Cohomological correspondence T_p

$f: X \to Y$ morphism of locally Noetherian scheme

Suppose f is finite flat

$\Rightarrow$ f^* f_* are exact

f_* has a right adjoint, say $f^!$

$$f^! F := \mathcal{H}om_{\mathcal{O}_Y}(f_* \mathcal{O}_X, F) \cong f^! \mathcal{O}_Y \otimes_{\mathcal{O}_X} f^* F$$

In the affine case

$\cancel{\mathrm{Hom}_A(B, N)}$

$\cong \mathrm{Hom}_A(B, A) \otimes_B B \otimes_A N$

If $Y = \mathrm{Spec}\, A$, $X = \mathrm{Spec}\, B$, $F = \widetilde{N}$

$$f^! F := \widetilde{\mathrm{Hom}_A(B, N)}$$

Adjointness is given by trivial duality

$$\mathrm{Hom}_A(M_A, N) \cong \mathrm{Hom}_B(M, \mathrm{Hom}_A(\ , N))$$

$$\mathrm{Tr}_f : f_* \mathcal{O}_X \to \mathcal{O}_Y$$

$$(\mathrm{Tr}: B \to A)$$

$\rightsquigarrow$ $\mathcal{O}_X \to f^! \mathcal{O}_Y$ in $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, f^! \mathcal{O}_Y)$

$\rightsquigarrow$ $f^* \to f^!$ natural isomorphism

$$f^* F \otimes_{\mathcal{O}_X} \mathcal{O}_X \to f^* F \otimes_{\mathcal{O}_X} f^! \mathcal{O}_Y$$

Given Y

A finite flat correspondence is

(X, p_1, p_2)

$p_1, p_2 : X \to Y$ finite flat

$$\begin{tikzcd} & X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ Y & & Y \end{tikzcd}$$

A finite flat cohomological correspondence (for $F \in Coh(Y)$)

is $\underline{T \in Hom_{\mathcal{O}_X}(p_2^* F \to p_1^! F)}$

$F \to p_{2*} p_2^* F$

$\rightsquigarrow \; R\Gamma(Y, F) \longrightarrow R\Gamma(Y, p_{2*} p_2^* F)$

$= R\Gamma(X, p_2^* F) \xrightarrow{T} R\Gamma(X, p_1^! F)$

$= R\Gamma(Y, p_{1*} p_1^! F) \to R\Gamma(Y, F)$

$p_{1*} p_1^! F \to F$

$\rightsquigarrow \; \underline{T \in End(R\Gamma(Y, F))}$

Let $N \geq 3$, $p \nmid N$

$Y/_{\mathbb{Z}_p}$, $Y = Y_1(N)$

Y is a smooth curve over $\mathbb{Z}_p$.

$Y_0(p)$ (level $\Gamma_1(N) \cap \Gamma_0(p)$)

$$\tilde{P}(S) := \underline{\{ (E, \underline{\eta_E}, \underline{F, \eta_F,} \varphi : E \to F) \}} / \sim$$

$E/_S$, $F/_S$ Elliptic curves

$\varphi : E \to F$ S-isogeny of degree p

η_E, η_F are $\Gamma_1(N)$-structure

$\varphi(\eta_E) = \eta_F$

$\tilde{P}$ is representable

$\tilde{P} \Longleftrightarrow ([\Gamma_1(N)], p\text{-isogeny})$

$\Updownarrow$

$([\Gamma_1(N) \cap \Gamma_0(p)])$

$$Y_0(p) \xrightarrow{p_1} Y$$

If $\varphi : E \to F$ degree p, given η_F

Want η_E s.t. $\varphi(\eta_E) = \eta_F$

Consider $\varphi^\vee : F \to E$

$$\varphi^\vee \psi_F = p \psi_E$$

$$\rightsquigarrow \psi_E := p^{-1} \psi_F \in E[N].$$

$p_2 : Y_0(p) \to Y$

$$\begin{tikzcd} & Y_0(p) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ Y & & Y \end{tikzcd} \qquad \begin{tikzcd} & (E, \psi_E, F, \psi_F, \varphi) \arrow[dl] \arrow[dr] & \\ (E, \psi_E) & & (F, \psi_F) \end{tikzcd}$$

$Y_0(p)$ is not smooth, but is

$Y_0(p) \to Y$ is finite etale neric fiber

$$\tilde{P}(S) = \{ (E, \psi_E, F, \psi_F, \varphi) \} / \sim$$

$$\varphi : F \to E$$

$$\varphi(\psi_F) = p \psi_E$$

$Y_0(p)_{\mathbb{F}_p} \xrightarrow{p_1} Y_{\mathbb{F}_p}$:

$Z \subset Y_{\mathbb{F}_p}$ a component

$Z^F, Z^V \to Z$

$Z^F \to Z$ is isomorphic

$Z^V \cong Z$ but $Z^V \to Z$ is purely inseparable of degree p.

