1 Eichler-Shimura Isomorphism

1.1 Cohomology of Fuchsian Groups

Let G be a group, R be a given ring, M be a $R[G]$-module. We define the group cohomology as

$$H^*(G,M) := \mathrm{Ext}^*_{R[G]}(R,M),$$

where R is endowed with the trivial G-action. In this way, $H^*(G,M)$ is endowed with natural R-module structure, while the underlying group itself is independent of the R, as it is the derived functor of $M \mapsto M^G$.

In terms of the non-homogeneous cochain, we define $C^n(G,M)$ as the R-module of all maps from $G^{\times n}$ to M, here R acts on M, with differential maps given by

$$du(g_1,\cdots,g_{n+1}) := g_1u(g_2,\cdots,g_{n+1}) + \sum_{i=1}^{n}(-1)^i u(g_1,\cdots,g_{i-1},g_ig_{i+1},g_{i+1},\cdots,g_{n+1})$$
$$+(-1)^{n+1}u(g_1,\cdots,g_n).$$

$H^*(G,M)$ is identified with the cohomology of $C^*(G,M)$. For degree 0, we have $H^0(G,M) = M^G$. For degree 1, we have

$$Z^1(G,M) = \{u : G \to M \mid u(g_1g_2) = g_1u(g_2) + u(g_1)\},$$

and

$$B^1(G,M) = \{dm_v : g \mapsto (g-1)m_v\}.$$

Let Q be a subset of G. We define $C^*_Q(G,M)$ as the sub-cochain of $C^*(G,M)$ given by

$$C^1_Q(G,M) := \{u : G \to M \mid u(g) \in (g-1)M \text{ for all } g \in Q\},$$

and $C^i_Q(G,M) = C^i(G,M)$ if $i \neq 1$. It is clear that $B^1(G,M) \subset C^1_Q(G,M)$. We define $H^*_Q(G,M)$ as the cohomology of $C^*_Q(G,M)$. In particular, $H^i_Q(G,M) = H^i(G,M)$ for $i \neq 1,2$, and

$$H^1_Q(G,M) = \ker(H^1(G,M) \to \prod_{g\in Q}(\langle g\rangle, M)).$$

Like the usual cohomology, $H^1_Q(G,M)$ has the following functoriality:

Lemma 1. Assume $Q \subset G$ is closed under conjugation and taking powers. Let $H \subset G$ be a subgroup of finite index and M be a H-module. Then the canonical isomorphism $H^1(G, \mathrm{Ind}_H^G(M)) \cong H^1(H, M)$ induces an isomorphism $H^1_Q(G, \mathrm{Ind}_H^G(M)) \cong H^1_{Q\cap H}(H, M)$.

Proof. Let S be a set of representatives of $H\backslash G/K$ and define $H_s := s^{-1}Hs \cap K$ for each $s \in S$, and M_s the H_s-module whose underlying space is M and $s^{-1}hs(m_s) := (hm)_s$. Then we have the canonical isomorphism

$$\mathrm{Res}\,|_K \mathrm{Ind}_H^G(M) \cong \bigoplus_{s\in S} \mathrm{Ind}_{H_s}^K(M_s).$$

To be explicit, the isomorphism is given by $\varphi \mapsto (\varphi_s : k \mapsto \varphi(sk))$.

For every $s \in S$, we have the restriction map

$$H^*(H, M) \to H^*(H_s, M_s)$$

induced by the pair of maps. $(s^{-1}hs \mapsto h, \mathrm{id}_M)$. Therefore, the map

$$H^*(G, \mathrm{Ind}_H^G(M)) \to H^*(H, M) \to H^*(H_s, M_s)$$

is induced by the pair of maps

$$(s^{-1}hs \mapsto h, \varphi \mapsto \varphi(1)).$$

On the other hand, $H^*(K, \mathrm{Ind}_H^G(M)) \to H^*(H_s, M_s)$ is induced by the pair of maps $(s^{-1}hs \mapsto s^{-1}hs, \varphi \mapsto \varphi(s))$. Hence

$$H^*(G, \mathrm{Ind}_H^G(M)) \to H^*(K, \mathrm{Ind}_H^G(M)) \to H^*(H_s, M_s)$$

is induced by the pair of maps

$$(s^{-1}hs \mapsto s^{-1}hs, \varphi \mapsto s\varphi(1)).$$

Hence we have a commutative diagram

$$\begin{array}{ccc} H^*(G, \mathrm{Ind}_H^G(M)) & \longrightarrow & H^*(K, \mathrm{Ind}_H^G(M)) \\ \downarrow & & \downarrow \\ H^*(H, M) & \longrightarrow & \bigoplus_{s\in S} H^*(H_s, M_s) \end{array},$$

whose vertical maps are isomorphisms. Let K runs through $\{\langle q\rangle\}_{q\in Q}$ and we get the result. $\square$

Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian group of the first kind and $s_1, \cdots, s_m$ be the set of cusps on $X(\Gamma)$. For every s_i we find a small open disk D_i on $X(\Gamma)$, centered at s_i. We can make $X(\Gamma) - \bigcup_{i=1}^m D_i$ a simplicial complex satisfying that

1. Each elliptic point is a 0-simplex.

2. For each cusp s_i, ∂D_i is a 1-simplex.

Let $\mathcal{H}_0$ be the preimage of $X(\Gamma) - \bigcup_{i=1}^m D_i$ under the projection map $\mathcal{H} \to Y(\Gamma)$. $\mathcal{H}_0$ can be chosen so that it has trivial homology. We pull-back the simplicial complex structure of $X(\Gamma) - \bigcup_{i=1}^m D_i$ and we make $\mathcal{H}_0$ a simplicial complex, say K. Let $C_*(K)$ be the simplicial chain complex with coefficient R. We have that there is a $R[\Gamma]$-action on $C_*(K)$ and $C_2(K), C_1(K)$ are free $R[\Gamma]$-modules. We define $C^*(K, M) := \mathrm{Hom}_{R[\Gamma]}(C_*(K), M)$ and $H^*(K, M)$ the cohomology of $C^*(K, M)$.

For every s_i we choose t_i a 1-simplex of K such that t_i is mapped to ∂D_i and define q_i as the starting point of t_i. Then $\partial t_i = (\pi_i - 1)[q_i]$ where π_i is a generator of Γ_{s_i}. Let $Q = \{\pi_1, \cdots, \pi_m\}$. We define $C_Q^*(K, M)$ as the subcochain complex of $C^*(K, M)$ given by

$$C_Q^1(K, M) := \{u \in \mathrm{Hom}_{R[\Gamma]}(C_1(K), M) \mid u(t_i) \in (\pi_i - 1)M \text{ for all } i\},$$

and $C_Q^1(K, M) = C^i(K, M)$ if $i \neq 1$. It is clear that $B^1(K, M) \subset C_Q^1(G, M)$. We define $H_Q^*(K, M)$ as the cohomology of $C_Q^*(K, M)$.

If Γ has no elliptic elements, $C_0(K)$ is also free over $R[\Gamma]$ and

$$C_*(K) \xrightarrow{a} R$$

is a free $R[\Gamma]$-resolution of R. In this case, $H^*(\Gamma, M)$ is canonically identified with $H^*(K, M)$. In general, we have to deal with those elliptic points of $X(\Gamma)$. Let $p_1, \cdots, p_r \in \mathcal{H}$ be a set of representatives of elliptic points of $X(\Gamma)$, $e_j := |\Gamma_{p_j}|$, and $E := \mathrm{lcm}\{e_j\}$. Let $C_*(\Gamma)$ be the homogeneous chain complex. We would like to define chain maps

$$f_* : C_*(K) \to C_*(\Gamma),\ g_* : C_*(\Gamma) \to C_*(K)$$

so that both $f_* \circ g_*$ and $g_* \circ f_*$ are homotopic to $E \cdot \mathrm{id}$. We define f_* as follows: Let $S \subset \mathcal{H}_0$ be a set of representatives of Γ-orbits on 0-simplices. We may assume that

S contains $p_1, \cdots, p_r, q_1, \cdots, q_m$. We define $f_0 : S \to C_0(\Gamma) = R[\Gamma]$ by

$$f_0(p_j) := \frac{E}{e_j} \sum_{g \in \Gamma_{p_j}} [g]$$

and $f_0(s) = E[e]$ if $s \in S - \{p_1, \cdots, p_r\}$. Then f_0 extend uniquely to a $R[\Gamma]$-homomorphism from $C_0(K)$ to $C_0(\Gamma)$ with the commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & C_2(K) & \longrightarrow & C_1(K) & \longrightarrow & C_0(K) & \xrightarrow{a} & R & \longrightarrow & 0 \\ & & & & & & \downarrow f_0 & & \downarrow E\cdot & & \\ \cdots & \longrightarrow & C_2(\Gamma) & \longrightarrow & C_1(\Gamma) & \longrightarrow & R[\Gamma] & \xrightarrow{a} & R & \longrightarrow & 0 \end{array}$$

Since $C_2(K), C_1(K)$ are free over $R[\Gamma]$ and $C_*(\Gamma)$ is exact, $(f_0, E\cdot)$ extends uniquely up to chain homotopy to a chain map f_*. Similarly, we pick an arbitrary 0-simplex p of K and define $g_0 : R[\Gamma] \to C_0(K)$ by evaluation at $[p]$. g_0 is extended uniquely up to chain homotopy to a chain map g_* such that g_* induces identity map at H_0. Now we see that $f_* \circ g_*$ induces $E\cdot$ on $H_0(C_*(\Gamma))$, so it's chain homotopic to $E \cdot \mathrm{id}$. For $g_* \circ f_*$, the only problem is that $f_0 \circ g_0(p_j)$ should be Γ_{p_j}-invariant, which is clear. Now we have

$$f^* : C^*(G, M) \to C^*(K, M),\ g^* : C^*(K, M) \to C^*(G, M),$$

such that both $f^* \circ g^*$ and $g^* \circ f^*$ induce multiplication by E on cohomology. In particular, if E is invertible in R, $H^*(G, M) \cong H^*(K, M)$.

To deal with the parabolic cohomology, we may assume that

1. $f_*(t_i) = E([1, \pi_i])$ and $g_*([1, \pi_i]) = t_i + (\pi_i - 1)b_i$ where b_i is a 1-chain with $\partial b_i = [p] - [q_i]$.

2. $p \in S$ but is not an elliptic point. In this way, we have that $f_0 \circ g_0 = E\cdot$.

3. The chain homotopy U_* between $f_* \circ g_*$ and $E \cdot \mathrm{id}_{C_0(\Gamma)}$ satisfies that $U_1([1, \pi_i]) \in (\pi_k - 1)C_2(\Gamma)$. We first take $U_0 = 0$. We have that

$$(f_1 \circ g_1)([1, \pi_i]) = E([1, \pi_i]) + (\pi_i - 1)f_1(b_i).$$

Since $\partial f_1(b_i) = f_0([p] - [q_i]) = 0$, U_1 can be chosen so that $U_1([1, \pi_i]) \in (\pi_i - 1)\partial^{-1}(f_1(b_i))$.

4. The chain homotopy V_* between $g_* \circ f_*$ and $E \cdot \mathrm{id}_{C_0(\Gamma)}$ takes 0 on t_i. Since $(g_0 \circ f_0)[q_i] = E[p]$, V_0 can be chosen so that $V_0([q_i]) = b_i$. Since

$$(g_1 \circ f_1)(t_i) - Et_i = V_0(\partial t_i),$$

V_1 can be chosen so that $V_0([t_i]) = 0$.

Now if $u \in C^1_Q(G, M)$, we have that

$$f^*(u)(t_i) = u(f_*(t_i)) = Eu([1, \pi_i]) = Eu(\pi_i) \in E(\pi_i - 1)M,$$

and if $u \in C^1_Q(K, M)$

$$g^*(u)(\pi_k) = g^*(u)([1, \pi_i]) = u(t_i) + (\pi_i - 1)u(b_i) \in (\pi_i - 1)M.$$

Hence $f^* : C^*(G, M) \to C^*(K, M)$, $g^* : C^*(K, M) \to C^*(G, M)$ and U_*, V_* remain chain homotopies. We obtain the same result as usual cohomology case.

Remark 1. Let P be the set of all parabolic elements of Γ, then every element $\pi \in P$ is conjugate to a power of some π_i. If $u(\pi_i) = (\pi_i - 1)x_i$, $u(g\pi_i^n g^{-1}) = (g\pi_i^n g^{-1} - 1)(gx - u(g))$. Hence $Z^1_P(\Gamma, M) = Z^1_Q(\Gamma, M)$ and $H^1_P(\Gamma, M) = H^1_Q(\Gamma, M)$.

Let R be a ring, G be a group, M be a $R[G]$-module, and S be a flat R-module endowed with trivial G-action. Then both $H^*(G, \cdot) \otimes_R S$ and $H^*(G, (\cdot) \otimes_R S)$ are cohomological delta functors from $\mathrm{Mod}_{R[G]}$ to Mod_R, or Mod_S when S is a R-algebra. Since $H^*(G, \cdot) \otimes_R S$ vanishes on injective $R[G]$-modules, $H^*(G, \cdot) \otimes_R S$ is an universal delta functor. At degree 0, we have the functorial map $M^G \otimes_R S \to (M \otimes_R S)^G$, which is injective. Hence we obtain a unique natural transformation t^* from $H^*(G, \cdot) \otimes_R S$ to $H^*(G, (\cdot) \otimes_R S)$. Alternatively $t^*(M)$ is induced by the obvious chain map $C^*(G, M) \otimes_R S \to C^*(G, M \otimes_R S)$.

t^* may not be a natural isomorphism of delta functors as tensor product does not commute with infinite product.

Lemma 2. If $R[G]$ is Noetherian, or G is cyclic, t^* is a natural isomorphism of delta functors.

Proof. In both cases, R has a finite free resolution. □

Lemma 3. If G is generated by finitely many elements, $t^0(M)$ is a natural isomorphism and $t^1(M)$ is injective for every M.

Proof. Let $g_1, \cdots, g_m$ be a generating set of G. Tensoring S on the exact sequence

$$0 \to M^G \to M \xrightarrow{\oplus g_i - 1} \bigoplus_{i=1}^{m} M$$

and we get the exact sequence $0 \to M^G \otimes_R S \to M \otimes_R S \xrightarrow{\oplus g_i - 1} \bigoplus_{i=1}^m M \otimes_R S$, hence the isomorphism $M^G \otimes_R S \cong (M \otimes_R S)^G$.

Consider a short exact sequence

$$0 \to M \to M' \to M'' \to 0$$

with injective M'. Apply the delta functors and use that t^0 is a natural isomorphism we have that $t^1(M)$ is injective. □

Suppose $t^1(M)$ is an isomorphism and $Q \subset G$ is a finite set. Then $H^1_Q(G, M) \otimes S = H^1_Q(G, M \otimes_R S)$.

Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian group of the first kind.

Lemma 4. $H^*(K, M) \otimes_R S \cong H^*(K, M \otimes_R S)$.

