Igusa tower : $(\rightsquigarrow W^{K^{un}} \simeq (\mathcal{O}_{Ig} \otimes \Lambda)^{\mathbb{Z}_p^*})$

$N \geq 3$, $(N, p) = 1$.

$Ig_n : R \in Nilp_{\mathbb{Z}_p}^{op} \to (E/R, \varphi_N, \varphi_{p^n}) / \simeq$

$\varphi_{p^n} : E[p^n]^0 \to \mu_{p^n}$.

Ig_n is representable by a formal affine schema.

Let $n \to \infty$.

$Ig : R \in Nilp_{\mathbb{Z}_p}^{op} \to (E/R, \varphi_N, \varphi_{p^\infty}) / \simeq$

$$\varphi_{p^\infty} : \underset{\hat{E}}{\underset{\wr|}{E[p^\infty]^0}} \longrightarrow \mu_{p^\infty} \simeq \hat{\mathbb{G}}_m$$

$$\mu_{p^\infty}(R) = (\underline{1 + \mathrm{Nil}_p(R)}, \times) > \hat{\mathbb{G}}_m(R).$$

x s.t. $(x-1)^{p^n} = 0$ for some n.

$p^N = 0$ in R $\Downarrow$

$(x-1)^m = 0$ for some m.

I_g is also represented by an affine formal scheme, pro-étale $\mathbb{Z}_p^\times$-torsor over $X^{ord}(N)$.

$$\mathbb{Z}_p^\times = \mathrm{Aut}(\mu_{p^\infty}) = \mathrm{Aut}(\mathbb{Z}_p)$$

acts on φ_p.

$f \in H^0(\mathfrak{X}^{ord}(N), \omega^k)$ $\qquad p : I_g \to \mathfrak{X}^{ord}(N)$.

$$\begin{tikzcd}
& \rightsquigarrow H^0(\mathcal{I}_g, p^*\omega^k) \arrow[d, "\wr"] & \ni & p^* f \arrow[d, mapsto] \\
\varphi_{p^\infty} \text{ induces} \arrow[ur] & \mathcal{O}_{\mathcal{I}_g} \cdot \left(\frac{dt}{t}\right)^k & \ni & \tilde{f}
\end{tikzcd}$$

a trivialization of invariant differentials by pulling back for $(\hat{\mathbb{G}}_m, \frac{dt}{t})$.

Weights translates into a character of $\mathbb{Z}_p^\times$.

$a \in \mathbb{Z}_p^\times \circlearrowright \hat{\mathbb{G}}_m$

$$t \mapsto t^a \overset{\text{Lie}}{\rightsquigarrow} a : \mathrm{Lie}\,\hat{\mathbb{G}}_m \to \mathrm{Lie}\,\hat{\mathbb{G}}_m$$

dualizing $\rightsquigarrow$ $a : \frac{dt}{t} \mapsto a\frac{dt}{t}$

$$a \cdot \hat{f} = a^{-k} \tilde{f}$$

$\chi : \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$ a character.

$$S_\chi^{ord}(N) := \{ f \in \mathcal{O}_{Ig} \mid a^{-1} \cdot f = \chi(a) f \}.$$

Serre's duality.

$X/\mathbb{Z}_p$ $\underset{\sim}{(N)}$ $\rightsquigarrow$ $H^1(X, \Omega_{X/\mathbb{Z}_p}) \to \mathbb{Z}_p$.

Want to construct

$$\langle , \rangle : H^0(\mathcal{X}^{ord}, \omega^{\kappa^{un}}) \times H_c^1(\mathcal{X}^{ord}, \omega^{2-\kappa^{un}}(-1))$$

$$\to H^1_c(\mathfrak{X}^{\mathrm{ord}}, \omega^2(-D)\otimes_{\mathbb{Z}_p}\Lambda) \longrightarrow \Lambda.$$

$$\longrightarrow : H^1_c(\mathfrak{X}^{\mathrm{ord}}, \omega^2(-D)\otimes_{\mathbb{Z}_p}\Lambda)$$

$$\to H^1(X, \omega^2(-D)\otimes\Lambda) \to H^1(X, \omega^2(-D))\otimes\Lambda$$

$$\xrightarrow{KS\otimes 1} H^1(X, \Omega_{X/\mathbb{Z}_p})\otimes\Lambda \xrightarrow{\mathrm{res}} \Lambda.$$

$$f \in H^0(\mathfrak{X}^{\mathrm{ord}}, \omega^{\kappa^{un}}) \Leftrightarrow \mathcal{O}_{\mathfrak{X}^{\mathrm{ord}}} \xrightarrow{f} \omega^{\kappa^{un}}.$$

$$\omega^{2-\kappa^{un}}(-D) := \omega^2(-D)\otimes \underline{\mathrm{Hom}}(\omega^{\kappa^{un}}, \Lambda\otimes\mathcal{O}_{\mathfrak{X}^{\mathrm{ord}}})$$

f induces $\omega^{2-\kappa^{un}}(-D) \to \omega^2(-D)\otimes\Lambda$

$$H^1_c(\mathfrak{X}^{\mathrm{ord}}, \omega^{2-\kappa^{un}}(-D)) \to H^1_c(\mathfrak{X}^{\mathrm{ord}}, \omega^2(-D)\otimes\Lambda).$$

