§1. $\mathbb{Z}_p$-extensions

$$\begin{tikzcd} K_\infty \arrow[d, dash, "\Gamma_n = (\Gamma)^{p^n}"'] \\ K_n \arrow[d, dash, "\Gamma/\Gamma_n \simeq \mathbb{Z}/p^n\mathbb{Z}"'] \\ K_0 = K \end{tikzcd}$$

$A_n := Cl(K_n) \otimes \mathbb{Z}_p$
p-part of class group

Def K number field

$\Gamma = Gal(K_\infty/K) \simeq \mathbb{Z}_p$
$\;\; = \langle \gamma \rangle$

① $p^n \mathbb{Z}_p$ are the only closed subgps
$\Rightarrow$ K_n are the only subextensions

② $I_v^{ab} \simeq (\text{finite}) \times (\text{pro-}\ell)$ if $v | \ell$
$\Rightarrow$ if a prime v is ramified, then $v | p$

③ Pick n_0 s.t. if a prime v of K_{n_0} is ramified in K_∞, it is tot. ramified
$\Rightarrow$ the same is true for $n \geq n_0$
note that the # of ramified primes in K_n are the same if $n \geq n_0$

eg $Gal(\mathbb{Q}(\zeta_{p^{n+1}})/\mathbb{Q}) \simeq \mu_{p-1} \times \mathbb{Z}/p^n\mathbb{Z}$, $\mathbb{Q}_n = \mathbb{Q}(\zeta_{p^{n+1}})^{\mu_{p-1}}$

For $n \gg 0$ then

$$\begin{tikzcd} & A_{n+1}\ L_{n+1} \arrow[dl, dash] \arrow[dr, dash] & \\ K_{n+1} \arrow[dr, dash, "\text{tot}"'] & & A_n\ L_n \arrow[dl, dash, "\text{unr}"] \\ & K_n & \end{tikzcd}$$

$$\begin{tikzcd} A_{n+1} \arrow[r, two heads, "\text{norm}"] & A_n \\ \Gamma/\Gamma_{n+1} \arrow[u, phantom, "\circlearrowleft"] \arrow[r] & \Gamma/\Gamma_n \arrow[u, phantom, "\circlearrowleft"] \end{tikzcd}$$

, take $X_\infty := \varprojlim A_n$
$\circlearrowleft$
$\Lambda = \varprojlim \mathbb{Z}_p[\Gamma/\Gamma_n] =: \mathbb{Z}_p[[\Gamma]]$
the Iwasawa algebra

Alternatively, $X_\infty \simeq Gal(L_\infty/K_\infty) \simeq \varprojlim Gal(K_\infty L_n/K_\infty)$
L_∞ = max. abelian pro-p unramified over K_∞

check: if $K_\infty(\alpha) \subset L_\infty$, $K_n(\alpha)/K_n$ is abelian unramified if $n \gg 0$

$$\begin{tikzcd} & L_\infty \arrow[dl, dash, "X_\infty"'] \arrow[ddr, dash, "G"] & \\ K_\infty \arrow[drr, dash, "\langle \gamma^{p^{n_0}} \rangle = \Gamma_{n_0}"'] & & \\ & & K_{n_0} \end{tikzcd}$$

Let $I_1, \cdots, I_k \subset G$ be the inertia gps of ramified primes

$I_i \cap X_\infty = \{1\}$. and $I_i|_{K_\infty} \simeq \Gamma_{n_0}$

$\sigma_i \longmapsto \gamma^{p^{n_0}} \qquad \sigma_i = x_i \sigma_1, \quad x_i \in X_\infty$

$$A_{n_0} = G \Big/ \overline{\langle [G.G], I_1, \cdots, I_k \rangle} = X_\infty \cdot I_1 \Big/ \sim \;\simeq X_\infty \Big/ \overline{\langle X_\infty^{\gamma^{p^{n_0}}-1}, x_2, \cdots, x_k \rangle} = X_\infty / Y$$

