M-SD p-adic L-functions

<u>Arithmetic object</u> Motive M

$E/\mathbb{Q}$ ell. curve

L-series
$L(M,s)$

<u>Analytic side</u>

$$\prod_{\ell \leq X} \frac{N_\ell(E)}{\ell} \overset{?}{\underset{+\infty}{\sim}} (\log X)^r$$

$$\text{''} \quad \overset{\shortparallel}{L(E,1)^{-1}} \quad \text{''}$$

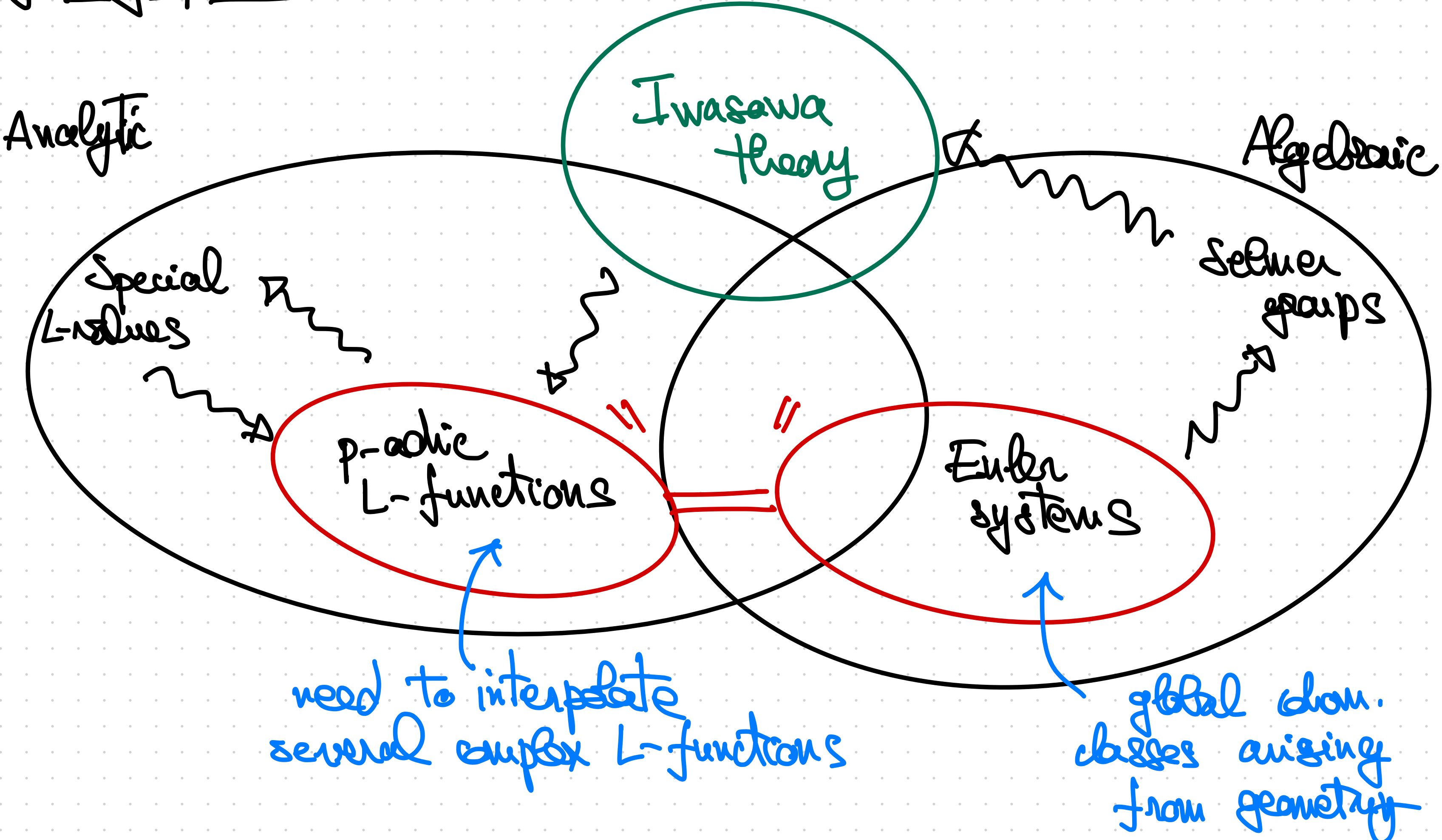
<u>Algebraic side</u>

$E(\mathbb{Q}) \simeq T \oplus \mathbb{Z}^r$

BK Selmer group
$H^1_f(\mathbb{Q}, M)$

$E(\mathbb{Q}) \longrightarrow E(\mathbb{F}_\ell)$ $\qquad N_\ell(E) := |E(\mathbb{F}_\ell)|$

§ Big picture


Analytic
Algebraic
Iwasawa theory
Special L-values
Selmer groups
p-adic L-functions
Euler systems
need to interpolate several complex L-functions
global cohom. classes arising from geometry

§ Examples

They're organized according to a p-adic family of global Galois repr.s $\mathbb{V} \circlearrowleft G_{\mathbb{Q}}$.

