


1. Mahler's Theorem

F is a finite extension of \mathbb{Q}_p , then for any closed subring

$A \subset F$, and $f \in C(\mathbb{Z}_p, A)$

$$\cdot a_n(f) = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} f(k) \in A$$

$$\lim_{n \rightarrow \infty} a_n(f) = 0$$

$$\cdot f(x) = \sum_{n=0}^{\infty} a_n(f) \binom{x}{n} \quad \text{on } \mathbb{Z}_p \quad (*)$$

by $a_n(f) \rightarrow 0$, this sequence converges uniformly

Sketch of the proof:

build up $(*)$ on \mathbb{Z} , and use the fact that

\mathbb{Z} is dense in \mathbb{Z}_p , the main difficulty is

to prove $\lim_{n \rightarrow \infty} a_n(f) = 0$

Remark: We can define a norm on $C(\mathbb{Z}_p, A)$, by

$$|f|_p := \sup_{x \in \mathbb{Z}_p} |f(x)|$$

then it's easy to see that:

$$\cdot |f|_p = \sup_n |a_n(f)|$$

• $C(\mathbb{Z}_p, A)$ is complete w.r.t. this norm

2. p -adic measures on \mathbb{Z}_p

$A \stackrel{\text{closed}}{\subset} F$, $\varphi: C(\mathbb{Z}_p, A) \rightarrow A$ is A -linear

φ is p -adic: $\exists B > 0$, s.t.

bounded measure $|\varphi(f)|_p \leq B \cdot \|f\|_p, \forall f$

$\text{Meas}(\mathbb{Z}_p; A) = \text{space of } p\text{-adic bounded measure}$
!!
 $\mathcal{M}(\mathbb{Z}_p, A)$ (with an obvious A -mod structure)

$\|\varphi\|_p = \sup_{\|f\|_p=1} |\varphi(f)|_p \rightarrow \mathcal{M}(\mathbb{Z}_p, A)$ is complete

• Structure of $\mathcal{M}(\mathbb{Z}_p, A)$

$f \in C(\mathbb{Z}_p, A)$ has an expansion $f = \sum_{n=0}^{\infty} a_n(f) \binom{x}{n}$

$f_m := \sum_{n=0}^m a_n(f) \binom{x}{n}, \quad f_m \Rightarrow f$

$$\begin{aligned} \varphi(f) &= \lim_{m \rightarrow \infty} \varphi(f_m) = \lim_{m \rightarrow \infty} \sum_{n=0}^m a_n(f) \varphi\left(\binom{x}{n}\right) \\ &= \sum_{n=0}^{\infty} a_n(f) \varphi\left(\binom{x}{n}\right) \quad \left| \binom{x}{n} \right|_p = 1 \end{aligned}$$

these equalities hold: $|\varphi\left(\binom{x}{n}\right)| \leq \|\varphi\|_p$ bounded

φ is totally determined by the bounded sequence

$$\left\{ \varphi\left(\binom{x}{n}\right) \right\}_{n=0}^{\infty}, \quad \text{and} \quad \|\varphi\|_p = \sup_n \left| \varphi\left(\binom{x}{n}\right) \right|$$

For any bounded sequence $\{b_n\}_{n=0}^{\infty} \rightsquigarrow \varphi_b \in \mathcal{M}(\mathbb{Z}_p, A)$

therefore $\mathcal{M}(\mathbb{Z}_p, A) \leftrightarrow$ bounded sequence space

- Power series associated to a measure

$$\varphi \in \mathcal{M}(\mathbb{Z}_p, A)$$

$$\Phi_\varphi(T) := \sum_{n=0}^{\infty} \left(\int \binom{x}{n} d\varphi \right) T^n$$

this gives: $\mathcal{M}(\mathbb{Z}_p, A) \longleftrightarrow A[[T]]$

If A is already bounded (i.e. $A \subset \mathcal{O}_F$)

then this is an isomorphism, $\mathcal{M}(\mathbb{Z}_p, \mathcal{O}_F) \xrightarrow{\sim} \mathcal{O}_F[[T]]$