Prop. 4.17 $(f,g)\in H^0(\mathfrak{X}^{ord},\omega^{\kappa^{un}})\times H^1_c(\mathfrak{X}^{ord},\omega^{2-\kappa^{un}}(-D))$, we have

$$\langle U_p f, g\rangle = \langle f, Fg\rangle.$$

Proof. **Step 1.**

$$\begin{tikzcd}
H^0(\mathfrak{X}^{ord},\omega^{\kappa^{un}})\times H^1_c(\mathfrak{X}^{ord},\omega^{2-\kappa^{un}}(-D)) \arrow[r] \arrow[d] & \Lambda \arrow[d] \\
\prod_{k\in\mathbb{Z}} H^0(\mathfrak{X}^{ord},\omega^{k})\times H^1_c(\mathfrak{X}^{ord},\omega^{2-k}(-D)) \arrow[r] & \prod_{k\in\mathbb{Z}}\mathbb{Z}_p
\end{tikzcd}$$

Step 2: Prove $\langle U_p f, g\rangle = \langle f, Fg\rangle$

for every $k\in\mathbb{Z}$.

Work with X_n^{ord}. By defn of

$$H^1_c(X_n^{ord}, \mathcal{F}_e) := \varprojlim H^1(X_n, I^n \tilde{\mathcal{F}}_e).$$

On H^1_c, F acts as

$$F : p_2^* I^{m\overset{s}{\cancel{k}}} \omega^{2-k}(-D) \to p_1^! I^{-l+s} \omega^{2-k}(-D)$$

Apply $R\mathcal{H}om(\ , \omega_{X_0(p)/\mathbb{Z}_p}) =: D$

$$D(F) : p_1^* I^{-s+l} \omega^k \longrightarrow p_2^! I^{-ms} \omega^k.$$

Lemma 4.14? $\underbrace{D(F) = U_p}$.

$\Uparrow$?

~~prop 3.6.~~ ? $\Omega_{X/\mathbb{Z}_p}(\log(SS+D))$.

$\uparrow$

$$\Omega_{X_0(p)/\mathbb{Z}_p}(\log D)[1]$$

$P(T_p) = T_p$

$$= \omega_{X_0(p)/\mathbb{Z}_p}.$$

Let $s \to \infty$, take H^0.

$$= \operatorname{colim} H^0(\;\;, \mathcal{I}^{-s} \hat{j}_* \mathcal{L})$$

$\square$

Prop 4.18. Restrict $\langle\,,\rangle$ to

$$e(U_p) H^0(\mathfrak{X}^{ord}, \omega^{\kappa^{un}}) \times e(F) H^1_c(\mathfrak{X}^{ord}, \omega^{2-\kappa^{un}}(-D)).$$

(1) $\langle\,,\rangle$ becomes a perfect pairing.

(2) $\langle U_p f, g\rangle = \langle f, F g\rangle$.

(3) Have commutative diagram

$$
\begin{tikzcd}
e(U_p) H^0(\mathfrak{X}^{ord}, \omega^k) \arrow[r, phantom, "\times"] & e(F) H^1_c(\mathfrak{X}^{ord}, \omega^{2-k}(-D)) \arrow[r] \arrow[d, "j"] & \mathbb{Z}_p \\
e(T_p) H^0(X, \omega^k) \arrow[u, "i"] \arrow[r, phantom, "\times"] \arrow[ur, phantom, "@"] & e(T_p) H^1(X, \omega^{2-k}(-D)) \arrow[ur, "\text{Serre's duality}"'] &
\end{tikzcd}
$$

Serre's duality.

<u>Proof</u>. (2) follows from 4.17.

(1) follows from (3): i & j are isom for $k \geq 3$. & <u>the same trick as step 1 in 4.17</u>.

Prove (3) for $k \geq 3$.

Oberserve that.

$$
\begin{tikzcd}
H^0(\mathfrak{X}^{ord}, \omega^k) \arrow[r, phantom, "\times"] & H^1_c(\mathfrak{X}^{ord}, \omega^{2-k}(-D)) \arrow[r] \arrow[d, "j"] & \mathbb{Z}_p \\
H^0(X, \omega^k) \arrow[u, "i"] \arrow[r, phantom, "\times"] \arrow[ur, phantom, "\circlearrowleft"] & H^1(X, \omega^{2-k}(-D)) \arrow[ur] &
\end{tikzcd}
$$

commutes.

For any $f \in e(T_p)H^0(X, \omega^k)$ and $g \in e(F)H^1_c(\mathfrak{X}^{ord}, \omega^{2-k}(-D))$.

$$\langle e(U_p) i(f), g \rangle = \langle i(f), e(F) g \rangle$$
$$= \langle i(f), g \rangle.$$

$$\langle f, e(T_p) j(g) \rangle = \langle e(T_p) f, j(g) \rangle$$

$$= \langle f, j(g) \rangle.$$

$$= \langle i(f), g \rangle.$$

$\square$.