$$1 \to X_\infty \to G \to \Gamma_{n_0} \to 1 \qquad 1 \to X_\infty \Big/ X_\infty^{\gamma^{p^{n_0}}-1} \to G \Big/ \overline{[G.G]} \to \Gamma_{n_0} \to 1$$

for $n \geq n_0$, define $\nu_n = \gamma^{p^n}-1 \Big/ \gamma^{p^{n_0}}-1$, then $A_n \simeq X_\infty / \nu_n Y$

$$(x\sigma)^{p^{n-n_0}} = x \cdot \underset{\substack{\| \\ x^{\gamma^{p^{n_0}}}}}{(\sigma x \sigma^{-1})} (\sigma^2 x \sigma^{-2}) \cdots (\quad) \sigma^{p^{n-n_0}} = x^{1+\gamma^{p^{n_0}} + \cdots + \gamma^{p^n - p^{n_0} - 1}} \sigma^{p^{n-n_0}}$$
$$= x^{\nu_n} \cdot \sigma^{p^{n-n_0}}$$

$$\Lambda = \varprojlim \mathbb{Z}_p[\Gamma/\Gamma_n] \xrightarrow{\sim} \varprojlim \mathbb{Z}_p[T] \Big/ (1+T)^{p^n}-1 \simeq \varprojlim \mathbb{Z}_p/p^m[T] \Big/ (1+T)^{p^n}-1 \simeq \mathbb{Z}_p[[T]]$$
$$\gamma \longmapsto (1+T)$$

① Λ is a regular local domain ($\Rightarrow$ UFD) of dim 2. $\mathfrak{m} = (p, T)$

② (p-adic Weierstrass) a power series $f(T) \in \Lambda$ can be written as

$f(T) = p^\mu \cdot p(T) \cdot u(T)$, $u(T) \in \Lambda^\times$ and $p(T) = T^\lambda + \cdots + a_\lambda$ s.t. $p \mid a_i$

call such polynomial distinguished of degree λ

③ A Λ-hom $\phi: M \to N$ is a pseudo-isom if $\ker\phi$ and $\operatorname{coker}\phi$ are finite

(an equivalence relation among f.g. torsion Λ-modules)

If X is a f.g. Λ-module, there exists a pseudo-isom

$$\phi : X \to E = \Lambda^{\oplus r} \oplus \bigoplus \Lambda/(f_i) \oplus \bigoplus \Lambda/(p^{\mu_j})$$

distinguished

Since $Y/\nu_n Y \subset A_n$. Y/mY is finite, pick Y' f.g s.t $Y' + mY = Y$

by cpt + tot. disconn of Y/Y', $m^n(Y/Y')$ is in an arbitrary small nbd if $n \gg 0$

$\Rightarrow Y/Y' = \bigcap m^n(Y/Y') = \{0\} \Rightarrow$ Y is f.g.

If $0 \to A \to Y \to E \to B \to 0$ is the pseudo-isom,

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\times \nu_n"'] & E \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\ Y \arrow[r] & E \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

so $E/\nu_n E$ is finite, therefore $r=0$ and f_j's coprime to ν_n

in particular, Y is torsion.

Define $Ch(Y) = \prod (f_j) \times \prod (p^{\mu_i})$, $\mu = \sum \mu_i$, $\lambda = \sum \deg f_j$

$$\begin{tikzcd} 0 \arrow[r] & \nu_n Y \arrow[r] \arrow[d, "\phi_n'"] & Y \arrow[r] \arrow[d, "\phi"] & Y/\nu_n Y \arrow[r] \arrow[d, "\phi_n''"] & 0 \\ 0 \arrow[r] & \nu_n E \arrow[r] & E \arrow[r] & E/\nu_n E \arrow[r] & 0 \end{tikzcd}$$

if $m \geq n$

a) $|\ker \phi_n'| \geq |\ker \phi_m'|$

b) $|\mathrm{coker}\, \phi_n'| \geq |\mathrm{coker}\, \phi_m'|$

c) $|\mathrm{coker}\, \phi| \geq |\mathrm{coker}\, \phi_m''| \geq |\mathrm{coker}\, \phi_n''|$

d) $|\ker \phi_n'| |\mathrm{coker}\, \phi| |\ker \phi_n''| = |\mathrm{coker}\, \phi_n'| |\ker \phi| |\mathrm{coker}\, \phi_n''|$