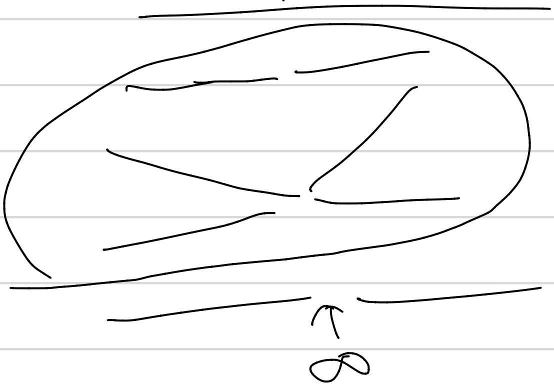
$j : Y \longrightarrow \mathrm{Spec}(\mathbb{Z}_p[j])$ finite flat

Define X as follows:

$X \longrightarrow \mathbb{P}^1_{\mathbb{Z}_p}$

$X|_{\mathbb{A}^1_{\mathbb{Z}_p}} = Y$

$X|_{\mathrm{Spec}\, \mathbb{Z}_p[\frac{1}{j}]}$ is the integral closure of $\mathrm{Spec}\, \mathbb{Z}_p[\frac{1}{j}]$



Also get $X_0(p)$, regular over $\mathbb{Z}_p$

$\Rightarrow$ Can extend

$$\begin{tikzcd} & X_0(p) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X' & & X \end{tikzcd}$$

$E \to Y$ universal curve

$\hat{E} \to X$ semi-abelian group scheme

Lemma

$G, H \to S$ semi-abelian group scheme

S noetherian

$U \subset S$ dense open

$\phi_U : G_U \to H_U$

$\rightsquigarrow \exists\, \phi : G \to H$ uniquely

Over $Y_0(p)$, $\exists\ \varphi : p_1^* E \to p_2^* E$

$X_0(p)\ \exists\ \varphi : p_1^* \hat{E} \to p_2^* \hat{E}$

Define $\omega := \omega_{\hat{E}/X} = e^* \Omega^1_{\hat{E}/X}$

$e : X \to \hat{E}$ unit section

Define $\pi_k : p_2^* \omega^k \to p_1^* \omega^k \quad \forall\, k \geq 0$

$\pi_k^\vee : p_1^* \omega^k \to p_2^* \omega^k$

$\pi_k, \pi_k^\vee$ are sum over generic fiber

π_k, $\pi_k^\vee$ $\forall\, \mathbb{Z}$ as on generic fiber

$$T^{naive}_{p,k} : p_2^* \omega^k \dashrightarrow p_1^! \omega^k$$

Def:

$$T^{naive}_{p,k} := \begin{array}{ccc} \pi_k : p_2^* \omega^k & \dashrightarrow & p_1^* \omega^k \\ \otimes & \otimes & \otimes \\ Tr_{p_1} : \mathcal{O}_{X_0(p)} & \longrightarrow & p_1^! \mathcal{O}_X \end{array}$$

normalization:

$$T_{p,k} := \frac{T^{naive}_{p,k}}{p^{\inf\{1,k\}}}$$

Prop. $T_{p,k}$ can be extended to

$$p_2^* \omega^k \longrightarrow \underline{p_1^! \omega^k}$$

Proof:

Both are invertible sheaves

$\Rightarrow$ only consider generic point on special fiber.

$\xi \longrightarrow p_1(\xi)$

ξ Verschiebung

$$\Rightarrow (p_2^* \omega^k)_\xi \xrightarrow{\cong} (p_1^* \omega^k)_\xi$$

$$\underline{\mathcal{O}_{X_0(p),\xi} \xrightarrow[\cong]{Tr} p_*(f^! \mathcal{O}_X)_\xi}$$

A, B discrete valuation ring

mix char, p is uniformizer,

residual extension is inseparable of degree p

$\Rightarrow P_2^* \omega^k \xrightarrow{\cong} p \cdot p_1^! \omega^k$

If ξ is Frobenius

$$(p_2^* \omega^k)_\xi \xrightarrow{\cong} p^k (p_1^* \omega^k)_\xi$$

$$Tr : \mathcal{O}_{X_0(p),\xi} \xrightarrow{\cong} (f^! \mathcal{O}_X)_\xi$$

$$\Rightarrow p_2^* \omega^k \xrightarrow{\cong} p^k p_1^! \omega^k$$

$\Rightarrow$ So we can divide $p^{\min\{1,k\}}$

$$T_{p,k}\left(\sum_n a_n q^n\right)$$

$$= \begin{cases} \sum_n a_{np} q^n + \chi(p) p^{k-1} \sum_n a_n q^{np} & \text{if } k \geq 1 \\ p^{1-k} \sum_n a_{np} q^n + \chi(p) \sum_n a_n q^{np} & \text{if } k \leq 1 \end{cases}$$

$D(T_{p,k}) = T_{p,2-k}$