The Igusa tower and the p-adic theory

August 17, 2021

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- $\omega =$ line bundle arising from universal elliptic curve as in previous talks

Igusa tower

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Hodge-Tate map $T_p(E)^{\text{ét}} \to \omega$ induces $T_p(E)^{\text{ét}} \otimes \mathcal{O}_{\mathfrak{X}^{\mathrm{ord}}} \xrightarrow{\sim} \omega$ induces

$$\begin{array}{ccc} \mathrm{Ig} & \longrightarrow & \{\text{trivializations of } \omega\} \\ & \searrow \quad \swarrow & \\ & \mathfrak{X}^{\mathrm{ord}} & \end{array}$$

The sheaf ω^κ

Set $\omega^\kappa = (\mathcal{O}_{\mathrm{Ig}} \widehat{\otimes} \Lambda)^{\mathbb{Z}_p^\times}$ with the diagonal action. For each $k \in \mathbb{Z}$, $x \mapsto x^k$ on $\mathbb{Z}_p^\times$ extends to $k : \Lambda \to \mathbb{Z}_p$.

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Lemma 1

As a sheaf of $\Lambda \otimes \mathcal{O}_{\mathfrak{X}^{\mathrm{ord}}}$-modules, ω^κ is invertible, and for each k we (canonically) recover $\omega^k \simeq \omega^\kappa \otimes_{\Lambda,k} \mathbb{Z}_p$.

Proof: compare sections and use the above diagram

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- Choose a coherent extension $\mathscr{F}_n$ of $\omega^\kappa/\mathfrak{m}_\Lambda^n$ to X_n, and let $\mathscr{I}$ be the ideal of $X_n \setminus X_n^{\text{ord}}$
- $H^1_c(X_n^{\text{ord}}, \omega^\kappa/\mathfrak{m}_\Lambda^n) := \lim_m H^1(X_n, \mathscr{I}^m \mathscr{F}_n)$ (independent of $\mathscr{F}_n$ by results of Hartshorne)

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Lift to $F : \text{Ig} \to \text{Ig}$ by composition:

$$(E, \psi : \mathbb{Z}_p \xrightarrow{\sim} T_p(E)^{\text{ét}}) \mapsto (E/\mu_p, \psi \xrightarrow{\sim} T_p(E)^{\text{ét}} \xrightarrow{\sim} T_p(E/\mu_p))$$

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Pullback $F^*\mathcal{O}_{\text{Ig}} \to \mathcal{O}_{\text{Ig}}$, trace $\text{tr} : F_*\mathcal{O}_{\text{Ig}} \to \mathcal{O}_{\text{Ig}}$

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Proof.

We have $\mathrm{tr}(F_*\mathcal{O}_{\mathfrak{X}^{\mathrm{ord}}}) \subseteq p\mathcal{O}_{\mathfrak{X}^{\mathrm{ord}}}$, and Ig is the fiber product

$$\begin{array}{ccc} \mathrm{Ig} & \xrightarrow{F} & \mathrm{Ig} \\ \downarrow & & \downarrow \\ \mathfrak{X}^{\mathrm{ord}} & \xrightarrow{F} & \mathfrak{X}^{\mathrm{ord}} \end{array}$$

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By equivariance define $F : F^*\omega^\kappa \to \omega^\kappa$ and $U_p : F_*\omega^\kappa \to \omega^\kappa$ by tensoring $F^*\mathcal{O}_{\mathrm{Ig}} \to \mathcal{O}_{\mathrm{Ig}}$ and $\frac{1}{p}\,\mathrm{tr}$ with Λ and taking $\mathbb{Z}_p^\times$-invariants.

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The universal isogeny $\pi : E \to E/\mu_p$ induces $\pi^* : F^*\omega \to \omega$, and similarly the dual induces $\hat{\pi}^* : \omega \to F^*\omega$.