$\Rightarrow$ every term stabilize. $|E/\nu_n E| \big/ |Y/\nu_n Y|$ is const for $n \gg 0$

Compute $|E/\nu_n E|$:

① $E = \Lambda/(p^\mu)$. $|E/\nu_n E| = p^{\mu \cdot \deg \nu_n} = p^{\mu(p^n - p^{n_0})}$

② $E = \Lambda/(f)$. $f \equiv T^\lambda \bmod p$. $\gamma^{p^n} = (1+T)^{p^n} \equiv 1 \bmod (p, f)$

$\gamma^{p^{n+1}} \equiv 1 \bmod (p^2, f)$

$$\nu_{n+2}/\nu_{n+1} = \gamma^{p^{n+2}} - 1 \big/ \gamma^{p^{n+1}} - 1 = 1 + \gamma^{p^{n+1}} + \cdots + \gamma^{p^{n+1}(p-1)} \equiv p \cdot (1 + p(\ast)) \mod f$$

$\Rightarrow \nu_{n+2} E = p \cdot \nu_{n+1} E$ and $|E/\nu_{n+2} E| = |E/p \cdot \nu_{n+1} E| = \underbrace{|E/pE|}_{= p^\lambda} \cdot |E/\nu_{n+1} E|$

Everything combined, there exists c indep of n s.t.

$$|A_n| = |X/\nu_n Y| = p^{c+\mu p^n+\lambda n} \quad \text{if } n \gg 0$$

Problem: How to compute any of them?

§2. Cyclotomic $\mathbb{Z}_p$-extension

Fix p odd prime $\quad \mathbb{Q}_n = \mathbb{Q}(\zeta_{p^{n+1}})^{\mu_{p-1}}$

N an integer, $p \nmid N \quad K_n = \mathbb{Q}(\zeta_{Np^n})$

$$\begin{tikzcd} & K_\infty \arrow[dl, dash] \arrow[dr, dash] & \\ \mathbb{Q}_\infty \arrow[dr, dash, "\Gamma"'] & & K_0 \arrow[dl, dash, "\Delta"] \\ & \mathbb{Q} & \end{tikzcd} \qquad \Gamma_K \simeq \Gamma \simeq \mathbb{Z}_p$$

$$\begin{tikzcd} G_{\mathbb{Q}} \arrow[r, "\varepsilon :"] \arrow[d, two heads] & \mathbb{Z}_p^\times \text{ cyclotomic} \\ \Gamma \arrow[r, "\sim"] & 1+p\mathbb{Z}_p \arrow[u, phantom, "\cup"] \\ \gamma \arrow[r, mapsto] & u \end{tikzcd}$$

Every K_n is quadratic over K_n^+ tot. real $\quad Cl(K_n) \xrightarrow{1+c} Cl(K_n^+)$

let $h_n = |Cl(K_n)|$, $h_n^+ = |Cl(K_n^+)|$, $h_n^- = h_n/h_n^+$

Also $A_n = A_n^+ \oplus A_n^- \circlearrowleft \langle c \rangle$. $\quad |A_n^+| = |\mathbb{Z}_p/h_n^+|$ and $|A_n^-| = |\mathbb{Z}_p/h_n^-|$

Minus part of class number formula:

$$|\mathbb{Z}_p/h_n^-| = \left|\mathbb{Z}_p \Big/ W_n \cdot \prod_{\chi \in \widehat{G(K_n/\mathbb{Q})} \text{ odd}} L(0,\chi)\right| \qquad W_n = \#\text{ of roots of unity in } K_n$$

eg. $A = Cl(\mathbb{Q}(\zeta_p)) \otimes \mathbb{Z}_p = A^+ \oplus A^- = \bigoplus_{j \text{ even}} A^{\omega^j} \oplus \bigoplus_{i \text{ odd}} A^{\omega^i}$

$$|\mathbb{Z}_p/h^-| = \cancel{|\mathbb{Z}_p/pL(0,\omega^{-1})|} \times \prod_{\substack{m=3\\ \text{odd}}}^{p-2} |\mathbb{Z}_p/L(0,\omega^{-m})| = \prod_m |\mathbb{Z}_p/B_{1,\omega^{-m}}|$$

