Classicality mod p.

Setup.

$N \geq 3$ integer, p prime, $(p, N) = 1$.

$X \xrightarrow[\text{smooth}]{\text{proper}} \operatorname{Spec} \mathbb{Z}_p$ compactified modular curve of level $\Gamma_1(N)$

$\mathcal{E} \to X$ (with section e) semi-abelian scheme extending universal elliptic curve

$\omega = e^* \Omega^1_{\mathcal{E}/X}$.

$X_0(p)$ compactified modular curve of level $\Gamma_1(N) \cap \Gamma_0(p)$

classifying $(E, i_E, E \to F$ deg p isogeny$)$.

$p_1, p_2 : X_0(p) \to X$

$$\begin{tikzcd} & (E, i_E, E \to F) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ (E, i_E) & & (F, i_F) \end{tikzcd}$$

$\pi : p_1^* \mathcal{E} \to p_2^* \mathcal{E}$ part of universal object above $X_0(p)$.

Recall T_p.

$\forall k$, $\quad \pi_k : p_2^* \omega^k \dashrightarrow p_1^* \omega^k$ deduced from pulling back on

differentials $\quad \pi : p_1^* \Sigma \longrightarrow p_2^* \Sigma$.

$\otimes$ with $\quad tr_{p_1} : \mathcal{O}_{X_0(p)} \longrightarrow p_1^! \mathcal{O}_X$

$\Rightarrow \quad T_{p,k}^{naive} : \quad p_2^* \Sigma^k \dashrightarrow p_1^! \Sigma^k$

Normalization : $\quad T_{p,k} = p^{-\inf\{1,k\}} \, T_{p,k}^{naive}$.

Prop. $\quad T_{p,k} : p_2^* \omega^k \longrightarrow p_1^! \Sigma^k$.

Sketch of proof.

Localize at generic points ξ of special fibre.

① $\xi = V$. $\quad (tr_{p_1})_\xi : (p_1^* \mathcal{O}_X)_\xi \xrightarrow{\sim} p\,(p_1^! \mathcal{O}_X)_\xi$ and

differential isom. $\Rightarrow (T_{p,k}^{naive})_\xi : (p_2^* \omega^k)_\xi \longrightarrow p\,(p_1^! \omega^k)_\xi$

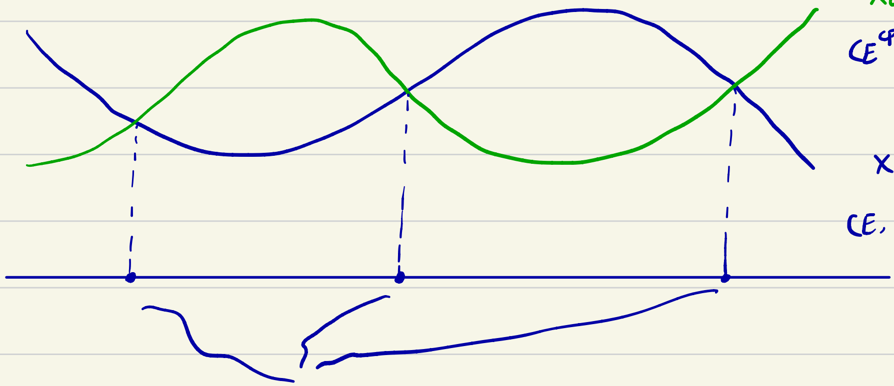
② $\xi = F$. $\quad$ differential map $\quad (p_2^* \omega)_\xi \xrightarrow{\sim} p\,(p_1^* \omega)_\xi$ and

$(tr_{p_1})_\xi$ isom. $\Rightarrow (T_{p,k}^{naive})_\xi : (p_2^* \omega^k)_\xi \longrightarrow p^k (p_1^! \omega^k)_\xi$.

Mod p theory : geometry of modular curves.