We will mainly focus on this case ($A = \mathcal{O}_F$)

then the ring structure of RHS gives multiplication on LHS

$$\varphi * \psi : f \mapsto \iint f(x+y) d\varphi(x) d\psi(y)$$

3. Riemann zeta as a measure

- Main result: for $\forall a \geq 2$, $(a, p) = 1$, $\exists!$ p -adic bounded measure ζ_a , s.t.

$$\int_{\mathbb{Z}_p} x^m d\zeta_a = (1 - a^{m+1}) \zeta(-m), \text{ for } \forall m \geq 0$$

Sketch of the proof:

every measure is determined by their effect on $\binom{x}{n}$, hence on x^n
therefore if ζ_a exists, then it is unique.

$$\binom{x}{n} = \sum_{k=0}^n C_{n,k} x^k \quad \text{be careful, } C_{n,k} \notin \mathbb{A} \text{ sometimes but the formula still holds}$$

we only need to show: $\int_{\mathbb{Z}_p} \binom{x}{n} d\zeta_a$ is bounded

$$\int_{\mathbb{Z}_p} \binom{x}{n} d\zeta_a = \sum_{k=0}^n C_{n,k} (1 - a^{n+1}) \zeta(-n) \quad \text{Chapter 1}$$

$$= \sum_{k=0}^n C_{n,k} \left(t \frac{d}{dt} \right)^n \psi_a(t) \Big|_{t=1}$$

$$= \frac{1}{n!} \frac{d^n}{dt^n} \psi_a \Big|_{t=1}$$

where
$$\psi_a(t) = \frac{t}{1-t} - \frac{at^a}{1-t^a} = - \frac{\sum_{b=1}^a \zeta(b) (1+t+\dots+t^{b-1})}{1+t+\dots+t^{a-1}}$$

Why we need pta ?

if we apply the same computation, we will finally

ψ_a has the same form as before, then $\frac{d^n}{dt^n} \psi_a \Big|_{t=1}$ will give

more p in the denominator.

By the formula

$$\int_{\mathbb{Z}_p} \binom{x}{n} d\zeta_a = \frac{1}{n!} \frac{d^n}{dt^n} \psi_a(t) \Big|_{t=1}, \in \mathbb{Z}_p$$

we get the corresponding power series:

$$\bar{\Phi}_{\zeta_a}(T) = \sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} \binom{x}{n} d\zeta_a \right) T^n = \psi_a(1+T) = a-1 - \frac{1}{T} + \frac{a}{(1+T)^{a-1}} \in \mathbb{Z}_p[[T]]$$

• Can we remove a ?

If there exists $\zeta(T)$ (i.e. ζ), s.t.

$$\int_{\mathbb{Z}_p} x^n d\zeta = \zeta(-n)$$

equivalently:

$$\left(t \frac{d}{dt} \right)^n \psi(t) \Big|_{t=1} = \zeta(-n) = -\frac{B_{n+1}}{n+1}$$

No! consider $\tilde{\psi}(s) = \psi(e^s)$, then $\frac{d^n}{ds^n} \tilde{\psi} \Big|_{s=0} = -\frac{B_{n+1}}{n+1}$

$$\tilde{\psi}(s) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n \tilde{\psi}}{ds^n}(0) s^n = \frac{1}{s} - \frac{1}{e^s - 1}$$

$$\psi(1+T) = \frac{1}{\log(1+T)} - \frac{1}{T} \notin \mathbb{Z}_p[[T]]$$

4. Interpolation with Dirichlet character

Given $\chi: (\mathbb{Z}/p^n\mathbb{Z})^* \simeq (\mathbb{Z}_p/p^n\mathbb{Z}_p)^* \rightarrow F^*$