Thm (Herbrand-Ribet) for $3 \le m \le p-2$ odd. $A^{\omega^m} \ne 0 \iff p \mid B_{1,\omega^{-m}} \equiv B_{p-m}$

Sketch of $\Leftarrow$ direction : Pick $4 \le k \le p+1$ even s.t. $k \equiv p-m \mod p-1$

consider Eisenstein series $E_k = \frac{\zeta(1-k)}{2} + \sum \sigma_{k-1}(n) q^n$ $\quad \zeta(1-k) = -\frac{B_k}{k}$

if $4a+6b=k$, $E_k - \frac{\zeta(1-k)}{2}(G_4)^a(G_6)^b \in S_k(SL_2(\mathbb{Z}), \mathbb{Z}_p)$

$$\left(G_k = 1 + \frac{2}{\zeta(1-k)} \sum \sigma_{k-1}(n) q^n \right)$$

let $\mathbb{T}_k \subset \mathrm{End}_{\mathbb{Z}_p}(S_k(\Gamma, \mathbb{Z}_p))$ generated by $T(n)$'s. the cusp form defines

$$\begin{aligned} \mathbb{T}_k &\longrightarrow \mathbb{Z}_p/p\mathbb{Z}_p \\ T(n) &\longmapsto \sigma_{k-1}(n) \mod p \end{aligned} \qquad \boxed{\text{if } p \mid B_{p-m} \equiv B_k}$$

Since $\mathbb{T}_k/\mathbb{Z}_p$ is flat, this lifts to $F \in S_k(\Gamma, \mathcal{O})$ s.t. $F \equiv E_k \mod \pi$

associated Galois repn $\rho_F : G_{\mathbb{Q}} \to GL_2(\mathcal{O})$ unramified away p

$$F \text{ is } p\text{-ordinary} \Rightarrow \rho_F|_{G_p} \sim \begin{pmatrix} \psi \varepsilon^{k-1} & * \\ & \psi \end{pmatrix}$$

One can find a stable lattice whose reduction gives

$$0 \to \mathbb{F} \to \bar{\rho}_F \to \mathbb{F}(\omega^{k-1}) \to 0$$ non-split, but splits locally

this gives a nonzero class in $H^1_{ur}(\mathbb{Q}, \mathbb{F}(\underset{\underset{m}{\parallel}}{1-k})) \simeq \mathrm{Hom}(A^{\omega^m}, \mathbb{F})$

$Q: |A^{\omega^m}| = |L(0, \omega^{-m})|$?

Rmk. Define $I = \langle T(n) - \sigma_{k-1}(n) \rangle \subset \mathbb{T}_k$ and

$$0 \neq J = \ker\left(\mathbb{Z}_p \twoheadrightarrow \mathbb{T}_k / I \right) \subset \mathbb{Z}_p$$

we see that $p \mid \zeta(1-k) \Rightarrow \mathbb{Z}_p / J \neq 0$

Compare to $f(z) = \sum a_n q^n$ an eigenform, $L = \langle f \rangle^{\perp} \subset S_k(N, \mathcal{O})$

$\mathbb{T} \subset \mathrm{End}_{\mathcal{O}}(L)$ generated by $T(n)$'s. $I = \langle T(n) - a_n \rangle \subset \mathbb{T}$