1) Kubota-Leopoldt $\qquad \chi \otimes \mathbb{Z}_p[[\mathbb{Z}_p^\times]] \qquad$ GL_1

χ Dirichlet

$\mathbb{Z}_p^\times = G(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$

2) $\boxed{\text{M-SD} \qquad V_p(E) \otimes \mathbb{Z}_p[[\mathbb{Z}_p^\times]]}$ $\qquad GL_2$

3) $K/\mathbb{Q}$ quad. imaginary

Anticyclotomic p-adic L-function (Bertolini-Darmon)

$V_p(E)|_{G_K} \otimes \mathbb{Z}_p[[G(K^{ac}/K)]] \qquad GL_2 \times GL_1$

4) $\mathcal{F}$ Hida family through E

GL$_2$ in families

$\mathbb{V}_{\mathcal{F}} \otimes \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$ Mazur-Kitagawa

⋮

§ Review of p-adic measures

G profinite

$\mathcal{M}(G, \mathbb{Z}_p)$

- Bounded lin. maps $\varphi : \mathcal{C}(G, \mathbb{Z}_p) \longrightarrow \mathbb{Z}_p$
- element in completed group ring $\mathbb{Z}_p[\![G]\!] = \varprojlim_{U} \mathbb{Z}_p[G/U]$
- finitely additive functions $\mu : \left\{ \begin{array}{l} \text{compact open} \\ \text{subsets of } G \end{array} \right\} \longrightarrow \mathbb{Z}_p$

Rmk

$$\mathrm{Hom}_{\mathbb{Z}_p\text{-lin}}(\mathbb{Z}_p[\![G]\!], \mathbb{C}_p) = \mathrm{Hom}_{\mathrm{cont}}(G, \mathbb{C}_p^{\times})$$

Upshot

Given $\mathcal{L}_p \in \mathbb{Z}_p[\![G]\!]$ then

$$\mathcal{L}_p(\chi) := \chi(\mathcal{L}_p)$$

makes sense $\forall\ \chi : G \longrightarrow \mathbb{C}_p^\times$.

§ Plan

For simplicity $E/\mathbb{Q}$ of good ordinary reduction at p

Using modularity construct $\mathcal{L}_p(E) \in \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$ s.t.

$$\mathcal{L}_p(E)(\chi) \doteq L(E, \overline{\chi}, 1)$$

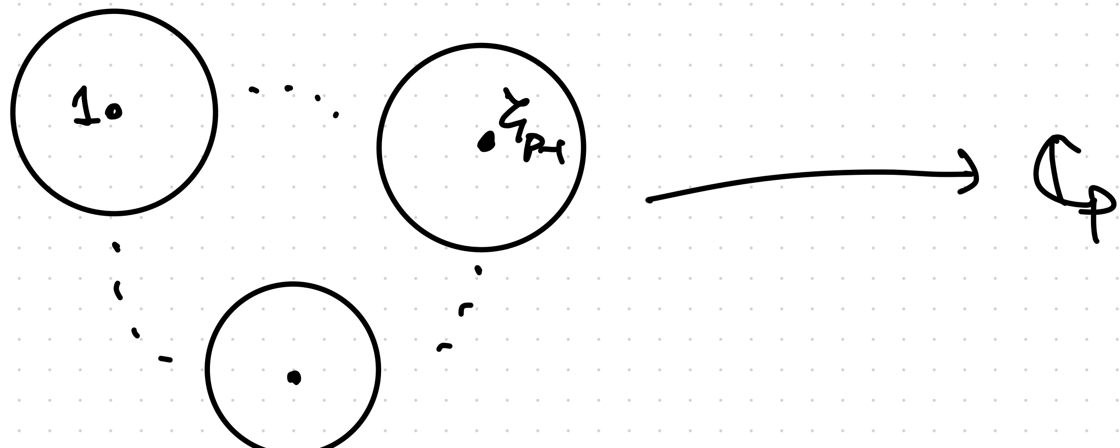
Rmk

$$\mathcal{L}_p(E,1) \doteq L(E,1)$$

What happens when $r_{an}(E/\mathbb{Q}) > 0$?

$$\mathbb{Z}_p[[\mathbb{Z}_p^\times]] \simeq \mathbb{Z}_p[\mu_{p-1}][[\Gamma]] \qquad \Gamma = 1 + p\mathbb{Z}_p$$

$p-1$ open unit disks



Want to focus around 1

$$\Theta_{1} = \frac{1}{p-1} \sum_{\zeta \in \mu_{p-1}} [\zeta] \in \mathbb{Z}_p[\mu_{p-1}]$$