$X_1 \longrightarrow \mathrm{Spec}\,\mathbb{F}_p$ special fibre, X_1^{ord} ordinary locus.

$$X_0(p)_1 = X_0(p)_1^F \cup X_0(p)_1^V$$



$X_0(p)_1^V$

$(E^{(p)}, i_E^{(p)}, E^{(p)} \xrightarrow{V} E)$

$X_0(p)_1^F$

$(E, i_E, E \xrightarrow{F} E^{(p)})$

SS : supersingular

$i^F : X_0(p)_1^F \hookrightarrow X_0(p)_1$, $p_i^F = p_i \circ i^F$.

$i^V : X_0(p)_1^V \hookrightarrow X_0(p)_1$, $p_i^V = p_i \circ i^V$.

$p_2^V : X_0(p)_1^V \xrightarrow{\sim} X_1$, $p_1^V : X_1 \xrightarrow{\sim} X_0(p)_1^V \xrightarrow{p_1^V} X_1$

$(E^{(p)}, E^{(p)} \xrightarrow{V} E) \longmapsto E$

$E \longmapsto (E^{(p)}, E^{(p)} \xrightarrow{V} E) \longmapsto E^{(p)}$

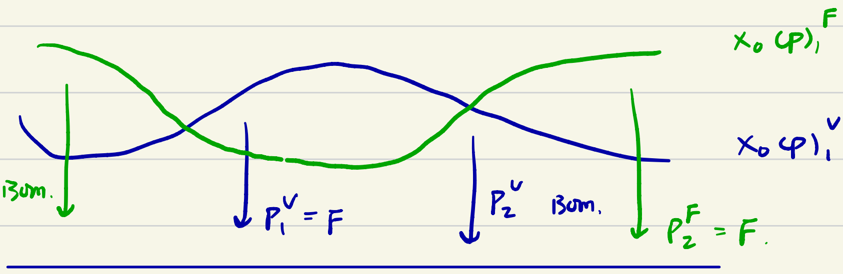
p_1^V is F (Frob of X_1).

$p_1^F : X_0(p)_1^F \xrightarrow{\sim} X_1$, $p_2^F : X_1 \xrightarrow{\sim} X_0(p)_1^F \xrightarrow{p_2^F} X_1$

$(E, E \xrightarrow{F} E^{(p)}) \longmapsto E$ $\qquad$ $E \longmapsto (E, E \xrightarrow{F} E^{(p)}) \longmapsto E^{(p)}$

p_2^F is F.

Picture



Mod p theory: analysis of T_p.

Lemma.
$$\begin{tikzcd} Y \arrow[r, hook, "i"] \arrow[dr, "g = fi"'] & X \arrow[d, "f"] \\ & Z \end{tikzcd}$$

i closed immersion

g, f finite flat

$\mathcal{F} \in \mathrm{Coh}(Z)$

$\Rightarrow i_* g^! \mathcal{F} = \Gamma_Y f^! \mathcal{F}$

subsheaf of sections killed by $\ker(\mathcal{O}_X \to i_* \mathcal{O}_Y)$.

Proof. $g^! = (fi)^! = i^! f^!$.

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Lemma.

If $k \geq 2$, we have factorization

$$\begin{tikzcd}
p_2^* \omega^k \arrow[r, "T_p"] \arrow[d] & p_1^! \omega^k \\
i_*^\vee p_2^{\vee,*} \omega^k \arrow[r] & i_*^\vee p_1^{\vee,!} \omega^k \arrow[u]
\end{tikzcd}$$

If $k \leq 0$, we have factorization

$$\begin{tikzcd}
p_2^* \omega^k \arrow[r, "T_p"] \arrow[d] & p_1^! \omega^k \\
i_*^F p_2^{F,*} \omega^k \arrow[r] & i_*^F p_1^{F,!} \omega^k \arrow[u]
\end{tikzcd}$$

Proof. $k \geq 2$

$$j^\vee : X_0(p)_1 \setminus X_0(p)_1^\vee \hookrightarrow X_0(p)_1$$

$$0 \to j_!^\vee j^{\vee,*} p_2^* \omega^k \to p_2^* \omega^k \to i_*^\vee \underbrace{i^{\vee,*} p_2^*}_{= \, p_2^{\vee,*}} \omega^k \to 0$$

ETS T_p vanish on $j_!^\vee j^{\vee,*} p_2^* \omega^k$.