$J = \ker(\mathcal{O} \twoheadrightarrow \mathbb{T}/I)$

we saw that $p \mid L^{alg}(k, \mathrm{ad} f) \Rightarrow \mathcal{O}/J \neq 0$

Recall p-adic L-function $L_p(1-k, \chi) = (1 - \chi\omega^{-k}(p) p^{k-1}) L(1-k, \chi\omega^{-k})$

if ψ is odd, there exists $g_\psi \in \Lambda_{\mathcal{O}}$, $h_\psi = \begin{cases} u(1+T) - 1 & \psi = \omega^{-1} \\ 1 & \text{o.w.} \end{cases}$

s.t. $\phi_{k,\zeta}(g_\psi / h_\psi) = L_p(-k, \psi \psi_\zeta \omega)$

where $\phi_{k,\zeta} : \Lambda_{\mathcal{O}} \to \mathcal{O}$, $\gamma \mapsto \zeta u^k$. $\zeta \in \mu_{p^\infty}$ $\quad \psi_\zeta : G \to \Gamma \to \mathcal{O}^\times$, $\gamma \mapsto \zeta$

Iwasawa main conjecture (proved by Mazur-Wiles)

If ψ is odd, $\mathrm{Ch}(X_\infty^{\psi}) = \mathrm{Ch}(Y^{\psi}) = (g_{\psi^{-1}})$

Consequence on $|A_n|$:

① $X_\infty^- = \varprojlim A_n^-$ has no nonzero finite Λ-submodule

②
$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & Y \arrow[r] \arrow[d, "\times \nu_n"'] & E \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Y \arrow[r] & E \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

$\Rightarrow |Y/\nu_n Y| = |E/\nu_n E| = |\Lambda/(\nu_n, \mathrm{Ch}(Y))|$

$$\left(0 \to \Lambda/(f) \xrightarrow{\times g} \Lambda/(fg) \to \Lambda/(g) \to 0 \right)$$

$$\text{Then } \left(|A_n^{\psi}| \Big/ |A_{n_0}^{\psi}| \right)^{[\mathcal{O}:\mathbb{Z}_p]} = \left| Y^{\psi} / \nu_n Y^{\psi} \right| = \left| \Lambda / (\nu_n, g_{\psi^{-1}}) \right| = \prod_{\zeta \in \mu_{p^n} \setminus \mu_{p^{n_0}}} \left| \mathcal{O} / L(0, (\psi\psi_\zeta)^{-1}) \right|$$

$$\left(0 \to \Lambda/\nu_n \to \prod \Lambda/_{T-(\zeta-1)} \to (\text{finite}) \to 0 \right)$$

Proof of ①:

<u>lemma</u> $A_n^- \to A_{n+1}^-$, $[\mathfrak{a}] \mapsto [\mathfrak{a} \cdot \mathcal{O}_{n+1}]$ is injective

if $\mathfrak{a} \cdot \mathcal{O}_{n+1} = (\alpha)$, then $\alpha^{\sigma}/\alpha \in \mathcal{O}_{n+1}^{\times}$ for $\sigma \in G(k_{n+1}/k_n)$

also $\mathfrak{a}\bar{\mathfrak{a}} = (\beta)$, if $\alpha\bar{\alpha} = \beta u$, consider $\mathfrak{a}^2 \mathcal{O}_{n+1} = \left(\frac{\alpha^2}{u} = \alpha_1\right)$

then $(\alpha_1^{\sigma}/\alpha_1)\left(\overline{\alpha_1^{\sigma}/\alpha_1}\right) = 1 \qquad \Rightarrow \alpha_1^{\sigma}/\alpha_1 \in W_{n+1} = \mu_{Np^n}$

but $H^1(k_{n+1}/k_n, W_{n+1}) = W_n / N \cdot W_{n+1} = \{1\}$

i. prime to p part : norm $= \times p$

ii. p-part : $\frac{1}{p^{n+1}} \sum_{a=1}^{p} (1 + ap^n) = \frac{1}{p^{n+1}} \left(p + \frac{p(p-1)}{2} p^n \right) \equiv \frac{1}{p^n} \mod \mathbb{Z}$

If $0 \neq X = (\cdots, a_{n+1}, a_n, \cdots)$ lies inside a finite submodule

then $\gamma^{p^n} \cdot X = X$ for $n \gg 0$

$\Rightarrow p \cdot X = (1 + \gamma^{p^n} + \cdots + \gamma^{p^n(p-1)}) \cdot X = (\cdots, \underset{\substack{\| \\ \text{image of } a_n \text{ in } A_{n+1}^-}}{N a_{n+1}}, p a_n, \cdots) \neq 0$

