The Hasse Invariant

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§1. Recall on modular forms

Given a moduli problem $\mathcal{P}$ which is finite étale over (Ell/S) and rep'ble, we get $\begin{matrix} E \\ \downarrow \\ \mathcal{M}(\mathcal{P}) \end{matrix}$, and we define $\omega = 0^* \Omega^1_{E/\mathcal{M}(\mathcal{P})}$, an inv. sheaf on $\mathcal{M}(\mathcal{P})$.

Classically, taking $S = \mathrm{Spec}(\mathbb{C})$ and $\mathcal{P} = [\Gamma_{0,1,\phi}(N)]$, get ω on $Y_{0,1,\phi}(N)$ $(N \gg 0)$. Then a modular form of wt k is a section in $H^0(Y_{0,1,\phi}(N), \omega^{\otimes k})$.

More algebraically, we can make this definition on the moduli problem itself:

A weakly mod. fm. of wt k and "level $\mathcal{P}$" for an R-scheme S is an assignment

$$F: \mathcal{P} \ni (E/S', \text{extra data}) \longmapsto F(E/S', \text{extra data}) \in H^0(S', \omega_{E/S}^{\otimes k})$$

which is functorial in E: for cartesian diagrams of S-sch's,

$$\begin{tikzcd} E_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E_1 \arrow[d] \\ S_2 \arrow[r] & S_1 \end{tikzcd}$$

preserving "extra data", we have $F(E_2/S_2, \ldots)$ is obtained from $F(E_1/S_1, \ldots)$ by pullback.

Fact For $R = \mathbb{Z}((q))$, $\mathbb{G}_m/q^{\mathbb{Z}} =: \mathrm{Tate}(q)$ defines an ell. curve$/\mathbb{Z}((q))$, called the Tate curve. It comes equipped w/ a canonical diff'l ω_{can}: We have an iso

$$\widehat{\mathrm{Tate}(q)} \xrightarrow[\phi_{can}]{\sim} \widehat{\mathbb{G}}_m = 1 + X \quad (\text{as formal groups})$$

and $\omega_{can} = \phi_{can}^*\left(\frac{dX}{X}\right)$

Def If $S = \mathrm{Spec}(R)$, F a weakly mod fm of wt k and level 1 for S, then the q-expansion of F is given by

$$F(\mathrm{Tate}(q)/\mathrm{Spec}(R((\mathbb{Z})))) = (q\text{-xpn}) \cdot \omega_{can}^{\otimes k}$$

This coincides w/ the classical notion of q-xpn for on $\mathbb{H}$.

F is a mod fm if its q-xpn is in $R[[q]]$.

§2 Hasse invariant

Recall Given $E/S/\mathbb{F}_p$, we had $F: E \to E^{(p)}$ the Frob. map, w/ dual $V: E^{(p)} \to E$

Def The Hasse invariant is

$$\begin{aligned} A = \mathrm{tg}(V) &\in \mathrm{Hom}_S(\mathrm{Lie}(E^{(p)}/S), \mathrm{Lie}(E/S)) \\ &= \mathrm{Hom}_S(\mathrm{Lie}(E/S)^{\otimes p}, \mathrm{Lie}(E/S)) \\ &= H^0(S, \omega_{E/S}^{\otimes(p-1)}) \end{aligned}$$

A is a mod fm of wt $p-1$ over $\mathbb{F}_p$.

Rmk Locally on S, if D is a derivation generating $\mathrm{Lie}(E/S)$, then $tg(F)D =: D^{(p)}$ generates $\mathrm{Lie}(E^{(p)}/S)$.

If $\omega \in H^0(S, \omega_{E/S})$ is dual to D, then

$$\begin{aligned}
\mathrm{Lie}(E/S) \ni D^p &= tg([p])D \\
&= tg(V)\,tg(F)D \\
&= tg(V)\,D^{(p)} \\
&= \underset{\substack{\cap \\ H^0(S,\,\omega_{E/S}^{\otimes(p-1)})}}{A(E)}\; \underset{\in \mathrm{Lie}(E/S)^{\otimes p}}{D^{\otimes p}} \\
&= A(E,\omega)\,\omega^{\otimes(p-1)} D^{\otimes p} \qquad (\text{where, for a mod fm } f, \text{ we write } f(E) = f(E,\omega)\omega^{\otimes k}.) \\
&= A(E,\omega)D.
\end{aligned}$$