(ift χ to a locally constant function $\chi: \mathbb{Z}_p \rightarrow F^*$ ($\chi(p\mathbb{Z}_p) = 0$)

We want to compute:

$$\int_{\mathbb{Z}_p} \chi(x) x^n d\zeta_a$$

• Theorem: set $\chi_0 = \chi$ if χ is non-trivial, $\chi_0 = 1$ if χ is trivial

$$\int_{\mathbb{Z}_p} \chi(x) x^n d\zeta_a = (1 - a^{n+1} \chi(a)) (1 - \chi_0(p) p^n) L(-n, \chi_0)$$

if χ is primitive, and χ is even, then

$$\int_{\mathbb{Z}_p} \chi(x) x^{-1} d\zeta_a = \begin{cases} (1 - p^{-1}) \log_p a & , \chi = 1 \\ -(1 - \chi(a)) p^{-n} G(\chi) \sum_{b \in (\mathbb{Z}/p^n\mathbb{Z})^*} \chi^{-1}(b) \log_p(1 - \zeta^b) & , \chi \neq 1 \end{cases}$$

here \log_p is p -adic logarithm defined over \mathbb{C}_p^*

ζ is primitive p^n -th root of unity

pf: for any locally constant $\phi \in C(\mathbb{Z}_p, A)$, it acts on ζ_a by:

$$\int f d([\phi]\zeta_a) := \int \phi f d\zeta_a$$

so we could derive an action formula of ϕ on $\Phi_{\zeta_a}(\tau) = \psi_0(t)$

$$([\phi] \psi_a)(t) = \sum_{n=1}^{\infty} \phi(n) t^n - a \sum_{n=1}^{\infty} \phi(an) t^{an}$$

Define L-function: $L(s, \phi - a^{k+1} \phi_a) = \sum_{n=1}^{\infty} (\phi(n) - a^{k+1} \phi(an)) n^{-s}$

By Mellin transform:

$$\begin{aligned} L(-n, \phi - a^{n+1} \phi_a) &= \left(t \frac{d}{dt} \right)^n ([\phi] \psi_a)(t) \Big|_{t=1} \\ &= \int_{\mathbb{Z}_p} \phi(x) x^n d\zeta_a \end{aligned}$$

replace ϕ by χ ,

$$\int_{\mathbb{Z}_p} \chi(x) x^n d\zeta_a = L(-n, \chi - a^{n+1} \chi_a) = (1 - a^{n+1} \chi(a)) (1 - \chi_0(p) p^n) L(-n, \chi_0)$$

Evaluation at -1 is more difficult

We have observed that: $[\chi] \leftrightarrow t \frac{d}{dt} = (1+\tau) \frac{d}{d\tau}$

intuitively, $[\chi^{-1}] \leftrightarrow [\chi]^{-1}$, and there do exist

$$\Phi_a(t) = \log \frac{1-t^a}{1-t}, \text{ s.t. } [\chi] \Phi_a = \psi_a$$

but χ^{-1} is not a well-defined continuous function over \mathbb{Z}_p

and coefficient of Φ_a is p -adically unbounded, $\Phi_a \in \mathcal{O}_F[[\tau]]$

We should combine χ and χ^{-1} together

$x \mapsto \chi(x) \chi^{-1}$ is continuous, 0 on $p\mathbb{Z}_p$

then we can verify: $[\chi(x) \chi^{-1}] \psi_a(t) = [\chi] \Phi_a(t) \left([\chi] \Phi_a \in \mathcal{O}_F[[\tau]] \right)$

$$\int_{\mathbb{Z}_p} \chi(x) \chi^{-1} d\zeta_a = ([\chi(x) \chi^{-1}] \psi_a)(1) = ([\chi] \Phi_a)(1)$$

By this theorem,

$$(1 - a^{n+1} \chi(a))^{-1} \int_{\mathbb{Z}_p} \chi(x) x^n d\xi_a \text{ is independent of } a$$