$\mathbb{Z}_p[\mu_{p-1}] \cong \bigoplus_{a=1}^{p-1} \mathbb{Z}_p \cdot \Theta_{\omega^a}$

get $\Theta_1 \mathbb{Z}_p(E) \in \mathbb{Z}_p[[\Gamma]] \simeq \mathbb{Z}_p[[X]]$

Augmentation ideal

$$I = \ker\left(\mathbb{Z}_p[[\Gamma]] \longrightarrow \mathbb{Z}_p\right)$$

$$\begin{array}{cc} I & \mathbb{Z}_p[[\Gamma]] \\ \text{\rotatebox{90}{$\cong$}} & \text{\rotatebox{90}{$\cong$}} \\ (X) & \mathbb{Z}_p[[X]] \end{array}$$

Definition

$$\mathrm{ord}_{\mathbb{1}}\, \mathcal{L}_p(E) := \max_r \left\{ \Theta_{\mathbb{1}} \cdot \mathcal{L}_p(E) \in I^r \right\}$$

$$\boxed{\begin{array}{l}\text{p-adic BSD-conjecture (MTT)} \\[1em] \qquad \mathrm{ord}_{\mathbb{1}}\, \mathcal{L}_p(E) = r_{\mathrm{alg}}(E/\mathbb{Q})\end{array}}$$

Rmk

E has good ordinary red. at p

Rmk
Kato proved $\geq$ using the Euler system of Siegel units.

§ Construction

$f_E \in S_2(\Gamma_0(N))$

$\Omega^{\pm}$ a positive gen of $\Lambda_E \cap \mathbb{R}$ $(\Lambda_E \cap i\mathbb{R})$

$$I_E : \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q}) \longrightarrow \mathbb{C}$$

$$(x, y) \longmapsto \int_x^y 2\pi i\, f_E(z)\, dz$$

Thm (Manin-Drinfeld)

$$\mathbb{Z}[I_E(x,y) \mid x,y \in \mathbb{P}^1(\mathbb{Q})] \subseteq \mathbb{C} \quad \text{is a lattice}$$

$$\boxed{I_E(x,y) = \{x,y\}^+ \cdot \Omega^+ + \{x,y\}^- \Omega^-}$$

rational with bounded denominators

Def. $\{x,y\}_E = \{x,y\}^+ + \{x,y\}^-$

Recall $X^2 - a_p(E) X + p = (X-\alpha)(X-\beta)$

where $\alpha \in \mathbb{Z}_p^\times$, $\beta \in p\mathbb{Z}_p$

$$\boxed{\{x,y\}_{f_\alpha} := \{x,y\}_E - \alpha^{-1}\{px,py\}_E} \quad \in \mathbb{Z}_p$$

<u>Fact</u> $\forall \gamma \in \Gamma_0(N_p)$ $\quad \{\gamma x, \gamma y\}_{f_\alpha} = \{x,y\}_{f_\alpha}$.

<u>Want</u> $\mu_E \in \mathcal{M}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$

A basis for the topology of $\mathbb{Z}_p^\times$ is given by

$$B(a,n) = \{x \in \mathbb{Z}_p^\times \mid x \equiv a \mod p^n\} \quad a \in \mathbb{Z},\ n \geq 0$$

Definition

$$\mu_E(B(a,n)) := \alpha^{-n} \left\{ \infty, \frac{a}{p^n} \right\}_{f_\alpha}$$

Proposition

$$\mu_E \in \mathcal{M}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$$

Proof. (Sketch)

$$B(a,n) = \coprod_{x \equiv_{p^n} a} B(x, n+1)$$

$$\left(\mathbb{Z}/p^{n+1}\mathbb{Z}\right)^\times \longrightarrow \left(\mathbb{Z}/p^n\mathbb{Z}\right)^\times$$

$$x \longmapsto a$$

· It suffices to check that

$$\sum_{x \equiv_{p^n} a} \left\{\infty, \frac{x}{p^{n+1}}\right\}_{f_\alpha} = \left\{\infty, \frac{a}{p^n}\right\}_{U_p f_\alpha} = \alpha \left\{\infty, \frac{a}{p^n}\right\}_{f_\alpha}.$$

$\square$

§ Interpolation formula

Facts
- $L(E,1) \in \mathbb{R}$
- $L(E,s) = \dfrac{(2\pi)^s}{\Gamma(s)} \displaystyle\int_0^\infty f_E(it)\, t^s \frac{dt}{t}$

Then $L(E,1) = I_E(\infty, 0)$

In particular

$$\frac{L(E,1)}{\Omega^+} = \{\infty, 0\}_E \quad \text{as} \quad \{\infty, 0\}^- = 0.$$

$$\mathbb{Z}_p^\times = B(1,0)$$

$$\mathcal{L}_p(E,\mathbb{1}) = \int_{\mathbb{Z}_p^\times} 1 \, d\mu_E(x) = \{\infty, 1\}_{f_\alpha}$$

$$\gamma = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$= \{\infty, 0\}_{f_\alpha}$$

$$= (1-\alpha^{-1}) \frac{L(E,1)}{\Omega^+}.$$