$\Rightarrow$ the finite submodule has no p-torsion $\rightarrow\!\!\leftarrow$

§3. Selmer groups

<u>Def</u> $\Psi : G_{\mathbb{Q}} \to \Gamma \to \Lambda^\times$ the tautological character

$(\text{or } G_K \to \Gamma_K \to \mathbb{Z}_p[\![\Gamma_K]\!]^\times)$

$\Lambda_{\mathcal{O}}^\vee = \mathrm{Hom}_{\mathcal{O}, cts}(\Lambda_{\mathcal{O}}, F/\mathcal{O})$ is isomorphic to the Pontryagin dual

Let $\Lambda_{\mathcal{O}}^\vee(\Psi^{-1}) = \mathrm{Hom}_{\mathcal{O}, cts}(\Lambda_{\mathcal{O}}(\Psi), F/\mathcal{O})$

$$H^1_{ur}(K, \Lambda^\vee(\Psi^{-1})) = \ker\left(H^1(K, \Lambda^\vee(\Psi^{-1})) \to \prod H^1(I_v, \Lambda^\vee(\Psi^{-1}))\right)$$

then
$$\begin{tikzcd}
0 \arrow[r] & H^1(\Gamma_K, \Lambda^\vee(\Psi^{-1})) \arrow[r] \arrow[d, equal] & H^1(K, \Lambda^\vee(\Psi^{-1})) \arrow[r] \arrow[d] & H^1(K_\infty, \Lambda^\vee(\Psi^{-1}))^{\Gamma_K} \arrow[r] \arrow[d, equal] & H^2(\Gamma_K, \Lambda^\vee(\Psi^{-1})) \arrow[d, equal] \\
& H^1(\Gamma_K, \Lambda^\vee(\Psi^{-1})) \arrow[r] & H^1(I_v, \Lambda^\vee(\Psi^{-1})) & \mathrm{Hom}_{\Gamma_K}(G_{K_\infty}, \Lambda^\vee(\Psi^{-1})) & 0
\end{tikzcd}$$
if $v | p$

$$\Rightarrow H^1_{ur}(K, \Lambda^\vee(\Psi^{-1})) \simeq \mathrm{Hom}_{\Gamma_K}(X_\infty, \Lambda^\vee(\Psi^{-1})) \simeq \left(X_\infty \otimes_{\mathbb{Z}_p[\![\Gamma_K]\!]} \Lambda(\Psi^{-1})\right)^*$$

<u>Def</u> $\Lambda_{\mathcal{O}}^\vee(\psi\Psi^{-1}) = \mathrm{Hom}(\Lambda(\Psi), F/\mathcal{O}(\psi))$

$\mathrm{Sel}_\infty(\psi) := H^1_{ur}(\mathbb{Q}, \Lambda_{\mathcal{O}}^\vee(\psi\Psi^{-1}))$. $X_\infty(\psi) = \mathrm{Sel}_\infty(\psi)^*$

Assume $p \nmid |\Delta|$, $\bigoplus_{\hat{\Delta}} \mathcal{O}(\psi) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}[\Delta], \mathcal{O})$

$$\Rightarrow \bigoplus \mathrm{Sel}_\infty(\psi) \simeq H^1_{ur}(\mathbb{Q}, \mathrm{Hom}(\mathcal{O}[\Delta], \Lambda_{\mathcal{O}}^\vee(\Psi^{-1}))) \simeq H^1_{ur}(K, \Lambda_{\mathcal{O}}^\vee(\Psi^{-1}))$$

$$\simeq (X_\infty \otimes \Lambda(\Psi^{-1}))^*$$

Taking ψ-comp. we get $X_\infty^\psi \simeq X_\infty(\psi)$

IMC: $\mathrm{Ch}(X_\infty^\psi) = (g_{\psi^{-1}})$

Control thm

Def $F/\mathcal{O}(\psi\varepsilon^{-k}) = \mathrm{Hom}(\mathcal{O}(\varepsilon^{k}), F/\mathcal{O}(\psi))$