Proof. We have to compare the cohomology of

$$\mathrm{Hom}_{R[\Gamma]}(C_*(K), M) \otimes_R S$$

to

$$\mathrm{Hom}_{R[\Gamma]}(C_*(K), M \otimes_R S).$$

Consider the natural transformation $\mathrm{Hom}_{R[\Gamma]}(\cdot, M) \otimes_R S \to \mathrm{Hom}_{R[\Gamma]}(\cdot, M \otimes_R S)$. Both functors commute with finite direct sum. If $N \cong R[\Gamma]/(g-1)$ for some $g \in \Gamma$, we have $\mathrm{Hom}_{R[\Gamma]}(N, M) \otimes_R S = M^g \otimes_R S$, $\mathrm{Hom}_{R[\Gamma]}(N, M \otimes_R S) = (M \otimes_R S)^g$, and the natural homomorphism is an isomorphism. Since each $C_*(K)$ is a finite direct sum of $R[\Gamma]$-modules of this form, we get an isomorphism of cochain complexes. □

Proposition 1. If $E \cdot : M \otimes_R S \to M \otimes_R S$ is an isomorphism and $t^1(M)$ is injective, $H^1(\Gamma, M) \otimes_R S \to H^1(\Gamma, M \otimes_R S)$ is an isomorphism.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} H^1(\Gamma, M) \otimes_R S & \xrightarrow{t^1} & H^1(\Gamma, M \otimes_R S) \\ \downarrow{\scriptstyle f^1 \otimes 1} & & \downarrow{\scriptstyle f^1} \\ H^1(K, M) \otimes_R S & \xrightarrow{\psi} & H^1(K, M \otimes_R S) \\ \downarrow{\scriptstyle g^1 \otimes 1} & & \downarrow{\scriptstyle g^1} \\ H^1(\Gamma, M) \otimes_R S & \xrightarrow{t^1} & H^1(\Gamma, M \otimes_R S). \end{array}$$

We already have that t^1 is injective, ψ is an isomorphism, and $f^1 \circ g^1 = g^1 \circ f^1 = E\cdot$. Now $(g^1 \otimes 1) \circ \psi^{-1} \circ (E^{-1} f^1)$ is the inverse of t^*. □

1.2 Eichler-Shimura Isomorphism

Let $R = \mathbb{R}$ or $\mathbb{C}$, R^2 be endowed with the standard $\mathrm{GL}_2(R)$-representation, and $\mathrm{Sym}^n(R^2)$ be the S_n-fixed subspace of $(R^2)^{\otimes n}$ with the natural $\mathrm{GL}_2(R)$-action. Let θ be the R-bilinear form on R^2 given by $(v, w) \to \det \begin{pmatrix} v & w \end{pmatrix}$. This is extended to Θ_n, the R-bilinear form on $\mathrm{Sym}^n(R^2)$, determined by

$$\Theta_n(v^{\otimes n}, w^{\otimes n}) = \Theta(v, w)^n.$$

We have $\Theta_n(v, w) = (-1)^n \Theta_n(w, v)$, and $\Theta_n(\alpha v, \alpha w) = \det(\alpha)^n \Theta_n(v, w)$. This makes $\mathrm{Sym}^n(R^2)$ a self-dual $\mathrm{SL}_2(R)$-module.

Let $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ be a Fuchsian group of the first kind, $\rho : \Gamma \to \mathrm{GL}(V)$ be a finite dimensional $\mathbb{C}$-representation with finite image, and k be a positive integer.

Definition 1. $S_k(\Gamma, \rho)$ is the space of holomorphic functions $f : \mathcal{H} \to V$ satisfying that

1. $f(\alpha z) j(\alpha, z)^{-k} = \rho(\alpha) f(z)$ for all $\alpha \in \Gamma$.

2. For every $\ell \in \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$, $\ell \circ f \in S_k(\ker(\rho))$.

Proposition 2. $S_k(\Gamma, \rho_1 \oplus \rho_2) = S_k(\Gamma, \rho_1) \oplus S_k(\Gamma, \rho_2)$. For another Fuchsian group of the first kind $\Gamma' \supset \Gamma$ with $[\Gamma : \Gamma'] < \infty$, there is a natural isomorphism

$$S_k(\Gamma, \rho) \cong S_k(\Gamma', \mathrm{Ind}_{\Gamma}^{\Gamma'}(\rho)).$$

Proof. The first assertion is trivial. For the second one, we define

$$\phi : S_k(\Gamma', \mathrm{Ind}_{\Gamma}^{\Gamma'}(\rho)) \to S_k(\Gamma, \rho),\ z \mapsto f(z)(1),$$

$$\psi : S_k(\Gamma, \rho) \to S_k(\Gamma', \mathrm{Ind}_{\Gamma}^{\Gamma'}(\rho)),\ z \mapsto (\alpha \mapsto f(\alpha z) j(\alpha, z)^{-k}).$$

They are well-defined $\mathbb{C}$-linear map that are inverse to each other. $\square$

Let $\overline{\rho} : \Gamma \to \mathrm{GL}(V)$ given by $\overline{\rho}(\alpha)(\overline{v}) := \overline{\rho(\alpha)(v)}$. For every $f \in S_k(\Gamma, \overline{\rho})$, we have $\overline{f(\alpha z)} j(\alpha, \overline{z})^{-k} = \overline{\overline{\rho}(\alpha) f(z)} = \rho(\alpha) \overline{f(z)}$.

Suppose $k \geq 2$. For every $f = (f_1, \overline{f_2}) \in S_k(\Gamma, \rho) \oplus \overline{S_k(\Gamma, \overline{\rho})}$, we define

$$\omega(f) \in H^0(\Omega^1(\mathcal{H}, V \otimes_{\mathbb{C}} \mathrm{Sym}^{k-2}(\mathbb{C}^2))),$$

$$\omega(f)(z) := f_1(z)(ze_1 + e_2)^{\otimes n} dz + \overline{f_2(z)(ze_1 + e_2)^{\otimes n} dz}.$$

In particular, $\omega(f)$ is a closed 1-form, and $\omega(f) \circ \alpha = \chi(\alpha)\omega(f)$, where χ is the representation $V \otimes \mathrm{Sym}^{k-2}(\mathbb{C}^2)$. Let F be a primitive of $\omega(f)$. Then F have the form

$$F(z) = \int_{z_0}^{z} \omega(f) + v$$

for some $z_0 \in \mathcal{H}$ and $v \in V \otimes_{\mathbb{C}} \mathrm{Sym}^{k-2}(\mathbb{C}^2)$. We define $u(f) \in Z^1(\Gamma, V \otimes_{\mathbb{C}} \mathrm{Sym}^{k-2}(\mathbb{C}^2))$ by

$$u(f)(\alpha) := F(\alpha z) - \chi(\alpha)F(z) = \int_{z_0}^{\alpha z_0} \omega(f) + (1 - \chi(\alpha))v.$$

Let $\pi \in \Gamma$ be a parabolic element and $s \in \mathbb{P}^1(\mathbb{R})$ be a cusp fixed by π. F can be extend to s and we obtain that

$$F(s) = F(\pi(s)) = \chi(\pi)F(s) + u(f)(\pi).$$

Hence $u(f) \in Z^1_P(\Gamma, V \otimes_{\mathbb{C}} \mathrm{Sym}^{k-2}(\mathbb{C}^2))$ and $[u(f)]$ is a well-defined class in $H^1_P(\Gamma, V \otimes_{\mathbb{C}} \mathrm{Sym}^{k-2}(\mathbb{C}^2))$, independent of the choice of F. We therefore obtain a $\mathbb{C}$-linear map

$$\Psi_\rho : S_k(\Gamma, \rho) \oplus \overline{S_k(\Gamma, \overline{\rho})} \to H^1_P(\Gamma, V \otimes_{\mathbb{C}} \mathrm{Sym}^{k-2}(\mathbb{C}^2)),\ f \mapsto [u(f)].$$

Lemma 5. Let $\Gamma' \supset \Gamma$ be a Fuchsian group of the first kind such that $[\Gamma : \Gamma'] < \infty$. Then we have the commutative diagram

$$\begin{array}{ccc} S_k(\Gamma', \mathrm{Ind}_{\Gamma}^{\Gamma'}(\rho)) \oplus \overline{S_k(\Gamma', \mathrm{Ind}_{\Gamma}^{\Gamma'}(\overline{\rho}))} & \xrightarrow{\Psi_{\mathrm{Ind}_{\Gamma}^{\Gamma'}(\rho)}} & H^1_P(\Gamma', \mathrm{Ind}_{\Gamma}^{\Gamma'}(V) \otimes_{\mathbb{C}} \mathrm{Sym}^{k-2}(\mathbb{C}^2)) \\ \downarrow \phi & & \downarrow \Phi \\ S_k(\Gamma, \rho) \oplus \overline{S_k(\Gamma, \overline{\rho})} & \xrightarrow{\Psi_\rho} & H^1_P(\Gamma, V \otimes_{\mathbb{C}} \mathrm{Sym}^{k-2}(\mathbb{C}^2)) \end{array}.$$

Note that $\mathrm{Ind}_\Gamma^{\Gamma'}(\overline{\rho}) = \overline{\mathrm{Ind}_\Gamma^{\Gamma'}(\rho)}$ and Φ is induced by the natural inclusion $\Gamma \to \Gamma'$ and $\mathrm{Ind}_\Gamma^{\Gamma'}(V) \otimes_{\mathbb{C}} \mathrm{Sym}^{k-2}(\mathbb{C}^2) \to V \otimes_{\mathbb{C}} \mathrm{Sym}^{k-2}(\mathbb{C}^2)$, $\varphi \otimes v \mapsto \varphi(1) \otimes v$. Here we use the isomorphism

$$\mathrm{Ind}_H^G(U \otimes \mathrm{Res}_H T) \cong \mathrm{Ind}_H^G(U) \otimes T.$$

Proof. An explicit computation shows that both $\Psi_\rho \circ \phi$ and $\Phi \circ \Psi_{\mathrm{Ind}_\Gamma^{\Gamma'}(\overline{\rho})}$ maps f to the class represented by

$$u : \alpha \mapsto \int_{z_0}^{z} \omega(f)(z)(1).$$

$\square$

Theorem 1.

$$\Psi_\rho : S_k(\Gamma, \rho) \oplus \overline{S_k(\Gamma, \overline{\rho})} \to H_P^1(\Gamma, V \otimes_{\mathbb{C}} \mathrm{Sym}^{k-2}(\mathbb{C}^2))$$

is an isomorphism.

Proof. By the additivity at ρ we may assume that ρ is a regular representation $\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\mathbb{C})$, where Γ_0 is the kernel of ρ, and the case is reduced to $\Gamma = \Gamma_0$ and ρ is the trivial representation.

Now we show that

$$\Psi_1 : S_k(\Gamma) \oplus \overline{S_k(\Gamma)} \to H_P^1(\Gamma, \mathrm{Sym}^{k-2}(\mathbb{C}^2))$$

is an isomorphism. Since both sides have the same dimension over $\mathbb{C}$, it suffices to show the injectivity. We define

$$(f, g) := \int_{\Gamma \backslash \mathcal{H}} \omega(f) \wedge \omega(g),$$

which is a nondegenerate $\mathbb{C}$-bilinear form on $S_k(\Gamma) \oplus \overline{S_k(\Gamma)}$. To be explicit, if $f = (f_1, \overline{f_2})$, $g = (g_1, \overline{g_2})$,

$$(f, g) = \int_{\Gamma \backslash \mathcal{H}} \left(f_1(z)\overline{g_2(z)} - g_1(z)\overline{f_2(z)} \right) (z - \overline{z})^{k-2} dz \wedge d\overline{z}.$$

Let F be a primitive of $\omega(f)$. If $\Psi_1(f) = 0$, F can be chosen so that $F(\alpha z) = \chi(\alpha)F(z)$ for all $\alpha \in \Gamma$. Let X be a fundamental domain of $X(\Gamma)$. $\partial X = \sum_i (\alpha_i - 1)s_i$

where s_i are 1-simplices. Then

$$(f,g) = \int_{\partial X} F \wedge \omega(g) = \sum_i \left(\int_{\alpha_i s_i} F \wedge \omega(g) - \int_{s_i} F \wedge \omega(g) \right) = 0.$$

for all g. Hence $f = 0$ and we get the injectivity. □

We similarly define

1. $M_k(\Gamma, \rho)$ is the space of holomorphic functions from $\mathcal{H}$ to V such that

$$f(\alpha z) j(\alpha, z)^{-k} = \rho(\alpha) f(z)$$

and for every $\ell \in \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$, $\ell \circ f \in M_k(\ker(\rho))$.

2.

$$\Psi_\rho : M_k(\Gamma, \rho) \oplus \overline{S_k(\Gamma, \overline{\rho})} \to H^1(\Gamma, \rho \otimes_{\mathbb{C}} \mathrm{Sym}^{k-2}(\mathbb{C}^2))$$

given by

$$f \mapsto \left[\left(\alpha \mapsto \int_{z_0}^{\alpha z_0} \omega(f) \right) \right].$$

Corollary 1.

$$\Psi_\rho : M_k(\Gamma, \rho) \oplus \overline{S_k(\Gamma, \overline{\rho})} \to H^1(\Gamma, \rho \otimes_{\mathbb{C}} \mathrm{Sym}^{k-2}(\mathbb{C}^2))$$

is an isomorphism.

Proof. By the same functoriality we reduce this to the case $\rho = 1$. Since $M_k(\Gamma) \oplus \overline{S_k(\Gamma)}$ and $H^1(\Gamma, \mathrm{Sym}^{k-2}(\mathbb{C}^2))$ have the same dimension, it suffices to show the injectivity. We consider the commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & S_k(\Gamma) \oplus \overline{S_k(\Gamma)} & \xrightarrow{\iota} & M_k(\Gamma) \oplus \overline{S_k(\Gamma)} & \longrightarrow & \mathrm{coker}(\iota) \\ & & \downarrow \Psi_1 & & \downarrow \Psi_1 & & \downarrow \\ 0 & \longrightarrow & H^1_P(\Gamma, \mathrm{Sym}^{k-2}(\mathbb{C}^2)) & \longrightarrow & H^1(\Gamma, \mathrm{Sym}^{k-2}(\mathbb{C}^2)) & \longrightarrow & \bigoplus_{i=1}^m H^1(\langle \pi_i \rangle, \mathrm{Sym}^{k-2}(\mathbb{C}^2)). \end{array}$$

We should show that $\mathrm{coker}(\iota) \to \bigoplus_{i=1}^m H^1(\langle \pi_i \rangle, \mathrm{Sym}^{k-2}(\mathbb{C}^2))$ is injective. Namely, $M_k(\Gamma) \to \bigoplus_{i=1}^m H^1(\langle \pi_i \rangle, \mathrm{Sym}^{k-2}(\mathbb{C}^2))$ has kernel $S_k(\Gamma)$. Let $\beta_i \in \mathrm{SL}_2(\mathbb{R})$ with $\beta_i \pi_i \beta_i^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ if s_i is regular, or $\beta_i \pi_i \beta_i^{-1} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$, if s_i is irregular. Let $f \in M_k(\Gamma)$ and $z_i := \beta z_0$. We have

$$\int_{z_0}^{\pi_i z_0} \omega(f) = \int_{z_i}^{\beta_i \pi_i \beta_i^{-1} z_i} \omega(f) \circ \beta_i^{-1} = \beta_i^{-1} \int_{z_i}^{\beta_i \pi_i \beta_i^{-1} z_i} f|[\beta^{-1}]_k(z)(ze_1 + e_2)^{\otimes k-2} dz.$$