To let n varies p -adically, we consider the decomposition

$$\mathbb{Z}_p^\times \simeq \mu_{q(p)} \times (1 + q\mathbb{Z}_p), \quad q = \begin{cases} 4, & p=2 \\ p, & p>2 \end{cases}$$

$$x \mapsto (\omega(x), \langle x \rangle)$$

Def: p -adic Dirichlet L -function with character χ

$$L_p(s, \chi) = (1 - \chi(a) \langle a \rangle^{1-s})^{-1} \int_{\mathbb{Z}_p} \chi \omega^{-1}(x) \langle x \rangle^{-s} d\xi_a$$

Rmk: the appearance of $\chi \omega^{-1}$ instead of χ is because of we use $\langle a \rangle$ instead of a

values at negative integer:

$$L_p(-n, \chi) = (1 - \chi(a) \langle a \rangle^{1+n})^{-1} \int_{\mathbb{Z}_p} \chi \omega^{-1}(x) \langle x \rangle^n d\xi_a$$

$$= (1 - (\chi \omega^{-n-1})(a) a^{n+1})^{-1} \int_{\mathbb{Z}_p} \chi \omega^{-n-1}(x) x^n d\xi_a$$

$$= (1 - (\chi \omega^{-n-1})_0(p) p^n) L(-n, (\chi \omega^{-n-1})_0)$$

$$n \equiv m \pmod{p^a(p-1)}, \quad \chi \omega^{-n-1} = \chi \omega^{-m-1}$$

• For nontrivial χ , $L_p(s, \chi)$ is continuous on \mathbb{Z}_p

For $\chi = 1$, $\zeta_p(s) := L_p(s, 1)$ is continuous on $\mathbb{Z}_p - \{1\}$

Moreover, $L_p(s, \chi)$ is p -adic analytic on \mathbb{Z}_p

$\zeta_p(s)$ is p -adic meromorphic on $\mathbb{Z}_p - \{1\}$

if χ is odd, $L_p(n, \chi) = 0$ if n is odd $\Rightarrow L_p(s, \chi) = 0$

χ is even, $L_p(1, \chi) = -p^{-n} \sum_{b \in (\mathbb{Z}_p^\times)^*} \chi^{-1}(b) \log_p(1 - \zeta^b)$

$\zeta_p(s)$ has a pole with residue $1-p^{-1}$

$$\int_{\mathbb{Z}_p^\times} \chi(x) \langle x \rangle^{-s} d\zeta_a$$

5. pseudo-measure \mathcal{L}_p

F/\mathbb{Q}_p finite extension, \mathcal{O}_F : integral closure of \mathbb{Z}_p in F

For any profinite group G , there is an isomorphism:

$$\mathcal{M}(G, \mathcal{O}_F) \simeq \varprojlim_{H \triangleleft G} \mathcal{O}_F[G/H] =: \Lambda(G)$$

\downarrow
Iwasawa algebra of G

$$\phi \mapsto \left(\sum_{a \in G/H} \phi(aH) aH \right)$$

$\forall g \in G$ can be viewed as an element in $\Lambda(G)$

$$(gH)_H$$

therefore it corresponds to a measure $\delta_g \in \mathcal{M}(G, \mathcal{O}_F)$

$$\delta_g(f) = \int_G f d\delta_g = f(g) \quad \text{Dirac measure}$$

Def: a pseudo measure on G is

- an element $\mu \in \mathcal{Q}(\Lambda(G))$ (total quotient ring)
- $(g-1)\mu \in \Lambda(G)$, $\forall g \in G$