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The maps $p^{-k}(\pi^*)^k$ *and* $F_*\omega^k \xrightarrow{(\hat{\pi}^*)^k} F_*F^*\omega^k \xrightarrow{\frac{1}{p}\operatorname{tr}} \omega^k$ *are the specializations of* F *and* U_p *respectively.*

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Proof.

We have a commutative diagram

$$\begin{array}{ccc} T_p(E)^{\text{ét}} & \longrightarrow & T_p(E/\mu_p)^{\text{ét}} \\ \downarrow & & \downarrow \\ \omega & \xrightarrow{\hat{\pi}^*} & F^*\omega \end{array}$$

so $F : F^*\omega^k \to \omega^k$ is given by $(\hat{\pi}^*)^{-k} = p^{-k}(\pi^*)^k$, and similarly for U_p. □

As cohomological correspondences

$X_0 \to \mathrm{Spa}(\mathbb{Z}_p, \mathbb{Q}_p)$ adic modular curve of level $\Gamma_0(p) \cap \Gamma_1(N)$, with projections $p_1, p_2 : X_0 \to X$

$\mathfrak{X}_0$, $\mathfrak{X}_0^{\mathrm{ord}}$, etc.

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On $\mathfrak{X}_0^{\mathrm{ord},F}$, p_1 is an isomorphism and $p_2 = F$, on $\mathfrak{X}_0^{\mathrm{ord},V}$ p_2 is an isomorphism and $p_1 = F$. Therefore on $\mathfrak{X}_0^{\mathrm{ord},F}$

$$F : F^*\omega^\kappa \simeq p_2^*\omega^\kappa \to \omega^\kappa \simeq p_1^!\omega^\kappa$$

and on $\mathfrak{X}_0^{\mathrm{ord},V}$

$$U_p : F_*\omega^\kappa \simeq p_{1*}p_2^*\omega^\kappa \to \omega^\kappa \quad \rightsquigarrow \quad U_p : p_2^*\omega^\kappa \to p_1^!\omega^\kappa$$

so we can view F and U_p as cohomological correspondences supported on $\mathfrak{X}_0^{\mathrm{ord},F}$, $\mathfrak{X}_0^{\mathrm{ord},V}$.

Comparison with T_p

Similarly let T_p^F, T_p^V be the projections onto $\mathfrak{X}_0^{\mathrm{ord},F}$, $\mathfrak{X}_0^{\mathrm{ord},V}$, and specialize all the operators to ω^k.

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Lemma 4

If $k \geq 1$, we have $T_p^F = p^{k-1}F$, $T_p^V = U_p$, and $T_p = p^{k-1}F + U_p$; if $k \leq 1$, we have $T_p^F = F$, $T_p^V = p^{1-k}U_p$, and $T_p = F + p^{1-k}U_p$.

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Proof.

$T_{p,k}^{\mathrm{naive},F}$, $T_{p,k}^{\mathrm{naive},V}$ are given by pullback by π and $\hat{\pi}$, so the claim follows from Lemma 3 by comparing normalizations. □

Main theorem

Theorem 5

There are locally finite actions of F and U_p on $H^1_c(\mathfrak{X}^{\mathrm{ord}}, \omega^\kappa)$ and $H^0(\mathfrak{X}^{\mathrm{ord}}, \omega^\kappa)$ respectively, and $e(F)H^1_c(\mathfrak{X}^{\mathrm{ord}}, \omega^\kappa)$ and $e(U_p)H^0(\mathfrak{X}^{\mathrm{ord}}, \omega^\kappa)$ are finite projective Λ-modules. Further for $k \leq -1$ we have

$$e(F)H^1_c(\mathfrak{X}^{\mathrm{ord}}, \omega^\kappa) \otimes_{\Lambda,k} \mathbb{Z}_p = e(T_p)H^1(X, \omega^k)$$

and for $k \geq 3$

$$e(U_p)H^0(\mathfrak{X}^{\mathrm{ord}}, \omega^\kappa) \otimes_{\Lambda,k} \mathbb{Z}_p = e(T_p)H^0(X, \omega^k).$$

The cases are similar, so we'll look at the statement for F.