Prop The q-xpn of A is $1 \in R((q))$ for any $\mathbb{F}_p$-alg R.

pf We must compute $A(\mathrm{Tate}(q), \omega_{can})$. We have

$$\widehat{\mathrm{Tate}(q)} \xrightarrow{\sim} \widehat{\mathbb{G}}_m$$

$$\omega_{can} \longleftrightarrow \frac{dX}{X}$$

$$\text{dual to}$$

$$D_{can} \longleftrightarrow X\frac{d}{dX}$$

So $\left(X\frac{d}{dX}\right)^p = A(\mathrm{Tate}(q), \omega_{can}) \cdot X\frac{d}{dX}$. But

$$\left(X\frac{d}{dX}\right)\left(\sum_{n=0}^{\infty} a_n X^n\right) = \sum_{n=0}^{\infty} n a_n X^n.$$

So

$$\left(X\frac{d}{dX}\right)^p\left(\sum_{n=0}^{\infty} a_n X^n\right) = \sum_{n=0}^{\infty} n^p a_n X^n = \sum_{n=0}^{\infty} n a_n X^n = \left(X\frac{d}{dX}\right)\left(\sum_{n=0}^{\infty} a_n X^n\right).$$

Thus

$$A(\mathrm{Tate}(q), \omega_{can})X\frac{d}{dX} = \left(X\frac{d}{dX}\right)^p = X\frac{d}{dX} \qquad \square.$$

Prop $\forall$ k-perfect field of char p, R-local Artin k-alg, E/R ell curve, TFAE

(1) $tg(V) = 0$ on $\mathrm{Lie}(E^{(p)}/\mathrm{Spec}\,R)$.

(2) $\exists$ E_0/k ss s.t. $E = E_0 \otimes_k R$.

Rmk This says A has zeros only at ss E, and these zeros are simple.

pf (2) $\Rightarrow$ (1) Omitted

(1) $\Rightarrow$ (2) R is a $\mathbb{Z}_p$-module, so $\mu_{p-1} \subseteq \mathbb{Z}_p^\times$ acts on R.

Let X- coord for $\hat{E}$ which linearizes action of μ_{p-1}. Then

$$V(X) = \sum a_n X^n \quad \text{w/ } a_n = 0 \text{ unless } n \equiv 1 \pmod{p-1}$$
$$= t_g(V) + a_p X^p + \dots$$
$$= a_p X^p + \dots$$

Now $(a_p \bmod m_R) \in k^\times$ since E is ss $\Rightarrow \ker(V) = \ker(F : E^{(p)}_{/k} \to E^{(p^2)}_{/k})$, and $\ker(F : E_{/k} \to E^{(p)}_{/k})$ doesn't kill X.
Thus

$$V(X) = X^p \cdot (\text{invertible})$$

$\Rightarrow$

$$\ker(V) = \ker(F : E^{(p)} \to E^{(p^2)}).$$

So

$$E \cong E^{(p)}/\ker(V) \cong E^{(p)}/\ker(F) \cong E^{(p^2)}$$

Iterating:

$$E \cong E^{(p^{2n})}.$$

But the p^{2n}th power map factors through k for $n \gg 0$. So

$$E \cong E^{(p^{2n})} \cong (E \otimes_k R)^{(p^{2n})} \otimes_k R.$$

<u>Def</u> $[\mathrm{ord}]$ is the moduli problem $\begin{cases} \{*\} & \text{if } E/S \text{ ord;} \\ \emptyset & \text{if } E/S \text{ ss.} \end{cases}$

<u>Prop</u> $[\mathrm{ord}]$ is rel. rep. and an open imm.

<u>pf</u> it is obtained locally by inverting A:

$$[\mathrm{ord}](S) = \left\{ \begin{array}{l} \text{open subscheme of } S \text{ where} \\ A \in H^0(S, \omega_{E/S}^{\otimes(p-1)}) \text{ is inv.} \end{array} \right\}.$$

So we can define, for any $\mathcal{P}$, $\mathcal{P}^{\mathrm{ord}} = \begin{cases} \text{level str's on } E \text{ def'd by } \mathcal{P} & \text{if } E \text{ ord} \\ \emptyset & \text{if } E \text{ ss} \end{cases}$