Let $x_i := \int_{z_i}^{\beta_i\pi_i\beta_i^{-1}z_i} f|[\beta^{-1}]_k(z)(ze_1+e_2)^{\otimes k-2}$. $\beta_i^{-1}x_i \in (\pi_i - 1)\operatorname{Sym}^{k-2}(\mathbb{C}^2)$ if and only if $x_i \in (\beta_i\pi_i\beta_i^{-1})\operatorname{Sym}^{k-2}(\mathbb{C}^2)$. Therefore, f is in the kernel if and only if for all regular cusps s_i,

$$\int_{z_i}^{\beta_i\pi_i\beta_i^{-1}z_i} f|[\beta_i^{-1}]_k(z)dz = 0.$$

Let $q = e^{2\pi iz}$. If s_i is irregular, $f|[\beta_i^{-1}]_k(z) \in q^{1/2}\,\mathbb{C}[[q]]$. If s_i is regular, $f|[\beta_i^{-1}]_k(z) \in \mathbb{C}[[q]]$, and the constant term is given by

$$\int_{z_i}^{\beta_i\pi_i\beta_i^{-1}z_i} f|[\beta_i^{-1}]_k(z)dz.$$

Hence the kernel of $M_k(\Gamma) \to \bigoplus_{i=1}^m H^1(\langle\pi_i\rangle, \operatorname{Sym}^{k-2}(\mathbb{C}^2))$ is exactly $S_k(\Gamma)$. □

1.3 Double Coset Operators

Let $\Gamma_1, \Gamma_2 \subset \mathrm{SL}_2(\mathbb{R})$ be two Fuchsian groups of the first kind. Let $\Delta \subset \mathrm{GL}_2^+(\mathbb{R})$ be a semi-group containing Γ_1, Γ_2, and for every $\alpha \in \Delta$, $\alpha\Gamma_1\alpha^{-1}$ and Γ_2 are commensurable. Consider the involution

$$\iota : \alpha \mapsto \det(\alpha)\alpha^{-1}.$$

Let X be a $R[\Delta^\iota]$-module. We define for every $\alpha \in \Delta$ a R-linear map

$$(\Gamma_1\alpha\Gamma_2)_X : H_P^1(\Gamma_1, X) \to H_P^1(\Gamma_2, X)$$

as follows: Let $\{\alpha_1, \cdots, \alpha_d\}$ be a set representatives of $\Gamma_1\backslash\Gamma_1\alpha\Gamma_2$. For every $\beta \in \Gamma_2$, we have $\alpha_i\beta = \gamma_i\alpha_j$ (or we write $\alpha_i\beta = \gamma_i^\beta\alpha_j$) for some $\gamma_i \in \Gamma_1$. $(\Gamma_1\alpha\Gamma_2)_X$ sends a 1-cocycle u to $v : \beta \mapsto \sum_{i=1}^d \alpha_i^\iota u(\gamma_i)$. This double coset operator actually defines a "corestriction" map on the category of $R[\Delta^\iota]$-modules. We first define a chain map α^* on homogeneous chains

$$\alpha^n : \widetilde{C}^n(\Gamma_1, X) \to \widetilde{C}^n(\Gamma_2, X)$$

by

$$\alpha(\widetilde{u})(g_0, \cdots, g_n) := \sum_{i=1}^d \alpha_i^\iota \widetilde{u}(\gamma_i^{g_0}, \cdots, \gamma_i^{g_n}).$$

Since

$$\alpha_i gh = \gamma_i^g \alpha_{ig} h = \gamma_i^g \gamma_{ig}^h \alpha_{igh},$$

$\gamma_i^{gh} = \gamma_i^g \gamma_{ig}^h$. Therefore,

$$\begin{aligned}\alpha(\widetilde{u})(gh_0, \cdots, gh_n) &= \sum_{i=1}^{d} \alpha_i^{\iota} \widetilde{u}(\gamma_i^g \gamma_{ig}^{h_0}, \cdots, \gamma_i^g \gamma_{ig}^{h_n}) = \sum_{i=1}^{d} \alpha_i^{\iota} \gamma_i^g \widetilde{u}(\gamma_{ig}^{h_0}, \cdots, \gamma_{ig}^{h_n}) \\ &= \sum_{i=1}^{d} g \alpha_{ig}^{\iota} \widetilde{u}(\gamma_{ig}^{h_0}, \cdots, \gamma_{ig}^{h_n}) = g\alpha(\widetilde{u})(h_0, \cdots, h_n).\end{aligned}$$

Moreover, α is clearly a chain map. Hence we obtain α^* on cohomology, and clearly is a homomorphism for delta functors.

Use $u(g_1, \cdots, g_n) = \widetilde{u}(1, g_1, g_1 g_2, \cdots, g_1 \cdots g_n)$ and we see that for H^1, α^1 is the double coset operator we defined.

At degree 0 we have $\alpha^0 : X^{\Gamma_1} \to X^{\Gamma_2}$, $x \mapsto \sum_{i=1}^{d} \alpha_i^{\iota} x$.

Let V be a finite dimensional $\mathbb{C}$-vector space and $\rho : \Delta^{\iota} \to \mathrm{GL}(V)$ be multiplicative such that $\rho(\Gamma_1)$, $\rho(\Gamma_2)$ are finite. Then Δ^{ι} acts on $V \otimes_{\mathbb{C}} \mathrm{Sym}^{k-2}(\mathbb{C}^2)$, denoted by χ. Suppose further that $\rho(-I_2) = (-1)^k$ if $-I_2 \in \Delta$. We define

$$f|[\Gamma_1 \alpha \Gamma_2]_{k,\rho} : z \mapsto \det(\alpha)^{k-1} \sum_{i=1}^{d} \rho(\alpha_i^{\iota}) f(\alpha_i z) j(\alpha_i, z)^{-k}.$$

Then $[\Gamma_1 \alpha \Gamma_2]_{k,\rho} : S_k(\Gamma_1, \rho) \to S_k(\Gamma_2, \rho)$ is a well-defined $\mathbb{C}$-linear map. We also define $[\Gamma_1 \alpha \Gamma_2]_{k,\rho} : \overline{S_k(\Gamma_1, \overline{\rho})} \to \overline{S_k(\Gamma_2, \overline{\rho})}$ by

$$\overline{f}|[\Gamma_1 \alpha \Gamma_2]_{k,\rho} := \overline{f|[\Gamma_1 \alpha \Gamma_2]_{k,\overline{\rho}}}.$$

This is also $\mathbb{C}$-linear.

Proposition 3. We have the commutative diagram

$$\begin{array}{ccc} S_k(\Gamma_1, \rho) \oplus \overline{S_k(\Gamma_1, \overline{\rho})} & \longrightarrow & H_P^1(\Gamma_1, V \otimes_{\mathbb{C}} \mathrm{Sym}^{k-2}(\mathbb{C}^2)) \\ \Big\downarrow {\scriptstyle [\Gamma_1 \alpha \Gamma_2]_{k,\rho}} & & \Big\downarrow {\scriptstyle [\Gamma_1 \alpha \Gamma_2]_{k,\rho}} \\ S_k(\Gamma_2, \rho) \oplus \overline{S_k(\Gamma_2, \overline{\rho})} & \longrightarrow & H_P^1(\Gamma_2, V \otimes_{\mathbb{C}} \mathrm{Sym}^{k-2}(\mathbb{C}^2)) \end{array}.$$

Proof. Let $f = (f_1, \overline{f_2}) \in S_k(\Gamma_1, \rho) \oplus \overline{S_k(\Gamma_1, \overline{\rho})}$. We have

$$\begin{aligned}\omega(f_1|[\Gamma_1 \alpha \Gamma_2]_{k,\rho}) &= \sum_{i=1}^{d} \rho(\alpha_i^{\iota}) f_1(\alpha_i z) j(\alpha_i, z)^{-k} \det(\alpha)^{k-1} (z e_1 + e_2)^{\otimes k-2} dz \\ &= \sum_{i=1}^{d} \rho(\alpha_i^{\iota}) f_1(\alpha_i z) \det(\alpha)^{k-1} \alpha_i^{-1} (\alpha_i z e_1 + e_2)^{\otimes k-2} d\alpha_i z \\ &= \sum_{i=1}^{d} \chi(\alpha_i^{\iota}) \omega(f_1) \circ \alpha_i,\end{aligned}$$

and

$$\begin{aligned}\omega(\overline{f_2}|[\Gamma_1\alpha\Gamma_2]_{k,\rho}) =&\overline{\omega(f_2|[\Gamma_1\alpha\Gamma_2]_{k,\rho})} = \overline{\sum_{i=1}^{d} \overline{\chi(\alpha_i^\iota)}\omega(f_2)\circ\alpha_i} \\ =&\overline{\sum_{i=1}^{d} \overline{\chi(\alpha_i^\iota)}\omega(f_2)\circ\alpha_i} = \sum_{i=1}^{d} \chi(\alpha_i^\iota)\omega(\overline{f_2})\circ\alpha_i.\end{aligned}$$

Therefore,

$$\omega(f|[\Gamma_1\alpha\Gamma_2]_{k,\rho}) = \sum_{i=1}^{d} \chi(\alpha_i^\iota)\omega(f)\circ\alpha_i.$$

We have that

$$\begin{aligned}\int_{z_0}^{\beta z_0} \omega(f|[\Gamma_1\alpha\Gamma_2]_{k,\rho}) =&\sum_{i=1}^{d}\int_{z_0}^{\beta z_0} \chi(\alpha_i^\iota)\omega(f)\circ\alpha_i = \sum_{i=1}^{d}\chi(\alpha_i^\iota)(F(\alpha_i\beta z_0) - F(\alpha_i z_0)) \\ =&\sum_{i=1}^{d}\chi(\alpha_i^\iota)(F(\gamma_i\alpha_j z_0) - F(\alpha_i z_0)) \\ =&\sum_{i=1}^{d}\chi(\alpha_i^\iota)(u(f)(\gamma_i) + \chi(\beta_i)F(\alpha_j z_0) - F(\alpha_i z_0)) \\ =&\sum_{i=1}^{d}\chi(\alpha_i^\iota)u(f)(\gamma_i) + \sum_{i=1}^{d}\left[\chi(\beta)\chi(\alpha_j^\iota)F(\alpha_j z_0) - \chi(\alpha_i)F(\alpha_i z_0)\right]\end{aligned}$$

and we get the commutativity. □

Similarly,

$$M_k \oplus \overline{S_k} \to H^1$$

is Hecke-equivariant.

1.4 Lattices and Duality

Let $\Gamma = \Gamma_1(N)$. Consider Diamond operators and Hecke operators:

$$\langle d\rangle := \left[\Gamma \begin{pmatrix} a & b \\ c & d\end{pmatrix} \Gamma\right]_k$$

for some $\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma_0(N)$, and

$$T_p := \left[\Gamma \begin{pmatrix} 1 & 0 \\ 0 & p\end{pmatrix} \Gamma\right]_k.$$

We denote by $H_k(N)$ and $h_k(N)$ $\mathbb{C}$-subalgebras of $\mathrm{End}_{\mathbb{C}}(M_k(N))$ and $\mathrm{End}_{\mathbb{C}}(S_k(N))$ generated by all Diamond operators and Hecke operators. There are both commutative $\mathbb{C}$-algebras.

For every Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ we define $M_k(N,\chi)$ and $S_k(N,\chi)$ as $M_k(N)[\chi]$ and $S_k(N)[\chi]$, respectively. That means, $\langle d\rangle(f) = \chi(d)f$ for all $(d,N)=1$. We take $\Gamma = \Gamma_0(N)$,

$$\Delta^\iota = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, N \mid c,\ (d,N)=1 \right\},$$

extend χ on Δ^ι by $\chi(g) = \chi(d)$, and define

$$T_p := \left[\Gamma \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma \right]_k .$$

We define $\mathcal{E}_k(N,\chi)$ as the orthogonal complement of $S_k(N,\chi)$ in $M_k(N,\chi)$ under the Petersson inner product. An explicit construction of a basis for $\mathcal{E}_k(N,\chi)$ when $k \geq 2$ is given as follows: Let ψ, φ be Dirichlet characters with conductor u, v, respectively and $(\psi\varphi)(-1) = (-1)^k$. Define

$$E_k^{\psi,\varphi}(q) := \delta(\psi)L(1-k,\varphi) + 2\sum_{n=1}^{\infty} \sigma_{k-1}^{\psi,\varphi}(n)q^n,$$

where

$$\sigma_{k-1}^{\psi,\varphi}(n) := \sum_{d\mid n} \psi(n/d)\varphi(d)d^{k-1}.$$

Define

$$E_k^{\psi,\varphi,t}(z) := \begin{cases} E_k^{\psi,\varphi}(tz), & (k,\psi,\varphi) \neq (2,1,1) \\ E_2^{1,1}(z) - tE_2^{1,1}(tz) & (k,\psi,\varphi) = (2,1,1) \end{cases}$$

Proposition 4. $\{E_k^{\psi,\varphi} : tuv \mid N,\ \psi\varphi = \chi\}$ is a basis for $\mathcal{E}_k(N,\chi)$.

Let R be a subring of $\mathbb{C}$ containing $\mathbb{Z}[\chi]$. We define $M_k(N,\chi;R)$, $S_k(N,\chi;R)$ as subspaces of $M_k(N,\chi)$, $S_k(N,\chi)$ consisting of forms whose q-expansions are in $R[[q]]$. We define $m_k(N,\chi;R)$ as the subspace of $M_k(N,\chi)$ of forms whose q-expansions are in $\mathrm{Frac}(R) + qR[[q]]$. Note that $M_k(N,\chi;R)$, $S_k(N,\chi;R)$, and $m_k(N,\chi;R)$ are all contained in finite free R-modules.

Lemma 6. Define $\mathcal{E}_k(N,\chi)$ has a basis with elements in $M_k(N,\chi;\mathbb{Q}(\chi))$.

Proof. Define $\mathcal{E}_k(N,\chi;R) := \mathcal{E}_k(N,\chi)\cap M_k(N,\chi;R)$. We should prove that

$$\mathcal{E}_k(N,\chi;\mathbb{Q}(\chi))\otimes_{\mathbb{Q}(\chi)}\mathbb{C} = \mathcal{E}_k(N,\chi).$$

Since all $E_k^{\psi,\varphi,t}$ are in $M_k(N,\chi;\mathbb{Q}(\zeta_N))$, we already have

$$\mathcal{E}_k(N,\chi;\mathbb{Q}(\zeta_N))\otimes_{\mathbb{Q}(\zeta_N)}\mathbb{C} = \mathcal{E}_k(N,\chi).$$

Let $G := \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\,\mathbb{Q}(\chi))$. Then $\mathcal{E}_k(N,\chi;\mathbb{Q}(\zeta_N))$ is a $\mathbb{Q}(\zeta_N)[G]$-module, where G acts on $\mathbb{Q}(\zeta_N)$ by its natural action. Therefore,

$$\mathcal{E}_k(N,\chi;\mathbb{Q}(\zeta_N)) = \mathcal{E}_k(N,\chi;\mathbb{Q}(\zeta_N))^G\otimes_{\mathbb{Q}(\chi)}\mathbb{Q}(\zeta_N).$$

Since $\mathcal{E}_k(N,\chi;\mathbb{Q}(\zeta_N))^G = \mathcal{E}_k(N,\chi;\mathbb{Q}(\zeta_N)^G) = \mathcal{E}_k(N,\chi;\mathbb{Q}(\chi))$, $\mathcal{E}_k(N,\chi;\mathbb{Q}(\chi))\otimes_{\mathbb{Q}(\chi)}\mathbb{C} = \mathcal{E}_k(N,\chi)$. $\square$

We also define $H_k(N,\chi)_R$, $h_k(N,\chi)_R$ as the R-subalgebra of $H_k(N,\chi)$, $h_k(N,\chi)$ generated by all Hecke operators. Then $H_k(N,\chi)_R$, acts on $m_k(N,\chi;R)$, $M_k(N,\chi;R)$, and $h_k(N,\chi)_R$ acts on $S_k(N,\chi;R)$. We define $H_k(N,\chi;R)$ and $h_k(N,\chi;R)$ as images of $H_k(N,\chi)_R$, $h_k(N,\chi)_R$ in $\mathrm{End}_R(m_k(N,\chi;R))$ and $\mathrm{End}_R(S_k(N,\chi;R))$, respectively. Note that if $h\in H_k(N,\chi;R)$ with $h(f)=0$ for all $f\in M_k(N,\chi;R)$, $h=0$. Therefore, $H_k(N,\chi;R)$ is also seen as the image of $H_k(N,\chi)_R$ in $\mathrm{End}_R(m_k(N,\chi;R))$.