Then

$$\begin{tikzcd}
T_p : p_2^* \omega^k \arrow[r] & i_*^\vee p_2^{\vee,*} \omega^k \arrow[r] \arrow[d] & p_1^! \omega^k \\
& \underline{\Gamma}_{X_0(p)_1^\vee} p_1^! \omega^k = i_*^\vee p_1^{\vee,!} \omega^k \arrow[ur] &
\end{tikzcd}$$

as image killed by $\ker(\mathcal{O}_{X_0(p)_1} \to i_*^\vee \mathcal{O}_{X_0(p)_1^\vee})$.

As $X_0(p)$, regular, enough to check $(T_p)_\xi$ at generic point ξ of $X_0(p)$, where ξ is of type F.

Since $k \geq 2$, $(T_{p,k}^{\text{naive}})_\xi : (p_2^* w^k)_\xi \longrightarrow p^k (p_1^! w^k)_\xi$

and $(T_{p,k})_\xi : (p_2^* w^k)_\xi \longrightarrow p^{k-1} (p_1^! w^k)_\xi$

$k-1 \geq 1$ so $T_{p,k}$ vanish at ξ.

Similar for $k \leq 0$. //

Prop.

For $k \geq 2$, T_p induces

$$p_2^*(w^k((np+k-2)SS)) \longrightarrow p_1^!(w^k(nSS))$$

for $k \leq 0$, T_p induces

$$p_2^*(w^k(-nSS)) \longrightarrow p_1^!(w^k((-np+k)SS)).$$

Proof. Assume $k \geq 2$.

By the lemma above, T_p is supported on $X_0(p)_1^\vee$.

The map $p_1^\vee$ can be viewed as $F: X_1 \longrightarrow X_1$

$$F(S): X_1(S) \longrightarrow X_1(S)$$

$$(E, i_E) \longmapsto (E^{(p)}, i_E^{(p)})$$

F is totally ramified of deg p $\Rightarrow$ $F^*(SS) = pSS$

Then

$$p_2^{\vee,*} \mathcal{O}_{X_1}(npSS) = \mathcal{O}_{X_0(p)_1^\vee}(npSS) = \mathcal{O}_{X_0(p)_1^\vee}(nF^*SS) = p_1^{\vee,*}\mathcal{O}_{X_1}(nSS)$$

tensor with $p_2^{\vee,*}\omega^\kappa \longrightarrow p_1^{\vee,!}\omega^\kappa$

get $p_2^{\vee,*}\omega^\kappa(npSS) \longrightarrow p_1^{\vee,!}\omega^\kappa(nSS)$

hence $T_p: p_2^*\omega^\kappa(npSS) \longrightarrow i_*^\vee p_2^{\vee,*}\omega^\kappa(npSS)$

$$\downarrow$$

$$i_*^\vee p_1^{\vee,!}\omega^\kappa(nSS)$$

$$\downarrow$$

$$p_1^!\omega^\kappa(nSS).$$

For $k \geq 3$, we can do more.

$p_2^{v,*} \omega^k \longrightarrow p_1^{v,!} \omega^k$ is tensor product of

$p_2^{v,*} \omega^2 \longrightarrow p_1^{v,!} \omega^2$ with $p_2^{v,*} \omega^{k-2} \longrightarrow p_1^{v,*} \omega^{k-2}$

The map $p_2^{v,*} \omega \longrightarrow p_1^{v,*} \omega$ comes from pullback of differentials of V: $p_2^{v,*} \varepsilon^{(p)} = p_1^{v,*} \varepsilon \longrightarrow p_2^{v,*} \varepsilon$

hence by definition is the Hasse invariant, which has zero locus at SS and simple zeros at SS.

Hence we get $p_2^{v,*} \omega(SS) \longrightarrow p_1^{v,*} \omega$.

i.e. $A \in H^0(p_2^{v,*} \omega^{p-1} (-SS))$.

Together we get

$$p_2^{*} (\omega^k ((np + k - 2) SS)) \longrightarrow p_1^{!} (\omega^k (n SS)).$$

Similar for $k \leq 0$.