Action on cohomology

First: define action of F on $H^1_c(\mathfrak{X}^{\mathrm{ord}}, \omega^\kappa) = \lim_n H^1_c(X_n, \omega^\kappa/\mathfrak{m}_\Lambda^n)$.

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We can extend to $X_{0,n}$ by choosing an extension $\mathscr{F}$ and taking (co)limits over $\mathscr{I}^{-m}\mathscr{F}$. As in Hung's talk this yields an endomorphism $F : H^1_c(X_n, \omega^\kappa/\mathfrak{m}_\Lambda^n)$, and taking the limit gives the desired action.

Finiteness: base case

Look at the case $n = 1$, i.e. mod p: sections of $\omega^\kappa/\mathfrak{m}_\Lambda = (\mathcal{O}_{\mathrm{Ig}}\widehat{\otimes}\Lambda)^{\mathbb{Z}_p^\times}/\mathfrak{m}_\Lambda$ are $\mathbb{Z}_p^\times$-invariant functions valued in $\mathbb{F}_p$ on pairs $(E, \psi : \mathbb{F}_p \to E[p])$, which decompose as

$$\omega^\kappa/\mathfrak{m}_\Lambda = \bigoplus_{j=-p+1}^{-1} \omega^j,$$

so

$$H_c^1(X_1^{\mathrm{ord}}, \omega^\kappa/\mathfrak{m}_\Lambda) = \bigoplus_{j=-p+1}^{-1} H_c(X_1^{\mathrm{ord}}, \omega^j).$$

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By Lemma 4, the action of F on the left is given by T_p on the right (since $j \leq -1$) so it follows from the mod p case that the action is locally finite and after applying projectors both sides are finite $\mathbb{F}_p$-vector spaces.

Finiteness: induction

$$0 \to \omega^\kappa \otimes (\mathfrak{m}_\Lambda^n/\mathfrak{m}_\Lambda^{n+1}) \to \omega^\kappa/\mathfrak{m}_\Lambda^{n+1} \to \omega^\kappa/\mathfrak{m}_\Lambda^n \to 0$$

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induces

$$0 \to H^1_c(X^{\mathrm{ord}}_{n+1}, \omega^\kappa \otimes (\mathfrak{m}_\Lambda^n/\mathfrak{m}_\Lambda^{n+1})) \to H^1_c(X^{\mathrm{ord}}_{n+1}, \omega^\kappa/\mathfrak{m}_\Lambda^{n+1}) \to H^1_c(X^{\mathrm{ord}}_{n+1}, \omega^\kappa/\mathfrak{m}_\Lambda^n) \to 0$$

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??

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so the result follows by induction plus a lemma for local finiteness.

Equalities

Finally,

$$e(F)H^1_c(\mathfrak{X}^{\mathrm{ord}}, \omega^\kappa) \otimes_{\Lambda,k} \mathbb{Z}_p = e(F)H^1_c(\mathfrak{X}^{\mathrm{ord}}, \omega^k),$$

and $X \to \mathfrak{X}$ induces $H^1_c(\mathfrak{X}^{\mathrm{ord}}, \omega^k) \to H^1(X, \omega^k)$.

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and $X \to \mathfrak{X}$ induces $H^1_c(\mathfrak{X}^{\text{ord}}, \omega^k) \to H^1(X, \omega^k)$. Applying projectors,

$$e(F)H^1_c(\mathfrak{X}^{\text{ord}}, \omega^k) \to e(T_p)H^1(X, \omega^k)$$

is a map of finite free $\mathbb{Z}_p$-algebras and is an isomorphism modulo p for $k \leq -1$ by Haodong's talk, so is an isomorphism.