Eichler-Shimura isomorphism gives the commutative diagram

$$\begin{array}{ccc} S_k(N,\chi)\oplus\overline{S_k(N,\overline{\chi})} & \longrightarrow & H_P^1(\Gamma_0(N),\mathrm{Sym}^{k-2}(\mathbb{C}^2)(\chi)) \\ \downarrow & & \downarrow \\ S_k(\Gamma_1(N))\oplus\overline{S_k(\Gamma_1(N))} & \longrightarrow & H_P^1(\Gamma_1(N),\mathrm{Sym}^{k-2}(\mathbb{C}^2)) \end{array},$$

which is Hecke-equivariant. Let h be in the Hecke algebra on $H_P^1(\Gamma_1(N),\mathrm{Sym}^{k-2}(\mathbb{C}^2))$. If $h=0$ on $S_k(\Gamma_1(N))$, $h=0$ on $S_k(\Gamma_1(N))$. Restricts this to the χ-isotypic part and we have that $h_k(N,\chi)$ acts on $H_P^1(\Gamma_0(N),\mathrm{Sym}^{k-2}(\mathbb{C}^2)(\chi))$. Define $L_P(k-2,\chi)$ as the image of

$$H_P^1(\Gamma_0,\mathrm{Sym}^{k-2}(\mathbb{Z}[\chi]^2)(\chi))\to H_P^1(\Gamma_0,\mathrm{Sym}^{k-2}(\mathbb{C}^2)(\chi)).$$

$L_P(k-2,\chi)$ is a Lattice of full-rank and equipped with $h_k(N,\chi)_R$-action. Similarly we get a lattice of full-rank $L(k-2,\chi) \subset H^1(\Gamma_0, \mathrm{Sym}^{k-2}(\mathbb{C}^2)(\chi))$ with $H_k(N,\chi)$-action.

Theorem 2. Suppose $k \geq 2$. For all $\mathbb{Z}[\chi] \subset R \subset \mathbb{C}$, there are natural isomorphisms

$$H_k(N,\chi)_R \cong H_k(N,\chi;R),\ \ h_k(N,\chi)_R \cong h_k(N,\chi;R),$$

$$H_k(N,\chi)_{\mathbb{Z}[\chi]} \otimes_{\mathbb{Z}[\chi]} R \cong H_k(N,\chi)_R,\ \ h_k(N,\chi)_{\mathbb{Z}[\chi]} \otimes_{\mathbb{Z}[\chi]} R \cong h_k(N,\chi)_R,$$

and

$$m_k(N,\chi;\mathbb{Z}[\chi]) \otimes_{\mathbb{Z}[\chi]} R = m_k(N,\chi;R),\ \ S_k(N,\chi;\mathbb{Z}[\chi]) \otimes_{\mathbb{Z}[\chi]} R = S_k(N,\chi;R).$$

Moreover, we have perfect pairings

$$H_k(N,\chi;R) \times m_k(N,\chi;R) \to R,\ \ h_k(N,\chi;R) \times S_k(N,\chi;R) \to R$$

given by $(h,f) \mapsto a_1(h(f))$.

Lemma 7. The duality is true if R is a field.

Proof. In this case, we are dealing with finite dimension R-vector spaces, so it suffices to prove the nondegeneracy of this R-bilinear pairing. If $(h,f) = 0$ for all h, $(T_n, f) = a_1(T_n(f)) = a_n(f) = 0$ for all $n \in \mathbb{N}$. Hence f is a constant. Since $k > 0$, $f = 0$. If $(h,f) = 0$ for all f, $(h, T_n(f)) = a_1(hT_n(f)) = a_1(T_n h(f)) = T_n(h(f)) = 0$. Hence $h(f) = 0$ for all f and we get that $h = 0$. $\square$

Lemma 8. The theorem is true for $R = \mathbb{C}$.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} 0 \longrightarrow & h_k(N,\chi)_{\mathbb{Z}[\chi]} \otimes_{\mathbb{Z}[\chi]} \mathbb{C} & \longrightarrow & \mathrm{End}_{\mathbb{Z}[\chi]}(L_P(k-2,\chi)) \otimes_{\mathbb{Z}[\chi]} \mathbb{C} \\ & \downarrow & & \downarrow \cong \\ 0 \longrightarrow & h_k(N,\chi) & \longrightarrow & \mathrm{End}_{\mathbb{C}}(H^1_P(\Gamma_0(N), \mathrm{Sym}^{k-2}(\mathbb{C}^2)(\chi))) \end{array}.$$

By definition, $h_k(N,\chi)_{\mathbb{Z}[\chi]} \otimes_{\mathbb{Z}[\chi]} \mathbb{C} \to h_k(N,\chi)$ is surjective. By diagram chasing, it is also injective, hence an isomorphism. Similarly we have that $H_k(N,\chi)_{\mathbb{Z}[\chi]} \otimes_{\mathbb{Z}[\chi]} \mathbb{C} \to H_k(N,\chi)$ is an isomorphism.

Consider the isomorphism

$$\mathrm{Hom}_{\mathbb{C}}(h_k(N,\chi),\mathbb{C}) \cong S_k(N,\chi),\ \phi \mapsto \sum_{n=1}^{\infty} \phi(T_n)q^n.$$

Since $h_k(N,\chi) = h_k(N,\chi)_{\mathbb{Z}[\chi]} \otimes_{\mathbb{Z}[\chi]} \mathbb{C}$, $\mathrm{Hom}_{\mathbb{C}}(h_k(N,\chi),\mathbb{C}) = \mathrm{Hom}_{\mathbb{Z}[\chi]}(h_k(N,\chi)_{\mathbb{Z}[\chi]},\mathbb{C})$. Since $h_k(N,\chi)_{\mathbb{Z}[\chi]}$ is finite projective, it is also $\mathrm{Hom}_{\mathbb{Z}[\chi]}(h_k(N,\chi)_{\mathbb{Z}[\chi]},\mathbb{Z}[\chi]) \otimes_{\mathbb{Z}[\chi]} \mathbb{C}$, and with the identification, $\mathrm{Hom}_{\mathbb{Z}[\chi]}(h_k(N,\chi)_{\mathbb{Z}[\chi]},\mathbb{Z}[\chi])$ is identified as the $\mathbb{Z}[\chi]$-submodule in $S_k(N,\chi)$ of elements f satisfying that $a_n(f) \in \mathbb{Z}[\chi]$ for all $n \in \mathbb{N}$. Hence $\mathrm{Hom}_{\mathbb{Z}[\chi]}(h_k(N,\chi)_{\mathbb{Z}[\chi]},\mathbb{Z}[\chi]) = S_k(N,\chi;\mathbb{Z}[\chi])$. Therefore,

$$S_k(N,\chi;\mathbb{Z}[\chi]) \otimes_{\mathbb{Z}[\chi]} \mathbb{C} = S_k(N,\chi),$$

$h_k(N,\chi)_{\mathbb{Z}[\chi]} \to \mathrm{End}_{\mathbb{Z}[\chi]}(S_k(N,\chi;\mathbb{Z}[\chi]))$ is isomorphic onto $h_k(N,\chi;\mathbb{Z}[\chi])$, and

$$h_k(N,\chi;\mathbb{Z}[\chi]) \otimes_{\mathbb{Z}[\chi]} \mathbb{C} = h_k(N,\chi).$$

For the duality part, we already have

$$\mathrm{Hom}_{\mathbb{Z}[\chi]}(h_k(N,\chi;\mathbb{Z}[\chi]),\mathbb{Z}[\chi]) = S_k(N,\chi;\mathbb{Z}[\chi]).$$

Apply $\mathrm{Hom}_{\mathbb{Z}[\chi]}(\cdot,\mathbb{Z}[\chi])$ and we have

$$\begin{aligned} &\mathrm{Hom}_{\mathbb{Z}[\chi]}(S_k(N,\chi;\mathbb{Z}[\chi]),\mathbb{Z}[\chi]) \\ =&\mathrm{Hom}_{\mathbb{Z}[\chi]}(\mathrm{Hom}_{\mathbb{Z}[\chi]}(h_k(N,\chi;\mathbb{Z}[\chi]),\mathbb{Z}[\chi]),\mathbb{Z}[\chi]) \\ =&h_k(N,\chi;\mathbb{Z}[\chi]). \end{aligned}$$

For modular forms, we should also prove that

Lemma 9.

$$m_k(N,\chi;R) = \{f \in M_k(N,\chi) \mid a_n(f) \in R \text{ for all } n > 0\}.$$

Proof. Since both $S_k(N,\chi)$ and $\mathcal{E}_k(N,\chi)$ have base with Fourier coefficients in $\mathbb{Q}(\chi)$, $M_k(N,\chi)$ has a basis with Fourier coefficients in $\mathbb{Q}(\chi)$. Therefore, if $f \in M_k(N,\chi)$ and $a_n(f) \in \mathrm{Frac}(R)$ for all $n \geq 0$, $a_0(f) \in \mathrm{Frac}(R)$. □

Now we similarly have

$$m_k(N,\chi;\mathbb{Z}[\chi]) = \mathrm{Hom}_{\mathbb{Z}[\chi]}(H_k(N,\chi)_{\mathbb{Z}[\chi]},\mathbb{Z}[\chi]),$$

$$M_k(N,\chi) = \mathrm{Hom}_{\mathbb{Z}[\chi]}(H_k(N,\chi)_{\mathbb{Z}[\chi]}, \mathbb{Z}[\chi]) \otimes_{\mathbb{Z}[\chi]} \mathbb{C} = m_k(N,\chi;\mathbb{Z}[\chi]) \otimes_{\mathbb{Z}[\chi]} \mathbb{C},$$

and

$$H_k(N,\chi) = \mathrm{Hom}_{\mathbb{Z}[\chi]}(m_k(N,\chi;\mathbb{Z}[\chi]), \mathbb{Z}[\chi]).$$

□

Now we prove the general case. Consider the natural map

$$h_k(N,\chi)_{\mathbb{Z}[\chi]} \otimes_{\mathbb{Z}[\chi]} R \to h_k(N,\chi)_R \to h_k(N,\chi;R).$$

By definition, this is surjective. If h is in the kernel, $h = 0$ on $S_k(N,\chi;\mathbb{Z}[\chi])$. Since $S_k(N,\chi;\mathbb{Z}[\chi]) \otimes_{\mathbb{Z}[\chi]} \mathbb{C} = S_k(N,\chi)$ and $h_k(N,\chi)_{\mathbb{Z}[\chi]} \otimes_{\mathbb{Z}[\chi]} R \to h_k(N,\chi)_{\mathbb{Z}[\chi]} \otimes_{\mathbb{Z}[\chi]} R \otimes_R \mathbb{C} = h_k(N,\chi)$ is injective for that $\mathbb{C}$ is R-flat, $h = 0$. Hence $h_k(N,\chi)_{\mathbb{Z}[\chi]} \otimes_{\mathbb{Z}[\chi]} R \to h_k(N,\chi)_R$ is injective. By definition, this is also surjective. We obtain that

$$h_k(N,\chi)_{\mathbb{Z}[\chi]} \otimes_{\mathbb{Z}[\chi]} R = h_k(N,\chi)_R \cong h_k(N,\chi;R).$$

The same argument for H_k and m_k gives

$$H_k(N,\chi)_{\mathbb{Z}[\chi]} \otimes_{\mathbb{Z}[\chi]} R = H_k(N,\chi)_R \cong H_k(N,\chi;R).$$

Since

$$h_k(N,\chi) = h_k(N,\chi)_R \otimes_R \mathbb{C},$$

$S_k(N,\chi) = \mathrm{Hom}_R(h_k(N,\chi)_R, \mathbb{C})$, and $\mathrm{Hom}_R(h_k(N,\chi)_R, R)$ is identified as $S_k(N,\chi;R)$. On the other hand, the isomorphism $h_k(N,\chi)_{\mathbb{Z}[\chi]} \otimes_{\mathbb{Z}[\chi]} R = h_k(N,\chi)_R$ gives that

$$\begin{aligned} S_k(N,\chi;R) &= \mathrm{Hom}_{\mathbb{Z}[\chi]}(h_k(N,\chi;\mathbb{Z}[\chi]), R) \\ &= \mathrm{Hom}_{\mathbb{Z}[\chi]}(h_k(N,\chi;\mathbb{Z}[\chi]), \mathbb{Z}[\chi]) \otimes_{\mathbb{Z}[\chi]} R \\ &= S_k(N,\chi;\mathbb{Z}[\chi]) \otimes_{\mathbb{Z}[\chi]} R. \end{aligned}$$

Similarly, $M_k(N,\chi) = \mathrm{Hom}_R(H_k(N,\chi)_R, \mathbb{C})$, $\mathrm{Hom}_R(H_k(N,\chi)_R, R)$ is identified as

$$\{f \in M_k(N,\chi) \mid a_n(f) \in R \text{ for all } n \in \mathbb{N}\} = m_k(N,\chi;R),$$

and

$$\mathrm{Hom}_R(H_k(N,\chi)_R, R) = \mathrm{Hom}_R(H_k(N,\chi)_{\mathbb{Z}[\chi]}, \mathbb{Z}[\chi]) \otimes_{\mathbb{Z}[\chi]} R = m_k(N,\chi;\mathbb{Z}[\chi]) \otimes_{\mathbb{Z}[\chi]} R.$$

Corollary 2. $M_k(N,\chi;\mathbb{Z}[\chi])\otimes_{\mathbb{Z}[\chi]} R = M_k(N,\chi;R)$.

