$ ($\mathcal{W}^{>r}$ called the "halo").

Let $\mathcal{C}_p^{>r}$ = preimage of $\mathcal{W}^{>r}$ in $\mathcal{C}_p$.

Then for $r \gg 0$, $\mathcal{C}_p^{>r}$ is an infinite disjoint union of components Z_i, each finite flat over $\mathcal{W}^{>r}$!

For any i, each $z \in Z_i$ has the same p-adic valuation for its U_p-eigenvalue, say α_i. Moreover, the set of all α_i's is a disjoint union of finitely many arithmetic progressions.

Rmk This is known if $p=2$, $N=1$ by Buzzard-Kilford. Computationally this has been verified in other cases.

Picture (Take 3)