Higher Hida Theory on the modular curve

Aim of (classical) Hida Theory

Construct p-adic analytic families of Hecke eigenforms.

Some great aplications: progress towards

- Langlands' program
  e.g. (Wiles) attached Galois repr. to Hilbert eigenforms of wt 1.

- BSD conjecture
  e.g. Bertolini-Darmon, Skinner-Urban

- Algebraicity of Stark-Heegner points
  e.g. Bertolini-Darmon

- Explicit class field theory
  Darmon-Pozzi-Vonk, Dasgupta-Kakde

Aim of higher Hida theory

Construct p-analytic families of Hecke eigenclasses in the coherent coh. of Shimura varieties.

Some great applications

- Potential modularity of abelian surfaces over tot. real $\mathbb{F}$ fields (Boxer - Calegari - Gee - Pilloni)

- Bloch-Kato conjecture beyond GL_2 (Loeffler - Zerbes, L - Pilloni - Skinner - Z.)

Goal of the seminar

Understand basic ideas of higher Hida theory in the simplest case of modular curves.

Set up $N \geq 3$ coprime to p

$X_{/\mathbb{Z}_p}$ compactified modular curve of level $\Gamma_1(N)$

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\pi"'] \\ X \arrow[u, bend right, "e"'] \end{tikzcd}$$

universal semi-abelian scheme

$\omega := e^* \Omega^1_{\mathcal{E}/X}$ line bundle

cotangent space at origin.

then

$$M_k(\Gamma_1(N)) = H^0(X, \omega^k)$$

$$S_k(\Gamma_1(N)) = H^0(X, \omega^k(-D))$$

where D is the boundary divisor.

Let $\Lambda = \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$ $\quad u \in \mathbb{Z}_p^\times$

$\forall k \in \mathbb{Z}$ $\quad k : \Lambda \longrightarrow \mathbb{Z}_p \quad [u] \longmapsto u^k$

Main Theorem

$\exists$ finite, projective Λ-modules M, N with prime to p Hecke action endowed with canonical, Hecke equiv. isomorphisms $\forall k \geq 3$

- $M \otimes_{\Lambda,k} \mathbb{Z}_p = e_p H^0(X, \omega^k)$
- $N \otimes_{\Lambda,k} \mathbb{Z}_p = e_p H^1(X, \omega^{2-k}(-D))$

Moreover,

$\exists\ M \times N \longrightarrow \Lambda$ perfect pairing interpolating classical Serre duality.

Comments

1) $e_p = \varinjlim T_p^{n!}$ ordinary projector

2) M, N have geometric construction
N isn't simply defined as M^*.

Question

Let $\mathbb{h}^{\text{ord}} \subseteq N$ be the Hecke algebra

Given $\lambda : \mathbb{h}^{\text{ord}} \longrightarrow \Lambda$ a Λ-algebra homomorphism,

how do we find $n_\lambda \in N$ s.t.

$$T \cdot n_\lambda = \lambda(T) \cdot n_\lambda \quad \forall\, T \in \mathbb{h}^{\text{ord}} ?$$

Strategy of the proof

1) Igusa tower construction

Let $\mathfrak{X}$ be the p-adic completion of $X/\mathbb{Z}_p$.

Concretely,

$$\mathfrak{X} = \varinjlim_n X_n \qquad \text{colimit in locally ringed spaces}$$

$$X_n := X \times_{\mathbb{Z}_p} \mathbb{Z}/p^n\mathbb{Z}$$

$\mathfrak{X}^{ord}$ its ordinary locus
(c.f. talk on Hasse invariant)

$$\exists \quad \begin{array}{c} Ig \\ \pi \downarrow \\ \mathfrak{X}^{ord} \end{array} \circlearrowright \mathbb{Z}_p^\times \qquad \text{profinite étale over}$$

(c.f. talk on p-adic theory)

and $\exists\ \Omega \subseteq \pi_{*}(\mathcal{O}_{\mathfrak{Ig}} \hat{\otimes}_{\mathbb{Z}_p} \Lambda)$ invertible $\mathcal{O}_{\mathfrak{X}^{ord}} \hat{\otimes} \Lambda$-module s.t.

$$\Omega \otimes_{\Lambda, k} \mathbb{Z}_p \simeq \omega^{k}|_{\mathfrak{X}^{ord}} .$$

<u>Definition</u>

$$M := e(U_p)\, H^0(\mathfrak{X}^{ord}, \Omega)$$

$$N := e(F)\, H^1_c(\mathfrak{X}^{ord}, \Omega(-D))$$

Now, it is easy to see that

- $M \otimes_{\Lambda, k} \mathbb{Z}_p = e(U_p)\, H^0(\mathfrak{X}^{ord}, \omega^{k})$
- $N \otimes_{\Lambda, k} \mathbb{Z}_p = e(F)\, H^1_c(\mathfrak{X}^{ord}, \omega^{2-k}(-D))$

2) Classicality

Need to prove that $\forall k \geq 3$

- $e_p H^0(X, \omega^k) = e(U_p) H^0(\mathfrak{X}^{ord}, \omega^k)$
- $e_p H^1(X, \omega^{2-k}(-D)) = e(F) H^1_c(\mathfrak{X}^{ord}, \omega^{2-k}(-D))$

Enough to prove it mod p
(i.e. on special fiber)

then use Nakayama.

Classicality mod p relies on careful study of T_p-correspondence

$$\begin{tikzcd} & X_0(p)_1 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X_1 & & X_1 \end{tikzcd} \qquad \begin{tikzcd} & (E_1 \to E_2) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ E_1 & & E_2 \end{tikzcd}$$

deg p isogeny

Here the basic idea is that

- $H^0(X_1^{ord}, \omega^k) = \varinjlim_n H^0(X_1, \omega^k(n \cdot SS))$
- $H^1_c(X_1^{ord}, \omega^{2-k}(-D)) = \varprojlim_n H^1(X_1, \omega^{2-k}(-D - nSS))$

and after applying ordinary projector

all transitions become isomorphisms.

Talks for the seminar

1) Geometry of $X_0(p)/\mathbb{Z}_p$ (Katz-Mazur)
David

2) Hasse invariant (Katz-Mazur, Katz)
Sam

3) T_p correspondence (ch. 3 Boxer-Pilloni)
Hung

4) Classicality mod p (ch. 4.1 BP)
Haodong

5) Igusa tower and p-adic theory (ch. 4.2 BP)
Avi

6) Main result and possibly duality (ch. 4.2.5 BP)
Jiaxi