Proof. Define $C(R) := m_k(N,\chi;R)/M_k(N,\chi;R)$, which is identified as a submodule of $\mathrm{Frac}(R)/R$ via a_0. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_k(N,\chi;\mathbb{Z}[\chi])\otimes_{\mathbb{Z}[\chi]} R & \longrightarrow & m_k(N,\chi;\mathbb{Z}[\chi])\otimes_{\mathbb{Z}[\chi]} R & \longrightarrow & C(\mathbb{Z}[\chi])\otimes_{\mathbb{Z}[\chi]} R & \longrightarrow & 0 \\ & & \downarrow & & \downarrow\cong & & \downarrow & & \\ 0 & \longrightarrow & M_k(N,\chi;R) & \longrightarrow & m_k(N,\chi;R) & \longrightarrow & C(R) & \longrightarrow & 0 \end{array}.$$

By snake lemma it suffices to show that $C(\mathbb{Z}[\chi])\otimes_{\mathbb{Z}[\chi]} R \to C(R)$ is injective. The map

$$C(\mathbb{Z}[\chi])\otimes_{\mathbb{Z}[\chi]} R \to C(R) \to \mathrm{Frac}(R)/R$$

is the same as the map

$$C(\mathbb{Z}[\chi])\otimes_{\mathbb{Z}[\chi]} R \to \mathbb{Q}(\chi)/\,\mathbb{Z}[\chi]\otimes_{\mathbb{Z}[\chi]} R \to \mathrm{Frac}(R)/R,$$

which is injective. □

The same method for $\Gamma_1(N)$ yields that for every subring $R\subset\mathbb{C}$ and $k\geq 2$ there are isomorphisms

$$H_k(N)_R \cong H_k(N;R),\ \ h_k(N)_R \cong h_k(N;R),$$

$$H_k(N)_{\mathbb{Z}}\otimes_{\mathbb{Z}} R \cong H_k(N,\chi)_R,\ \ h_k(N)_{\mathbb{Z}}\otimes_{\mathbb{Z}} R \cong h_k(N)_R,$$

$$m_k(N;\mathbb{Z})\otimes_{\mathbb{Z}} R = m_k(N;R),\ \ S_k(N;\mathbb{Z})\otimes_{\mathbb{Z}} R = S_k(N;R),$$

and perfect pairings

$$H_k(N;R)\times m_k(N;R)\to R,\ \ h_k(N;R)\times S_k(N;R)\to R$$

given by $(h,f)\mapsto a_1(h(f))$. In particular, $M_k(N;\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C} = M_k(N)$. Let $\mathrm{Aut}(\mathbb{C}\,/\,\mathbb{Q})$ acts on $M_k(N)$ by acting on coefficients of q-expansions. Then $\mathrm{Aut}(\mathbb{C}\,/\,\mathbb{Q})$ commute with $H_k(N;\mathbb{Z})$. Therefore, for all $\sigma\in\mathrm{Aut}(\mathbb{C}\,/\,\mathbb{Q})$ and $f\in M_k(N)$, $f^\sigma\in M_k(N)$, and if $f\in M_k(N;\chi)$, $f^\sigma\in M_k(N;\chi^\sigma)$. The same result holds for $S_k(N)$.

1.5 Dimension Computation

Let R be a field, and M be a finite dimensional R-vector space. If E is invertible in R, we can compute the parabolic cohomology in terms of simplicial cohomology.

Proposition 5. $H^0_Q(K,M) = M^G$. This is easily seen by $H^0_Q(K,M) = H^0(K,M)$ and $\mathcal{H}_0$ is connected.

Proposition 6. $H^2_Q(K,M) = M/\sum_{g\in\Gamma}(g-1)M = H_0(\Gamma, M)$.

Now we compute $H^1_Q(K,M)$ via Euler characteristic. We have that

$$\chi_Q(K,M) = \dim_R(C^0(K,M)) - \dim_R(C^1_Q(K,M)) + \dim_2(C^1(K,M)).$$

Let N_i be the number of Γ-orbits of i-simplices in K. We have that

$$\dim_R(C^0(K,M)) = N_0 \dim_R(M) - \sum_{j=1}^{r}(\dim_R(M) - \dim_R(M^{\Gamma_{p_j}})),$$

$$\dim_R(C^1(K,M)) = N_1 \dim_R(M) - \sum_{i=1}^{m}(\dim_R(M) - \dim_R((\pi_i - 1)M)),$$

and $\dim_R(C^2(K,M)) = N_2 \dim_R(M)$. Let g be the genus of $X(\Gamma)$. We have

$$N_0 - N_1 + N_2 + m = 2 - 2g.$$

Let $\epsilon_0 := \dim_R(M^G)$, $\epsilon_2 := \dim_R(M/\sum_{g\in\Gamma}(g-1)M)$. We have that

$$\begin{aligned}\dim_R(H^1_Q(K,M)) =& \epsilon_0 + \epsilon_2 - \chi_Q(K,M) = (2g-2)\dim_R(M) + \epsilon_0 + \epsilon_2 \\ &+ \sum_{i=1}^{m}(\dim_R((\pi_i - 1)M)) + \sum_{j=1}^{r}(\dim_R(M) - \dim_R(M^{\Gamma_{p_j}})).\end{aligned}$$

For modular forms, we should also compute $\dim_R(H^1(K,M))$. If Γ has cusps, $H^2(K,M) = 0$. Hence

$$\dim_R(H^1(K,M)) = (2g-2+m)\dim_R(M) + \epsilon_0 + \sum_{j=1}^{r}(\dim_R(M) - \dim_R(M^{\Gamma_{p_j}})).$$

Let $\overline{\Gamma}$ be the image of Γ in $\mathrm{PSL}_2(\mathbb{R})$. Let M be a Γ-module. If $-I_2 \in \Gamma$, we use the Hochschild–Serre spectral sequence

$$H^p(\overline{\Gamma}, H^q(\{\pm I_2\}, M)) \Rightarrow H^{p+q}(\Gamma, M).$$

Suppose 2 is invertible in $\mathbb{R}$. $H^q(\{\pm I_2\}, M) = 0$ for $q \geq 1$, hence the isomorphism

$$H^*(\overline{\Gamma}, M^H) \cong H^*(\Gamma, M).$$

Now $M = \mathrm{Sym}^{k-2}(\mathbb{C}^2)$. If k is odd, $-I_2 \notin \Gamma$. Hence we always have $H^*(\overline{\Gamma}, \mathrm{Sym}^{k-2}(\mathbb{C}^2)) = H^*(\Gamma, \mathrm{Sym}^{k-2}(\mathbb{C}^2))$. We have to show that

$$\begin{aligned} &2\dim_{\mathbb{C}}(S_k(\Gamma)) = \dim_{\mathbb{C}}(H^1_P(\Gamma, \mathrm{Sym}^{k-2}(\mathbb{C}^2))) \\ =&(2g-2)(k-1) + \dim_{\mathbb{C}}(\mathrm{Sym}^{k-2}(\mathbb{C}^2)^{\Gamma}) + \dim_{\mathbb{C}}(\mathrm{Sym}^{k-2}(\mathbb{C}^2)/\sum_{g\in\Gamma}(g-1)\,\mathrm{Sym}^{k-2}(\mathbb{C}^2)) \\ &+\sum_{i=1}^{m}((\pi_i - 1)\,\mathrm{Sym}^{k-2}(\mathbb{C}^2)) + \sum_{j=1}^{r}\dim_{\mathbb{C}}(\mathrm{Sym}^{k-2}(\mathbb{C}^2)/\,\mathrm{Sym}^{k-2}(\mathbb{C}^2)^{\Gamma_{p_j}}). \end{aligned}$$

and that

$$\begin{aligned} &\dim_{\mathbb{C}}(M_k(\Gamma)) - \dim_{\mathbb{C}}(S_k(\Gamma)) \\ =&\sum_{i=1}^{m}(\mathrm{Sym}^{k-2}(\mathbb{C}^2)^{\pi_i}) - \dim_{\mathbb{C}}(\mathrm{Sym}^{k-2}(\mathbb{C}^2)/\sum_{g\in\Gamma}(g-1)\,\mathrm{Sym}^{k-2}(\mathbb{C}^2)). \end{aligned}$$

Theorem 3. If $k = 2$, $\dim_{\mathbb{C}}(S_2(\Gamma)) = g$. If $k > 2$,

$$\dim_{\mathbb{C}}(S_k(\Gamma)) = \begin{cases} (k-1)(g-1) + \dfrac{k-2}{2}m + \sum_{j=1}^{r}\left\lfloor \dfrac{k(e_j-1)}{2e_j} \right\rfloor, & k \text{ is even} \\ (k-1)(g-1) + \dfrac{k-2}{2}m_1 + \dfrac{k-1}{2}m_2 + \sum_{j=1}^{r}\left\lfloor \dfrac{k(e_j-1)}{2e_j} \right\rfloor, & k \text{ is odd} \end{cases},$$

here m_1, m_2 are numbers of regular and irregular cusps, respectively. Moreover,

$$\dim_{\mathbb{C}}(M_k(\Gamma)) - \dim_{\mathbb{C}}(S_k(\Gamma)) = \begin{cases} m-1, & k=2 \\ m, & k \geq 4,\ k \text{ is even} \\ m_1, & k \text{ is odd} \end{cases}.$$

We first consider cusp forms.

1. $k = 2$: $\mathbb{C}^{\Gamma} = \mathbb{C}$, $\mathbb{C}\,/\sum_{g\in\Gamma}(g-1)\,\mathbb{C} = \mathbb{C}$, $(\pi_k - 1)\,\mathbb{C} = 0$, $\mathbb{C}^{\Gamma_{p_j}} = \mathbb{C}$. We get $\dim_{\mathbb{C}}(H^1_P(\Gamma, \mathbb{C})) = 2g$.

2. $k > 2$: $(\pi_k - 1)\,\mathrm{Sym}^{k-2}(\mathbb{C}^2)$ has dimension $k-1$ if s_k is an irregular cusp, otherwise it has dimension $k-2$. Let σ_j be a generator of Γ_{p_j}. Let e'_j be the order of σ_j. Then σ_j has tow eigenvalues ω_j, ω_j^{-1} where ω_j is a primitive e'_jth root

of unity. σ_j acts on $\operatorname{Sym}^{k-2}(\mathbb{C}^2)$ with eigenvalues $\omega_j^{k-2}, \omega_j^{k-4}, \cdots, \omega_j^{4-k}, \omega_j^{2-k}$. Therefore, $\dim_{\mathbb{C}}(\operatorname{Sym}^{k-2}(\mathbb{C}^2)/\operatorname{Sym}^{k-2}(\mathbb{C}^2)^{\Gamma_{p_j}})$ is twice the numbers of positive integers $a \in \{1, \cdots, k-2\}$ such that $a \equiv k \pmod 2$ and $e'_j \nmid a$. We should show that the number of such a is $\left\lfloor \frac{k(e_j - 1)}{2e_j} \right\rfloor$.

(a) If e'_j is even, $-I_2 \in \Gamma_{p_j}$. We have that k is even and $e'_j = 2e_j$. Let $\ell := \frac{k}{2}$. We have to verify the identity

$$\ell - 1 - \left\lfloor \frac{\ell - 1}{e_j} \right\rfloor = \left\lfloor \frac{\ell(e_j - 1)}{e_j} \right\rfloor,$$

or equivalently,

$$\ell - 1 = \left\lfloor \frac{\ell - 1}{e_j} \right\rfloor + \left\lfloor \frac{\ell(e_j - 1)}{e_j} \right\rfloor.$$

This is true for that $\ell - 1 + \ell(e_j - 1) = \ell e_j - 1$.

(b) If e'_j is odd, $e_j = e'_j$. If k is even write $\ell = \frac{k}{2}$. Then $e_j \mid k - 2i$ if and only if $e_j \mid \ell - i$. Hence we reduce the case to the previous one. Suppose k is odd, say $k = 2\ell + 1$. We have to verify that

$$2\ell - 1 - \left\lfloor \frac{2\ell - 1}{e_j} \right\rfloor - \left(\ell - 1 - \left\lfloor \frac{\ell - 1}{e_j} \right\rfloor\right) = \left\lfloor \frac{(2\ell + 1)(e_j - 1)}{2e_j} \right\rfloor.$$

Since $\left\lfloor \frac{(2\ell+1)(e_j-1)}{2e_j} \right\rfloor = \ell - \left\lfloor \frac{2\ell + e_j}{2e_j} \right\rfloor = \ell - \left\lfloor \frac{\ell}{e_j} + \frac{1}{2} \right\rfloor$, we have to show that

$$\left\lfloor \frac{\ell}{e_j} + \frac{1}{2} \right\rfloor + \left\lfloor \frac{\ell - 1}{e_j} \right\rfloor = \left\lfloor \frac{2\ell - 1}{e_j} \right\rfloor.$$

Since ℓ is a period of both $\left\lfloor \frac{\ell}{e_j} + \frac{1}{2} \right\rfloor + \left\lfloor \frac{\ell - 1}{e_j} \right\rfloor - \frac{2\ell}{e_j}$ and $\left\lfloor \frac{2\ell - 1}{e_j} \right\rfloor - \frac{2\ell}{e_j}$, and e_j is odd, it suffices to show the equation for $1 \leq \ell < e_j/2$ and $e_j/2 < \ell \leq e_j$, both of which are clear.

Suppose $x \in \operatorname{Sym}^{k-2}(\mathbb{C}^2)$. Define

$$p(z) := \Theta_{k-2}(x, (ze_1 + e_2)^{\otimes k-2}).$$

Then $p(z)$ is a polynomial in z of degree at most $k - 2$. For every $\alpha \in \Gamma$,

$$p(\alpha z) j(\alpha, z)^{k-2} = \Theta_{k-2}(x, \alpha(ze_1 + e_2)^{\otimes k-2}) = p(z).$$

For every cusp s_k, let $g_k \in \mathrm{SL}_2(\mathbb{R})$ such that $g_k \pi_k g_k^{-1} = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$. Then

$$p|[g_k^{-1}]_{2-k}$$

is a polynomial in z and $p|[g_k^{-1}]_{2-k}(z) = p|[g_k^{-1}]_{2-k}(z+2h)$. This gives that $p|[g_k^{-1}]_{2-k}$ is a constant. Hence $p(z) \in M_{2-k}(\Gamma)$. Since $2-k<0$, $p = 0$. This gives that $x = 0$. We use the duality between H^0 and H_0 to get $\mathrm{Sym}^{k-2}(\mathbb{C}^2)/\sum_{g\in\Gamma}(g-1)\,\mathrm{Sym}^{k-2}(\mathbb{C}^2) = 0$.

Since $\dim_{\mathbb{C}}(\mathrm{Sym}^{k-2}(\mathbb{C}^2)/\sum_{g\in\Gamma}(g-1)\,\mathrm{Sym}^{k-2}(\mathbb{C}^2))$ is 0 if $k>2$, is 1 if $k=2$, and $\sum_{i=1}^{m}(\mathrm{Sym}^{k-2}(\mathbb{C}^2)^{\pi_i})$ is the number of regular primes, the case for modular forms follows.

2 $\mathcal{O}$-adic Modular Forms

2.1 Basic Definitions

Let $K/\mathbb{Q}_p$ be a finite extension, $\mathcal{O} \subset K$ be the ring of integers, and $\varpi \in \mathcal{O}$ be a uniformizer. Let $q = p$ if $p > 2$ and $q = 4$ if $p = 2$. Let $\omega : (\mathbb{Z}/q\mathbb{Z})^\times \to \mathcal{O}^\times$ be the Teichmüller character and $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathcal{O}^\times$ be a Dirichlet character. We define

$$M_k(N,\chi;\mathcal{O}) := M_k(N,\chi;\mathbb{Z}[\chi]) \otimes_{\mathbb{Z}[\chi]} \mathcal{O},\ S_k(N,\chi;\mathcal{O}) := S_k(N,\chi;\mathbb{Z}[\chi]) \otimes_{\mathbb{Z}[\chi]} \mathcal{O}.$$

We similarly have $H_k(H,\chi;\mathcal{O})$, $h_k(N,\chi;\mathcal{O})$, and we endow all spaces with p-adic topology.