Example:

$$\cdot G = \mathbb{Z}_p$$

$$\Lambda(\mathbb{Z}_p) \simeq \mathcal{O}_F[[T]], \quad \mathcal{Q}(\Lambda(\mathbb{Z}_p)) \simeq \mathcal{O}_F((T)) \otimes_{\mathcal{O}_F} F$$

$$\forall a \in \mathbb{Z}_p, \delta_a \text{ corresponds to } (1+T)^a \in \mathcal{O}_F[[T]]$$

$$\mu \in \mathcal{O}_F((T)) \otimes_{\mathcal{O}_F} F \text{ is a pseudo-measure} \Leftrightarrow \mu \cdot ((1+T)^a - (1+T)) \in \mathcal{O}_F[[T]], \forall a \in \mathbb{Z}_p$$
$$\Rightarrow \mu \in \frac{1}{T(1+T)} \mathcal{O}_F[[T]]$$

$$\cdot G = \mathbb{Z}_p^\times$$

there is an *inclusion* (but multiplication are different)

$$i: \Lambda(\mathbb{Z}_p^\times) \xrightarrow{\quad} \Lambda(\mathbb{Z}_p)$$

$$\mu \longmapsto i(\mu): f \mapsto \int_{\mathbb{Z}_p^\times} f|_{\mathbb{Z}_p^\times} d\mu$$

image of $\Lambda(\mathbb{Z}_p^\times)$ can be identified with

$$\Phi \in \mathcal{O}_F[[T]], \text{ s.t. } \sum_{\xi \in \mu_p} \Phi(\xi(1+T)-1) = 0$$

example: $\Phi_a(T) = (1+T)^a, a \in \mathbb{Z}_p^\times \rightsquigarrow$ evaluation at a

then

$\mu \in \mathcal{Q}(\Lambda(\mathbb{Z}_p^\times)) \subset F((T))$ is a pseudo-measure

$$\Leftrightarrow \underbrace{\mu \cdot (a-1)}_{\downarrow} \in \Lambda(\mathbb{Z}_p^\times)$$

this multiplication shouldn't
be viewed as in $F((T))$

We have already constructed ζ_a , s.t.

$$\int_{\mathbb{Z}_p} x^m d\zeta_a = (1-a^{m+1})\zeta(-m)$$

- $\zeta_a |_{\mathbb{Z}_p^\times} : x^m \rightarrow (1-a^{m+1})(1-p^m)\zeta(-m) \in \Lambda(\mathbb{Z}_p^\times)$
- $\theta_a = [x] \cdot (1-a) \in \Lambda(\mathbb{Z}_p^\times)$ not a zero-divisor

$$\zeta_p := \frac{\zeta_a}{\theta_a}$$

independence of $a \in \mathbb{Z}_p^\times$ is easy to check $\Rightarrow \zeta_p$ is pseudo-measure

the integral of $\text{Hom}_{\text{cts}}(G, \mathcal{O}_L^\times)$ w.r.t pseudo-measure

is well-defined, therefore we can consider $\int_{\mathbb{Z}_p^\times} x^n d\zeta_p, \int_{\mathbb{Z}_p^\times} \chi(x) x^n d\zeta_p$

$$\int_{\mathbb{Z}_p^\times} x^n d\zeta_p = (1-p^n)\zeta(-n) \quad \begin{array}{l} \text{integral on } \mathbb{Z}_p^\times \\ \text{instead of on } \mathbb{Z}_p \end{array}$$

if χ is primitive mod $p^n, n \geq 1$, then

$$\int_{\mathbb{Z}_p^\times} \chi(x) x^n d\zeta_p = L(-n, \chi)$$

and $\zeta_p(s) = L_p(s, 1) = \int_{\mathbb{Z}_p^\times} \omega^{-s}(x) \langle x \rangle^{-s} d\zeta_p$

$$\int_{\mathbb{Z}_p^\times} \omega^{-s}(x) \langle x \rangle^n d\zeta_p = \frac{\int_{\mathbb{Z}_p^\times} \omega^{-s-n}(x) x^n d\zeta_a}{\int \omega^{-s-n}(x) \langle x \rangle^n d\theta_a} = \frac{(1-p^n) \int_{\mathbb{Z}_p} \omega^{-s-n}(x) x^n d\zeta_a}{(a)^{n+1} - 1} = \zeta_p(-n)$$