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Cor.

(1) T_p acts on $R\Gamma(X_1, \omega^k(nss))$, $\forall\, n \geq 0$, $k \geq 2$

and the maps $R\Gamma(X_1, \omega^k(nss)) \longrightarrow R\Gamma(X_1, \omega^k(n'ss))$ are

equivariant for $0 \leq n \leq n'$.

(2) We have commutative diagram for $k \geq 0$, $n \geq 2$

$$\begin{tikzcd}
R\Gamma(X_1, \omega^k((np+k-2)ss)) \arrow[r, "T_p"] \arrow[dr, squiggly] & R\Gamma(X_1, \omega^k((np+k-2)ss)) \\
R\Gamma(X_1, \omega^k(nss)) \arrow[u] \arrow[r, "T_p"] & R\Gamma(X_1, \omega^k(nss)) \arrow[u]
\end{tikzcd}$$

(3) T_p acts on $R\Gamma(X_1, \omega^k(nss))$, $\forall\, n \leq 0$, $k \leq 0$

and the maps $R\Gamma(X_1, \omega^k(nss)) \longrightarrow R\Gamma(X_1, \omega^k(n'ss))$ are

equivariant for $n \leq n' \leq 0$.

(4) We have commutative diagrams for all $n \le 0$, $k \le 0$

$$\begin{tikzcd} R\Gamma(X_1, \omega^k(-nss)) \arrow[r, "T_p"] & R\Gamma(X_1, \omega^k(-nss)) \\ R\Gamma(X_1, \omega^k((-np+k)ss)) \arrow[u] \arrow[r, "T_p"] \arrow[ur, phantom, ""] & R\Gamma(X_1, \omega^k((-np+k)ss)) \arrow[u] \end{tikzcd}$$

For any k, let $H^i_c(X_1^{ord}, \omega^k) = \lim\limits_{n} H^i(X_1, \omega^k(-nss))$

and $H^i(X_1^{ord}, \omega^k) = \operatorname{colim}\limits_{n} H^i(X_1, \omega^k(nss))$.

$H^i_c(X_1^{ord}, \omega^k)$ profinite $\mathbb{F}_p$ - vector space

$H^i(X_1^{ord}, \omega^k)$ colim of finite $\mathbb{F}_p$ - vector space

Fact. V finite $\mathbb{F}_p$ - vector space, $T \in \operatorname{End}_{\mathbb{F}_p} V$

then $\{T^{n!}\}$ converges to a projector $e(T)$.

Also true for profinite $\mathbb{F}_p$ - vector space

and colim of finite $\mathbb{F}_p$ - vector space.

Cor.

(1) $k \geq 2$, T_p acts on $H^i(X_1^{ord}, \omega^k)$ and $\{T_p^{n!}\}$ converges to a projector $e(T_p)$.

$k \leq 0$, T_p acts on $H_c^i(X_1^{ord}, \omega^k)$ and $\{T_p^{n!}\}$ converges to a projector $e(T_p)$.

(2) $k \geq 3$, $e(T_p) H^i(X_1^{ord}, \omega^k) = e(T_p) H^i(X_1, \omega^k)$

$k = 2$, $e(T_p) H^i(X_1^{ord}, \omega^2) = e(T_p) H^i(X_1, \omega^2(SS))$.

(3) $k \leq -1$, $e(T_p) H_c^i(X_1^{ord}, \omega^k) = e(T_p) H^i(X_1, \omega^k)$

$k = 0$, $e(T_p) H_c^i(X_1^{ord}, \mathcal{O}_{X_1}) = e(T_p) H^i(X_1, \omega^k(-SS))$.

Proof. $k \geq 3$, $\forall\, n > 0$, $\exists\, 0 \leq N < n$ s.t. $n \leq Np + k - 2$ then

$$\begin{array}{ccc} H^i(X_1, \omega^k(nSS)) & \longrightarrow & H^i(X_1, \omega^k((Np+k-2)SS)) \\ & & \downarrow T_p \\ & & H^i(X_1, \omega^k(NSS)). \end{array}$$

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