2.2 Ordinary Forms

Lemma 10. Let A be $\mathcal{O}$-algebra which is finite as a $\mathcal{O}$-module. Then for every $x \in A$, the limit $\lim_{n\to\infty} x^{n!}$ exists under p-adic topology and is an idempotent.

Proof. Since A is finite over $\mathcal{O}$, A is p-adically complete, and for every $m \in \mathbb{N}$, A/p^mA is finite. There are $a(m), b(m) \in \mathbb{N}$ such that

$$x^{a(m)} \equiv x^{a(m)+b(m)} \pmod{p^mA}.$$

Hence for every $n \geq a(m)$, $x^n \equiv x^{n+b(m)} \pmod{p^m A}$. Take $n(m) := \max\{a(m), b(m)\}$ and we have that for every $n \geq n(m)$,

$$x^{(n+1)!} \equiv x^{n!} \equiv x^{2(n!)} \pmod{p^m A}.$$

Hence $\lim_{n\to\infty} x^{n!}$ exists in $A/p^m A$, which is $x^{n(m)!} \pmod{p^m A}$, which is an idempotent. Let $e_m := x^{n(m)!} \pmod{p^m A}$. Then $(e_m)_{m\in\mathbb{N}}$ defines an element in $\varprojlim_{m\in\mathbb{N}} A/p^m A = A$, which is an idempotent. □

Definition 2. The ordinary projector e is defined as $\lim_{n\to\infty} T_p^{n!} \in H_k(N, \chi; \mathcal{O})$. By definition, $e(f) = \lim_{n\to\infty} T_p^{n!}(f)$ under p-adic topology. $f \in M_k(N, \chi; \mathbb{C}_p)$ is called ordinary if $e(f) = f$. Equivalently, $f \in eM_k(N, \chi; \mathbb{C}_p)$.

Example 1. Assume $p \mid N$, $k \geq 2$. In this case, $a_n(T_p(f)) = a_{pn}(f)$. For every $(t, p) = 1$ we define $V_{\psi,\varphi,t}$ as the subspace generated by

$$E_k^{\psi,\varphi,t},\ E_k^{\psi,\varphi,pt}, \cdots$$

in $\mathcal{E}_k(N, \psi, \varphi)$. Assume that $V_{\psi,\varphi,t} \neq 0$ and we compute $eV_{\psi,\varphi,t}$. Note that if $E_k^{\psi,\varphi}(p^{\alpha+1}tz) \in \mathcal{E}_k(N, \psi, \varphi)$, $T_p E_k^{\psi,\varphi}(p^{\alpha+1}tz) = E_k^{\psi,\varphi}(p^\alpha tz)$.

1. $\psi(p) = 0$: $T_p E_k^{\psi,\varphi,t} = \varphi(p)p^{k-1}E_k^{\psi,\varphi,t}$. In this case, $eV_{\psi,\varphi,t} = 0$.

2. $\psi(p) \neq 0$ but $\varphi(p) = 0$: $T_p E_k^{\psi,\varphi,t} = \psi(p)E_k^{\psi,\varphi,t}$. In this case, $eV_{\psi,\varphi,t} = \mathbb{C}\, E_k^{\psi,\varphi,t}$.

3. $\psi(p)\varphi(p) \neq 0$: Let $\alpha := v_p(N) > 0$. Suppose $(k, \psi, \varphi) \neq (2, 1, 1)$. We consider another basis

   $$\{E_k^{\psi,\varphi,t} - \varphi(p)p^{k-1}E_k^{\psi,\varphi,pt}, \cdots, E_k^{\psi,\varphi,p^{\alpha-1}t} - \varphi(p)p^{k-1}E_k^{\psi,\varphi,p^\alpha t},\ E_k^{\psi,\varphi,t} - \psi(p)E_k^{\psi,\varphi,pt}\}.$$

   $E_k^{\psi,\varphi,t} - \varphi(p)p^{k-1}E_k^{\psi,\varphi,pt}$, $E_k^{\psi,\varphi,t} - \psi(p)E_k^{\psi,\varphi,pt}$ are T_p-eigenvectors of eigenvalues $\psi(p)$, $\varphi(p)p^{k-1}$, respectively. Therefore, $eV_{\psi,\varphi,t} = \mathbb{C}(E_k^{\psi,\varphi,t} - \varphi(p)p^{k-1}E_k^{\psi,\varphi,pt})$. If $(k, \psi, \varphi) = (2, 1, 1)$, we similarly have $eV_{\psi,\varphi,t} = \mathbb{C}(E_k^{\psi,\varphi}(tz) - 2E_k^{\psi,\varphi}(2tz))$.

e preserves $S_k(N, \chi; \mathcal{O})$ as $S_k(N, \chi; \mathcal{O})$ is a complete subspace of $M_k(N, \chi; \mathcal{O})$. We define

$$H_k^{\text{ord}}(N, \chi; \mathcal{O}) := eH_k(N, \chi; \mathcal{O}),\ h_k^{\text{ord}}(N, \chi; \mathcal{O}) := eh_k(N, \chi; \mathcal{O}),$$

$$M_k^{\mathrm{ord}}(N,\chi;\mathcal{O}) := eM_k(N,\chi;\mathcal{O}),\ S_k^{\mathrm{ord}}(N,\chi;\mathcal{O}) := eS_k(N,\chi;\mathcal{O}),$$

and

$$m_k^{\mathrm{ord}}(N,\chi;\mathcal{O}) := em_k(N,\chi;\mathcal{O}).$$

We still have the duality

$$\mathrm{Hom}_{\mathcal{O}}(H_k^{\mathrm{ord}}(N,\chi;\mathcal{O}),\mathcal{O}) \cong m_k^{\mathrm{ord}}(N,\chi;\mathcal{O}),$$

$$\mathrm{Hom}_{\mathcal{O}}(h_k^{\mathrm{ord}}(N,\chi;\mathcal{O}),\mathcal{O}) \cong S_k^{\mathrm{ord}}(N,\chi;\mathcal{O}).$$

Note that $e(f)$ may be a cusp form even if f is not a cusp form. The example on Eisenstein series shows that

$$\dim_{\mathbb{C}_p}(M_k^{\mathrm{ord}}(N,\chi\omega^a)) - \dim_{\mathbb{C}_p}(S_k^{\mathrm{ord}}(N,\chi\omega^a))$$

is independent of a and $k \geq 2$.

Lemma 11. Suppose $(p,N)=1$, $\alpha > 0$, and χ is a Dirichlet character modulo Np^α. Then T_p sends $M_k(Np^{\alpha+1},\chi)$ to $M_k(Np^\alpha,\chi)$.

Proof. It suffices to show that if $f \in M_k(Np^{\alpha+1},\chi)$, $T_p(f)$ is $\Gamma_1(Np^\alpha)$-invariant. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(Np^\alpha)$.

$$\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}^{-1} = \begin{pmatrix} a+cj & B \\ pc & d-cj \end{pmatrix}$$

where $B \in \dfrac{(d-a)j - cj^2 + b}{p}$. Hence if $p \mid b$, $T_p(f)|[g]_k = T_p(f)$. Since $T_p(f)$ is $\begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix}$-invariant, $T_p(f)$ has level Np^α. □

Let p^α be the p-part of the conductor of χ. We see that if $\alpha > 0$ and f is ordinary of Nebentypus χ and tame level N, then f has level Np^α.

2.3 Constant Rank

Suppose $(N,p)=1$.

Theorem 4. Let $\chi : (\mathbb{Z}/Np^\alpha\mathbb{Z})^\times \to \mathcal{O}^\times$ be a Dirichlet character for some $\alpha > 0$. Let $\epsilon : (\mathbb{Z}/Np^\alpha\mathbb{Z})^\times \to \mu_{p^\infty}(\mathcal{O}^\times)$ be a finite order character. Then

$$\dim(M_k^{\mathrm{ord}}(Np^\alpha, \epsilon\chi\omega^{-k})) = \dim(M_2^{\mathrm{ord}}(Np^\alpha, \chi\omega^{-2})),$$

$$\dim(S_k^{\mathrm{ord}}(Np^\alpha, \epsilon\chi\omega^{-k})) = \dim(S_2^{\mathrm{ord}}(Np^\alpha, \chi\omega^{-2})).$$

For $\Gamma_1(Np^\alpha)$ we have

$$\dim(M_k^{\mathrm{ord}}(\Gamma_1(Np^\alpha))) = \dim(M_2^{\mathrm{ord}}(\Gamma_1(Np^\alpha))),$$

$$\dim(S_k^{\mathrm{ord}}(\Gamma_1(Np^\alpha))) = \dim(S_2^{\mathrm{ord}}(\Gamma_1(Np^\alpha))).$$

Proof. Let $\Gamma := \Gamma_0(Np^\alpha)$ or $\Gamma_1(Np^\alpha)$ and define $L(k-2, R) := \mathrm{Sym}^{k-2}(R^2)$. For our purpose we may assume that $\alpha >> 0$. Suppose that Γ has a subgroup $H \subset \Gamma$ of finite index with $p \nmid [\Gamma : H]$ and H has no torsion elements other than $\{\pm I_2\}$. For example, if $\Gamma = \Gamma_1(Np^\alpha)$, for our purpose we may assume $Np^\alpha > 3$ and hence Γ has no torsion element. For Γ_0, if $p > 3$, we may take $H = \Gamma_1(p) \cap \Gamma$, and if $p = 2, 3$, we may assume $\alpha \geq 2$ as $\Gamma_0(4)$ and $\Gamma_0(9)$ have no torsion points other than $\{\pm I_2\}$ and take $H = \Gamma$. Now we have $H^*(H, M) = H^*(\Gamma, M)$ for all $\mathcal{O}[\Gamma]$-module M. In particular, $H^2(\overline{H}, M) = 0$. This gives $H^2(\Gamma, M) = 0$ for all $\mathcal{O}[\Gamma]$-module M except the case $\Gamma = \Gamma_0(N2^\alpha)$.

Let $\mathbb{F}$ be the residue field of $\mathcal{O}$. Consider the short exact sequence

$$0 \to L(k-2, \mathcal{O})(\epsilon\chi\omega^{-k}) \xrightarrow{\varpi\cdot} L(k-2, \mathcal{O})(\epsilon\chi\omega^{-k}) \to L(k-2, \mathbb{F})(\epsilon\chi\omega^{-k}) \to 0$$

and the corresponding long exact sequence. Since

$$T_p^2|_{L(k-2,\mathbb{F})(\epsilon\chi\omega^{-k})} = 0,$$

$eH^0(\Gamma, L(k-2, \mathbb{F})(\epsilon\chi\omega^{-k})) = 0$. Since the image of $H^0(\Gamma, L(k-2, \mathbb{F})(\epsilon\chi\omega^{-k}))$ in $H^1(\Gamma, L(k-2, \mathcal{O})(\epsilon\chi\omega^{-k}))$ is $H^1(\Gamma, L(k-2, \mathcal{O})(\epsilon\chi\omega^{-k}))[\varpi]$, $eH^1(\Gamma, L(k-2, \mathcal{O})(\epsilon\chi\omega^{-k}))[\varpi] = 0$. Therefore,

$$eH^1(\Gamma, L(k-2, \mathcal{O})(\epsilon\chi\omega^{-k}))$$

is finite free, and

$$\begin{aligned}&\dim_K(eH^1(\Gamma, L(k-2,K)(\epsilon\chi\omega^{-k})))\\=&\dim_{\mathbb{F}}(eH^1(\Gamma, L(k-2,\mathcal{O})(\epsilon\chi\omega^{-k}))\otimes_{\mathcal{O}}\mathbb{F})\\=&\dim_{\mathbb{F}}(eH^1(\Gamma, L(k-2,\mathbb{F})(\chi\omega^{-k}))).\end{aligned}$$

when we are not in the case $\Gamma = \Gamma_0(N2^\alpha)$. If $\Gamma = \Gamma_0(N2^\alpha)$, we show that $eH^2(\Gamma, L(k-2,\mathcal{O})(\epsilon\chi\omega^{-k})) = 0$ and we also have the formula above. Consider the spectral sequence

$$E_2^{p,q} = H^p(\overline{\Gamma}, H^q(\{\pm I_2\}, M)) \Rightarrow H^{p+q}(\Gamma, M).$$

where M is a $\mathcal{O}[\Gamma]$-module, which is finite free over $\mathcal{O}$ with trivial $\{\pm I_2\}$-action. Then $H^1(\{\pm I_2\}, M) = M[2] = 0$. Since $H^p(\overline{\Gamma}, \cdot)$ vanishes for $p \geq 2$, the spectral sequence gives the isomorphism $H^2(\Gamma, M) \cong H^2(\{\pm I_2\}, M)^{\overline{\Gamma}}$.

Now we compute $H^2(\{\pm I_2\}, M)$. Let φ_2 be a inhomogeneous 2-cochain. The condition that it is a cocycle is that

$$\varphi_2(I_2, I_2) = \varphi_2(I_2, -I_2) = \varphi_2(-I_2, I_2).$$

Let φ_1 be a 1-cocycle. Say $\varphi(I_2) = a$ and $\varphi(-I_2) = b$. Then

$$(\delta\varphi_1)(I_2, I_2) = (\delta\varphi_1)(-I_2, I_2) = (\delta\varphi_1)(I_2, -I_2) = a, (\delta\varphi_1)(-I_2, -I_2) = 2b - a.$$

Therefore,

$$[\varphi_2] \mapsto \overline{\varphi_2(-I_2, -I_2) - \varphi_2(I_2, I_2)}$$

induces an $\overline{\Gamma}$-equivariant isomorphism from $H^2(\{\pm I_2\}, M)$ to $M/2M$. Hence we obtain an isomorphism $H^2(\Gamma, M) \to H^0(\Gamma, M/2M)$. Take $M = L(k-2,\mathcal{O})(\epsilon\chi\omega^{-k})$. The isomorphism is compatible with Hecke operators for that a class $[u]$ in $H^2(\Gamma, M)$ is uniquely determined by values of u on $\{\pm I_2\}^2$ and $\{\pm I_2\}$ is in the center of Γ. Since $eH^0(\Gamma, M/2M) = 0$, $eH^2(\Gamma, M) = 0$.

Consider the Γ-equivariant map

$$\iota : L(k-2,\mathbb{F})(\chi\omega^{-k}) \to \mathbb{F}(\chi\omega^{-2}),\ e_1 \mapsto 0,\ e_2 \mapsto 1.$$

This gives the long exact sequence

$$eH^*(\Gamma, \ker(\iota)) \to eH^*(\Gamma, L(k-2, \mathbb{F})(\chi\omega^{-k})) \to eH^*(\Gamma, \mathbb{F}(\chi\omega^{-2})) \xrightarrow{+1} .$$

Since $\begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix}$ vanishes on $\ker(\iota)$, $eH^*(\Gamma, \ker(\iota)) = 0$ and we get

$$eH^1(\Gamma, L(0, \mathbb{F})(\chi\omega^{-2})) = eH^1(\Gamma, \mathbb{F}(\chi\omega^{-2})) \cong eH^1(\Gamma, L(k-2, \mathbb{F})(\chi\omega^{-k})).$$

□

3 Hida Family

3.1 Λ-adic Modular Forms

Let $\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty / \mathbb{Q}) = 1 + q\,\mathbb{Z}_p$ and $u \in \Gamma = 1 + q\,\mathbb{Z}_p$ be a fixed geometric generator. Define

$$\Lambda = \mathcal{O}[[\Gamma]] := \varprojlim_k \mathcal{O}[\Gamma/\Gamma^{p^k}] \cong \varprojlim_k \mathcal{O}[T]/\langle (1+T)^{p^k} - 1\rangle$$

where the last isomorphism is given by $\gamma_0 \mapsto 1+T$. We will show that $\varprojlim_k \mathcal{O}[T]/\langle (1+T)^{p^k} - 1\rangle = \mathcal{O}[[T]]$.