$Sel(\psi\varepsilon^{-k}) := H^1_{ur}(\mathbb{Q}, F/\mathcal{O}(\psi\varepsilon^{-k}))$, $\quad X(\psi\varepsilon^{-k}) = Sel(\psi\varepsilon^{-k})^*$

$$0 \to \Lambda(\Psi) \xrightarrow{\gamma-\zeta u^k} \Lambda(\Psi) \to \mathcal{O}(\psi_\zeta\omega^{-k}\varepsilon^{k}) \to 0$$

$$0 \to F/\mathcal{O}(\psi\psi_\zeta^{-1}\omega^{k}\varepsilon^{-k}) \to \Lambda^*(\psi\Psi^{-1}) \to \Lambda^*(\psi\Psi^{-1}) \to 0$$

$\sim$

$$\begin{tikzcd}
0 \arrow[r] & \Lambda^*(\psi\Psi^{-1})^{G_\mathbb{Q}}/(\gamma-\zeta u^k) \arrow[r] \arrow[d] & H^1(\mathbb{Q}, F/\mathcal{O}(\cdot)) \arrow[r] \arrow[d] & H^1(\mathbb{Q}, \Lambda^*(\cdot))[\gamma-\zeta u^k] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Lambda^*(\cdot)^{I_v}/(\gamma-\zeta u^k) \arrow[r] & H^1(I_v, F/\mathcal{O}(\cdot)) \arrow[r] & H^1(I_v, \Lambda^*(\cdot))[\gamma-\zeta u^k] \arrow[r] & 0
\end{tikzcd}$$

a. $\Lambda^*(\psi\Psi^{-1})^{\Gamma} \simeq F/\mathcal{O}(\psi)^{\Delta} = \{0\}$

b. if $v = \ell \neq p$, either $\psi|_{I_\ell}$ not a p-power $\Rightarrow \Lambda^*(\cdot)^{I_v} = 0$

or $\psi|_{I_\ell} = 1 \Rightarrow \Lambda^*/(\gamma-\zeta u^k) = 0$

c. $\Lambda^*(\psi\Psi^{-1})^{\Gamma} = 0$ unless $\psi|_{I_p}$ factor through Γ

$\Rightarrow$ Assume $p \nmid |\Delta|$ and $\psi|_{I_p}$ doesn't factor through Γ, then

$$Sel(\psi\psi_\zeta^{-1}\omega^k\varepsilon^{-k}) \simeq Sel_\infty(\psi\Psi^{-1})[\gamma-\zeta u^k]$$

$$\Leftrightarrow \quad X_\infty(\psi)/_{\gamma-\zeta u^k} \simeq X(\psi\psi_\zeta^{-1}\omega^k\varepsilon^{-k})$$

$$\Rightarrow |X(\psi\psi_\zeta^{-1}\omega^k\varepsilon^{-k})| = |\Lambda/(\gamma-\zeta u^k, g_{\psi^{-1}})| = |\mathcal{O}/L(-k, \psi^{-1}\psi_\zeta)|$$

Final Remark

① In general, given a p-adic representation ρ and $\rho_\Lambda = \rho \otimes \Lambda^*$

Let $X = (Sel(\mathbb{Q}, \rho_\Lambda))^*$ for suitable local conditions

Ideally, there is a control thm connecting $X/_{\gamma - \zeta u^k}$ to interesting finite gps from which X is f.g. torsion.

② Mazur-Wiles proved the main conj by a refined study of Eisenstein congruence. Alternatively, one can prove it using cyclotomic Euler system. Both relies on class number formula

In general, to prove $Ch(X) = (\mathcal{L})$. one expect to combine

i) construct cong $\Rightarrow$ $Ch(X) \subset (\mathcal{L})$

ii) Euler system, which bounds Selmer gps $\Rightarrow$ $(\mathcal{L}) \subset Ch(X)$