Definition 3. Let $P \in \mathcal{O}[T]$. P is called a distinguished polynomial if P is non-constant, monic, and $P \equiv T^{\deg(P)} \pmod{\varpi}$.

Proposition 7 (Division Algorithm)**.** Suppose $P = a_0 + a_1 T + \cdots \in \mathcal{O}[[T]]$, $P \not\equiv 0 \pmod{\varpi}$, and $n = \min\{k \in \mathbb{N} \mid a_k \in \mathcal{O}^\times\}$. Then for every $f \in \mathcal{O}[[T]]$ there exists a unique pair (Q, R) where $Q \in \mathcal{O}[[T]]$ and $R \in \mathcal{O}[T]$ has degree smaller than n, such that

$$f = QP + R.$$

Theorem 5 (Weierstrass Preparation)**.** For every $f \in \mathcal{O}[[T]]$ there exists a unique triple $(u, U(T), P(T))$ where $u \in \mathbb{Z}_{\geq 0}$, $U \in \mathcal{O}[[T]]^\times$, and $P(T)$ is a distinguished polynomial, such that

$$f = \varpi^u P U.$$

It is easily seen that $\mathcal{O}[[T]]$ is a UFD of dimension 2 and its irreducible elements are ϖ and all irreducible distinguished polynomials.

Theorem 6. Let $P_1, P_2, \cdots$ be a sequence of distinguished polynomials such that $P_k \in (\varpi, T)^k$ and $P_k \mid P_{k+1}$ for all $k \in \mathbb{N}$. We endow $\mathcal{O}[[T]]$ with the $\mathfrak{m}$-adic topology and $\mathcal{O}[[T]]/(P_k)$ the p-adic topology. Then the natural map

$$\varphi : \mathcal{O}[[T]] \to \varprojlim_k \mathcal{O}[[T]]/(P_k)$$

is an isomorphism both algebraically and topologically.

Proof. Since $\mathcal{O}[[T]]/(P_k)$ is p-adically complete, it is isomorphic to $\varprojlim_\ell \mathcal{O}[[T]]/(P_k, \varpi^\ell)$ with each object endowed with discrete topology. Hence

$$\varprojlim_k \mathcal{O}[[T]]/(P_k) = \varprojlim_{k,\ell} \mathcal{O}[[T]]/(P_k, \varpi^\ell) = \varprojlim_k \mathcal{O}[[T]]/(P_k, \varpi^k),$$

where each $\mathcal{O}[[T]]/(P_k, \varpi^k)$ is given discrete topology. Since $(P_k, \varpi^k) \subset \mathfrak{m}^k$, it suffices to show that for every $k \in \mathbb{N}$ is a ℓ such that $\mathfrak{m}^\ell \subset (P_k, \varpi^k)$. This is true as the radical of (P_k, ϖ^k) is $\mathfrak{m}$ and hence $\mathcal{O}[[T]]/(P_k, \varpi^k)$ is Artinian. □

Let $P_k := (1+T)^{p^k} - 1$. Since $\mathcal{O}[T]/(P_k) \to \mathcal{O}[[T]]/(P_k)$ is an isomorphism, $\varprojlim_k \mathcal{O}[T]/\langle (1+T)^{p^k} - 1 \rangle = \mathcal{O}[[T]]$.

Definition 4. Let $\chi : (\mathbb{Z}/Np^\alpha\mathbb{Z})^\times \to \mathcal{O}^\times$ be a Dirichlet character for some $\alpha \geq 1$. We say $F \in \Lambda[[q]]$ is a (cusp,ordinary) Λ-adic modular form if $F(u^k - 1) \in \mathcal{O}[[q]]$ is a (cusp,ordinary) modular form in $M_k(Np^\alpha, \chi\omega^{-k}, \mathcal{O})$ for all $k >> 0$. We define $\mathbb{M}(\chi;\Lambda)$ ($\mathbb{M}^{\mathrm{ord}}(\chi;\Lambda)$, $\mathbb{S}(\chi;\Lambda)$, $\mathbb{S}^{\mathrm{ord}}(\chi;\Lambda)$) as the space of Λ-adic (ordinary, cusp, ordinary cusp) modular forms.

Example 2. Let ψ, φ be two primitive Dirichlet characters modulo u, v, respectively, with value in $\mathcal{O}^\times$. Suppose $\psi(p) \neq 0$. Then

$$\frac{1}{2}\left(E_k^{\psi,\varphi}(z) - \varphi(p)p^{k-1}E_k^{\psi,\varphi}(pz)\right)$$

is ordinary. The q-expansion of the ordinary Eisenstein series is

$$n \mapsto \sum_{\substack{d \mid n \\ p \nmid d}} \psi(n/d)\varphi(d)d^{k-1}$$

and the constant term is

$$\frac{1}{2}\delta(\psi)L_p(1-k,\varphi)$$

We define $A_{n,\psi,\varphi}$ as

$$\sum_{\substack{d|n\\ p\nmid d}}\psi(n/d)\varphi(d)d^{-1}\langle d\rangle$$

and $A_{0,\psi,\varphi}$ as the element in $\mathrm{Frac}(\Lambda)$ with

$$A_{0,\psi,\varphi}(\epsilon(u)u^s-1)=\frac{1}{2}\delta(\psi)L_p(1-s,\epsilon\varphi)$$

for all $|s|_p<qp^{-1/(p-1)}$ and ϵ any finite order character on $1+q\,\mathbb{Z}_p$. If φ is odd or $\psi\neq 1$, $A_{0,\psi,\varphi}=0$. If $\psi=1$, φ is nontrivial and even, then $A_{0,\psi,\varphi}\in\Lambda$. If $\psi=\varphi=1$, $A_{0,\psi,\varphi}\in\frac{\Lambda}{T}$. Define

$$E^{\psi,\varphi}:=A_{0,\psi,\varphi}+\sum_{n=1}^{\infty}A_{n,\psi,\varphi}q^n.$$

When $(\varphi,\psi)\neq(1,1)$,

$$E^{\psi,\varphi}\in\mathbb{M}^{\mathrm{ord}}(\psi\varphi;\Lambda)$$

with suitable level, $E^{1,1}\in T^{-1}\mathbb{M}^{\mathrm{ord}}(1;\Lambda)$, and

$$E^{\psi,\varphi}(\epsilon(u)u^k-1)\in\mathcal{M}_k^{\mathrm{ord}}(Np^{\alpha},\epsilon\psi\varphi;\mathbb{Q}_p[\epsilon])$$

for all $k\geq 2$ with suitable N,α.

Definition 5. For every $k\geq 2$ we define

$$P_k:=T-(u^k-1).$$

More generally, for every finite order character $\epsilon:1+2p\,\mathbb{Z}_p\to\mathbb{C}_p^{\times}$, we define $P_{k,\epsilon}$ as the minimal polynomial of $\epsilon(u)u^k-1$ over $\mathcal{O}$.

3.2 Ordinary Hida Families

Theorem 7. $\mathbb{M}^{\mathrm{ord}}(\chi;\Lambda)$ and $\mathbb{S}^{\mathrm{ord}}(\chi;\Lambda)$ are free of finite rank over Λ.

Proof. Let M' be a finite free submodule of $\mathbb{M}^{\mathrm{ord}}$, say $F_1,\cdots,F_n$ be a basis. Then there exists $b_1,\cdots,b_n\in\mathbb{N}$ such that $D:=\det(a(b_j,F_i))\neq 0$. Therefore, for all $k >> 0$, $\{F_i(u^k - 1)\} \subset M_k^{\mathrm{ord}}(Np^\alpha, \chi\omega^{-k}; \mathcal{O})$ and generates a free $\mathcal{O}$-module of rank n. Therefore, $n \leq \mathrm{rank}_{\mathcal{O}}(M_k^{\mathrm{ord}}(Np^\alpha, \chi\omega^{-k}; \mathcal{O}))$ for all $k >> 0$. Since $\mathrm{rank}_{\mathcal{O}}(M_k^{\mathrm{ord}}(Np^\alpha, \chi\omega^{-k}; \mathcal{O}))$ is bounded independent of k, n is bounded independent of M. Therefore, there is a $n_0 \in \mathbb{Z}_{\geq 0}$ such that n_0 is the maximal possible rank of free submodules of $\mathbb{M}^{\mathrm{ord}}$.

Let $F_1, \cdots, F_{n_0} \subset \mathbb{M}^{\mathrm{ord}}$ be a basis of a free submodule M' of rank n_0 of $\mathbb{M}^{\mathrm{ord}}$. Let $L := \mathrm{Frac}(\Lambda)$. Let $F \in \mathbb{M}^{\mathrm{ord}}$. There are $\lambda_1, \cdots, \lambda_{n_0} \in L$ such that

$$\lambda_1 F_1 + \cdots + \lambda_{n_0} F_{n_0} = F.$$

Consider linear equations

$$\lambda_1 a(n_j, F_1) + \cdots + \lambda_{n_0} a(n_j, F_{n_0}) = a(n_j, F)$$

and we have that $D\lambda_j \in \Lambda$ for all j. Hence $\dfrac{M'}{D} \supset \mathbb{M}^{\mathrm{ord}}$, and $\mathbb{M}^{\mathrm{ord}}$ is finitely generated. Therefore, there is a $a \in \mathbb{N}$ such that for all $k \geq a$ and $F \in \mathbb{M}^{\mathrm{ord}}$, $F(u^k - 1) \in M_k^{\mathrm{ord}}(Np^\alpha, \chi\omega^{-k}; \mathcal{O})$. Let $k \geq a$. If $F(u^k - 1) = 0$, then $F = P_k F'$ for some $F' \in \Lambda[[q]]$ and $F'(u^r - 1) = F(u^r - 1)/(u^r - u^k) \in M_r^{\mathrm{ord}}(Np, \chi\omega^{-r}; \mathcal{O})$ for all $r > k$. Hence $F \in P_k \mathbb{M}^{\mathrm{ord}}$. We have that

$$\mathbb{M}^{\mathrm{ord}} / P_k \mathbb{M}^{\mathrm{ord}} \to M_k^{\mathrm{ord}}(Np^\alpha, \chi\omega^{-k}; \mathcal{O})$$

is injective. Let $f_1, \cdots, f_n$ be a $\mathcal{O}$-basis of the image and $F_1, \cdots, F_n$ be their liftings. By Nakayama's lemma, $\mathbb{M}^{\mathrm{ord}}$ is generated by $F_1, \cdots, F_n$. If $\lambda_1, \cdots, \lambda_n \in \Lambda$ such that

$$\lambda_1 F_1 + \cdots + \lambda_n F_n = 0,$$

$P_k \mid \lambda_i$ for all i. By infinite descent method $\lambda_1 = \cdots = \lambda_n = 0$. Namely, $\mathbb{M}^{\mathrm{ord}}$ is free and $\{F_1, \cdots, F_n\}$ is a basis.

The proof for $\mathbb{S}^{\mathrm{ord}}$ is identical. □

We define Hecke operators on $\mathbb{M}$ as follows:

$$a(m, T_n F) := \sum_{\substack{d \mid (m,n) \\ (d,Np)=1}} \chi(d) \langle d \rangle d^{-1} a(mn/d^2, F).$$

Since $(T_n F)(u^k - 1) = T_n(F(u^k - 1))$ for $k >> 0$, $T_n \in \mathrm{End}_\Lambda(\mathbb{M})$, preserving subspaces of ordinary and cusp forms.

We would like to define an ordinary projector $e : \mathbb{M} \to \mathbb{M}^{\mathrm{ord}}$, which should be

$$eF = \lim_{n\to\infty} T_p^{n!} F$$

under the $\mathfrak{m}$-adic topology. This is done circuitously. Given an $F \in \mathbb{M}$. Let $a \in \mathbb{N}$ such that $F(u^k - 1) \in M_k(Np, \chi\omega^{-k}; \mathcal{O})$ for all $k \geq a$. We define

$$\mathbb{M}_{a,j} := \{F \in \mathbb{M} \mid F(u^k - 1) \in M_k(Np, \chi\omega^{-k}; \mathcal{O}) \ \forall\ k \in [a, j]\}.$$

Let $\Omega_j := \prod_{k=a}^{j} P_k(T)$ where $P_k(T) := T - (u^k - 1)$. Then

$$\mathbb{M}_{a,j} \to \bigoplus_{k=a}^{j} M_k(Np, \chi\omega^{-k}; \mathcal{O})$$

has kernel $\Omega_j[[q]] \cap \mathbb{M}_{a,j}$. Since T_p preserves $\mathbb{M}_{a,j}$ and $\Omega_j[[q]] \cap \mathbb{M}_{a,j}$, the image of $\mathbb{M}_{a,j} \to \bigoplus_{k=a}^{j} M_k(Np, \chi\omega^{-k}; \mathcal{O})$ is a T_p-invariant subspace. Hence $\lim_{n\to\infty} T_p^{n!}$ is defined on $\dfrac{\mathbb{M}_{a,j}}{\Omega_j[[q]] \cap \mathbb{M}_{a,j}}$, denoted by e_j. Then we have the commutative diagram

$$\begin{array}{ccc} \dfrac{\mathbb{M}_{a,j+1}}{\Omega_{j+1}[[q]] \cap \mathbb{M}_{a,j+1}} & \longrightarrow & \dfrac{\mathbb{M}_{a,j}}{\Omega_j[[q]] \cap \mathbb{M}_{a,j}} \\ \Big\downarrow{\scriptstyle e_{j+1}} & & \Big\downarrow{\scriptstyle e_j} \\ \dfrac{\mathbb{M}_{a,j+1}}{\Omega_{j+1}[[q]] \cap \mathbb{M}_{a,j+1}} & \longrightarrow & \dfrac{\mathbb{M}_{a,j}}{\Omega_j[[q]] \cap \mathbb{M}_{a,j}} \end{array}.$$

On the other hand, $\dfrac{\mathbb{M}_{a,j}}{\Omega_j[[q]] \cap \mathbb{M}_{a,j}}$ is a subspace of $(\Lambda/\Omega_j)[[q]]$. Therefore,

$$\varprojlim_j \frac{\mathbb{M}_{a,j}}{\Omega_j[[q]] \cap \mathbb{M}_{a,j}} \subset \varprojlim_j (\Lambda/\Omega_j)[[q]] = \Lambda[[q]],$$

and the image is clearly M_a. We thus define $e := \varprojlim_j e_j$ on M_a. Since $\varprojlim_j (\Lambda/\Omega_j)$ with p-adic topology on each Λ/Ω_j is Λ with $\mathfrak{m}$-adic topology,

$$eF = \lim_{n\to\infty} T_p^{n!} F$$

under the $\mathfrak{m}$-adic topology, and $(eF)(u^k - 1) = e(F(u^k - 1))$ for all $k \geq a$. Hence e is an idempotent from $\mathbb{M}$ onto $\mathbb{M}^{\mathrm{ord}}$, mapping cusp forms to cusp forms.

Proposition 8. For every $a \geq 0$ and $f \in M_a(Np^\alpha, \chi\omega^{-a}; \mathcal{O})$ there is a $F \in \mathbb{M}(\chi; \Lambda)$ such that $F(u^a - 1) = f$. If f is cusp (ordinary), F can be taken to be cusp (ordinary).

Proof. We first consider $E^{1,1} \in \mathbb{M}^{\mathrm{ord}}(1, \Lambda)$. The T^{-1}-term of $E^{1,1}$ is

$$\lim_{s\to 0} \frac{(u^s-1)L_p(1-s)}{2} = 2^{-1}(p^{-1}-1)\log_p(u) \in \mathbb{Z}_p^\times .$$

Define

$$E' := \frac{TE^{1,1}}{2^{-1}(p^{-1}-1)\log_p(u)} \in \mathbb{M}^{\mathrm{ord}}(\chi;\Lambda)$$

and

$$E(T) := E'(u^{-a}T + (u^{-a}-1)),\ \ F := fE.$$

Then for all $k \geq a$, $F(u^k-1) \in M_k(Np^\alpha, \chi\omega^{-k};\mathcal{O})$, and $F(u^a-1) = fE'(0) = f$.

If f is cusp, F is cusp. If f is ordinary, we take $F := e(fE)$ instead. □

From this we can write down a basis for $\mathbb{M}^{\mathrm{ord}}$ ($\mathbb{S}^{\mathrm{ord}}$) as follows: We first take a $a \in \mathbb{N}$ such that $F(u^k-1) \in M_k^{\mathrm{ord}}(Np^\alpha, \chi\omega^{-k};\mathcal{O})$ ($S_k^{\mathrm{ord}}(Np^\alpha, \chi\omega^{-k};\mathcal{O})$) for all $F \in \mathbb{M}^{\mathrm{ord}}$ ($\mathbb{S}^{\mathrm{ord}}$) and $k \geq a$. Let $f_1, \cdots, f_n$ be a basis of $M_a^{\mathrm{ord}}(Np^\alpha, \chi\omega^{-a};\mathcal{O})$ ($S_a^{\mathrm{ord}}(Np^\alpha, \chi\omega^{-a};\mathcal{O})$) and $F_i := e(fE')$. Then $\{F_1, \cdots, F_n\}$ is a Λ-basis of $\mathbb{M}^{\mathrm{ord}}$ ($\mathbb{S}^{\mathrm{ord}}$). This shows that for all finite order characters $\epsilon : 1 + q\,\mathbb{Z}_p \to \mathbb{C}_p^\times$ and $k \geq a$,

$$F(\epsilon(u)u^k - 1) \in M_k^{\mathrm{ord}}(Np^\alpha, \epsilon\chi\omega^{-k};\mathcal{O}[\epsilon])\ (S_k^{\mathrm{ord}}(Np^\alpha, \epsilon\chi\omega^{-k};\mathcal{O}[\epsilon])).$$

We define

$$\epsilon_* : \mathbb{M}(\chi;\Lambda) \to \mathbb{M}(\epsilon\chi;\Lambda[\epsilon]),\ \ (\epsilon_* F))(T) := F(\epsilon T + (\epsilon - 1)).$$

Since $\epsilon_*^{-1} \circ \epsilon_* = \mathrm{id}$, when ϵ takes value in $\mathcal{O}^\times$,

$$\epsilon_* : \mathbb{M}(\chi;\Lambda) \cong \mathbb{M}(\epsilon\chi;\Lambda).$$

Theorem 8. For every $k \geq 2$ and every $F \in \mathbb{M}^{\mathrm{ord}}$ ($\mathbb{S}^{\mathrm{ord}}$), $F(\epsilon(u)u^k - 1) \in \mathbb{M}^{\mathrm{ord}}(Np^\alpha, \epsilon\chi\omega^{-k};\mathcal{O}[\epsilon])$ ($\mathbb{S}^{\mathrm{ord}}(Np^\alpha, \epsilon\chi\omega^{-k};\mathcal{O}[\epsilon])$). Moreover, there are isomorphisms

$$\mathbb{M}^{\mathrm{ord}}(\chi;\Lambda)/P_{k,\epsilon}\mathbb{M}^{\mathrm{ord}}(\chi;\Lambda) \cong M_k^{\mathrm{ord}}(Np^\alpha, \epsilon\chi\omega^{-k};\mathcal{O}),$$

and

$$\mathbb{S}^{\mathrm{ord}}(\chi;\Lambda)/P_{k,\epsilon}\mathbb{S}^{\mathrm{ord}}(\chi;\Lambda) \cong S_k^{\mathrm{ord}}(Np^\alpha, \epsilon\chi\omega^{-k};\mathcal{O}).$$

for all $k \geq 2$. In particular, $\mathrm{rank}_{\mathcal{O}}(M_k^{\mathrm{ord}}(Np^\alpha, \epsilon\chi\omega^{-k};\mathcal{O}))$ and $\mathrm{rank}_{\mathcal{O}}(S_k^{\mathrm{ord}}(Np^\alpha, \epsilon\chi\omega^{-k};\mathcal{O}))$ are constant for all $k \geq 2$.

Proof. We first show the case $\epsilon = 1$.

From the previous proposition we have that the image of

$$\mathbb{M}^{\text{ord}}(\chi;\Lambda)/P_k\mathbb{M}^{\text{ord}}(\chi;\Lambda) \hookrightarrow \mathcal{O}[[q]]$$

contains $M_k^{\text{ord}}(Np^\alpha, \chi\omega^{-k}; \mathcal{O})$ for all $k \geq 0$. For $k >> 0$, $\mathbb{M}^{\text{ord}}(\chi;\Lambda)/P_k\mathbb{M}^{\text{ord}}(\chi;\Lambda) \subset M_k^{\text{ord}}(Np^\alpha, \chi\omega^{-k}; \mathcal{O})$ and the equality holds.

Since $\text{rank}_{\mathcal{O}}(M_k^{\text{ord}}(Np^\alpha, \chi\omega^{-k}; \mathcal{O}))$ is a constant for all $k \geq 2$, then $\mathbb{M}^{\text{ord}}(\chi;\Lambda)/P_k\mathbb{M}^{\text{ord}}(\chi;\Lambda)$ and $\text{rank}_{\mathcal{O}}(M_k^{\text{ord}}(Np^\alpha, \chi\omega^{-k}; \mathcal{O}))$ have the same rank. Therefore,

$$\begin{aligned} M_k^{\text{ord}}(Np^\alpha, \chi\omega^{-k}; \mathcal{O}) &= (M_k^{\text{ord}}(Np^\alpha, \chi\omega^{-k}; \mathcal{O}) \otimes_{\mathcal{O}} K) \cap \mathcal{O}[[q]] \\ \supset & \mathbb{M}^{\text{ord}}(\chi;\Lambda)/P_k\mathbb{M}^{\text{ord}}(\chi;\Lambda) \supset M_k^{\text{ord}}(Np^\alpha, \chi\omega^{-k}; \mathcal{O}). \end{aligned}$$

For general ϵ we first consider $\mathbb{M}^{\text{ord}}(\chi;\Lambda')$ where $\Lambda' := \mathcal{O}[\epsilon][[T]]$. Since $\epsilon_* : \mathbb{M}^{\text{ord}}(\chi;\Lambda') \cong \mathbb{M}^{\text{ord}}(\chi;\epsilon\Lambda')$,

$$\begin{aligned} &\mathbb{M}^{\text{ord}}(\chi;\Lambda')/(T - (\epsilon(u)u^k - 1))\mathbb{M}^{\text{ord}}(\chi;\Lambda') \\ &\cong \mathbb{M}^{\text{ord}}(\chi;\epsilon\Lambda')/(T - (\epsilon(u)u^k - 1))\mathbb{M}^{\text{ord}}(\epsilon\chi;\Lambda') \\ &\cong M_k(Np^\alpha, \epsilon\chi\omega^{-k}; \mathcal{O}[\epsilon]). \end{aligned}$$

for all $k \geq 2$. Every $F \in \mathbb{M}^{\text{ord}}(\chi, \Lambda')$ can be written as a finite sum

$$F = \sum_i F_i \epsilon(u)^i$$

where each $F_i \in \mathbb{M}^{\text{ord}}(\chi;\Lambda)$. Given $k \geq 2$. Define

$$F' := \sum_i F_i \frac{(1+T)^i}{u^{ik}} \in \mathbb{M}^{\text{ord}}(\chi;\Lambda).$$

Then

$$F'(\epsilon(u)u^k - 1) = F(\epsilon(u)u^k - 1)$$

and therefore,

$$\mathbb{M}^{\text{ord}}(\chi;\Lambda)/P_{k,\epsilon}\mathbb{M}^{\text{ord}}(\chi;\Lambda) \cong \mathbb{M}^{\text{ord}}(\chi;\Lambda')/(T - (\epsilon(u)u^k - 1))\mathbb{M}^{\text{ord}}(\chi;\Lambda').$$

The proof for cusp forms is identical. □

3.3 Duality and Lifting

We define Hecke algebras $H^{\mathrm{ord}}(\chi;\Lambda)$ and $h^{\mathrm{ord}}(\chi;\Lambda)$ as the Λ-subalgebra of $\mathrm{End}_\Lambda(\mathbb{M}^{\mathrm{ord}}(\chi;\Lambda))$ and $\mathrm{End}_\Lambda(\mathbb{S}^{\mathrm{ord}}(\chi;\Lambda))$, respectively. Moreover generally, for every Λ-algebra A, We define $H^{\mathrm{ord}}(\chi;A) = \mathrm{End}_A(\mathbb{M}^{\mathrm{ord}}(\chi;A)) = H^{\mathrm{ord}}(\chi;\Lambda)\otimes_\Lambda A$ an similarly define $h^{\mathrm{ord}}(\chi;A)$.

Theorem 9 (Duality)**.** The pairing

$$(h,f)\mapsto a_1(h(f))$$

defines a perfect pairing between $h^{\mathrm{ord}}(\chi;A)$, $\mathbb{S}^{\mathrm{ord}}(\chi;A)$, and $H^{\mathrm{ord}}(\chi;A)$, $m^{\mathrm{ord}}(\chi;A)$.

Proof. It suffices to prove the case $A=\Lambda$. The pairing gives a map $h^{\mathrm{ord}}(\chi;\Lambda)\to \mathrm{Hom}_\Lambda(\mathbb{S}^{\mathrm{ord}}(\chi;\Lambda);\Lambda)$. If h is in the kernel, we have for all f and n,

$$0=(h,T_nf)=a_1(hT_nf)=a_1(T_nhf)=a_n(h(f)),$$

so $h=0$. Let N be the cokernel of the map. We tensor $\Lambda/(P_k)$ on the short exact sequence

$$0\to h^{\mathrm{ord}}(\chi;\Lambda)\to \mathrm{Hom}_\Lambda(\mathbb{S}^{\mathrm{ord}}(\chi;\Lambda),\Lambda)\to N\to 0.$$

Since $\mathbb{S}^{\mathrm{ord}}(\chi;\Lambda)$ is finite free, the middle term is

$$\begin{aligned}\mathrm{Hom}_\Lambda(\mathbb{S}^{\mathrm{ord}}(\chi;\Lambda),\Lambda)\otimes_\Lambda \Lambda/(P_k) &\cong \mathrm{Hom}_\Lambda(\mathbb{S}^{\mathrm{ord}}(\chi;\Lambda)/P_k\,\mathbb{S}^{\mathrm{ord}}(\chi;\Lambda),\mathcal{O})\\ \cong \mathrm{Hom}_{\mathcal{O}}(S_k^{\mathrm{ord}}(\chi\omega^{-k};\mathcal{O}),\mathcal{O}) &\cong h_k^{\mathrm{ord}}(\chi\omega^{-k};\mathcal{O}).\end{aligned}$$

The image from the first term is the $\mathcal{O}$-subalgebra generated by $\{T_n\}_{n\in N}$, which is $h_k^{\mathrm{ord}}(\chi\omega^{-k};\mathcal{O})$ itself. Hence $N=P_kN$. By Nakayama's lemma, $N=0$. Hence we have

$$h^{\mathrm{ord}}(\chi;\Lambda)\cong \mathrm{Hom}_\Lambda(\mathbb{S}^{\mathrm{ord}}(\chi;\Lambda),\Lambda).$$

In particular, $h^{\mathrm{ord}}(\chi;\Lambda)$ is finite free over Λ. Take dual on both sides and we have

$$\mathrm{Hom}_\Lambda(h^{\mathrm{ord}}(\chi;\Lambda),\Lambda)=\mathrm{Hom}_\Lambda(\mathrm{Hom}_\Lambda(\mathbb{S}^{\mathrm{ord}}(\chi;\Lambda),\Lambda),\Lambda)\cong \mathbb{S}^{\mathrm{ord}}(\chi;\Lambda).$$

The proof for H and m are identical. □

Now we have that for every Λ-algebra A, $\mathrm{Hom}_\Lambda(h^{\mathrm{ord}}(\chi;\Lambda), A) \cong \mathbb{S}^{\mathrm{ord}}(\chi; A)$. Moreover, $\varphi \in \mathrm{Hom}_\Lambda(h^{\mathrm{ord}}(\chi;\Lambda), A)$ is a Λ-algebra homomorphism if and only if $F_\varphi := \sum_{n=1}^{\infty} \varphi(T_n) q^n$ is a normalized Hecke eigenform with coefficients in A.

Let $k \geq 2$ and $f \in S_k^{\mathrm{ord}}(\chi\omega^{-k}; \mathcal{O})$. f induces an $\mathcal{O}$-algebra homomorphism $h_k^{\mathrm{ord}}(\chi\omega^{-k}; \mathcal{O}) \to \mathcal{O}$. Since $h_k^{\mathrm{ord}}(\chi\omega^{-k}; \mathcal{O}) \cong h^{\mathrm{ord}}(\chi;\Lambda) \otimes_\Lambda \Lambda/(P_k)$, we obtain a unique Λ-algebra homomorphism from $h^{\mathrm{ord}}(\chi;\Lambda) \to \mathcal{O}$. Since Λ is a complete local ring and hence henselian, $h^{\mathrm{ord}}(\chi;\Lambda)$ decomposes into a finite product of Λ-algebras, which are again henselian local rings. Let P_f be the kernel of $h_k^{\mathrm{ord}}(\chi\omega^{-k}; \mathcal{O}) \to \mathcal{O}$ and $\mathfrak{m}_f$ be the maximal ideal lying over P_f. Then the ring homomorphism factors through $h^{\mathrm{ord}}(\chi;\Lambda) \to h^{\mathrm{ord}}(\chi;\Lambda)_{\mathfrak{m}_f}$.

We may lift f to an normalized Hecke eigenform over a Λ-algebra with better algebraic properties. Let Q_f be a minimal prime ideal of $h^{\mathrm{ord}}(\chi;\Lambda)$ contained in P_f, $I' := h^{\mathrm{ord}}(\chi;\Lambda)/Q_f$ and define I as the integral closure of I'. Then I/Λ is finite. We see that I is a complete local ring, and the topology coincide with the $\mathfrak{m}_\Lambda$-adic topology. Let P_f' be any prime ideal of I over $\overline{P_f} \subset I'$. Then I/P_f' is finite over $\mathcal{O}$ and hence $h^{\mathrm{ord}}(\chi;\Lambda) \to I/P_f'$ defines an normalized Hecke eigenform with coefficient in $\overline{\mathbb{Z}_p}$ which is exactly f.

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