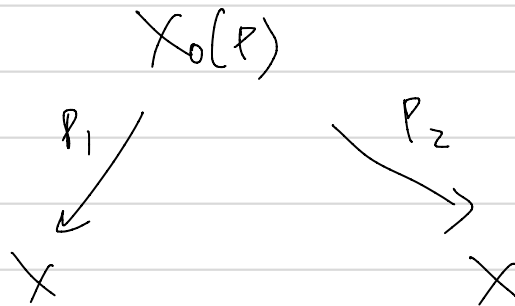


Hecke and diamond operators

$$N \geq 3, \quad \Gamma \backslash N, \quad X = X(N)/\mathcal{O}_p,$$

$\Sigma \rightarrow X$   
 semi-abelian  
 scheme, extending  
 universal curve



$$(E, \mathcal{F}_N, \varphi: E \rightarrow F)$$



$$\exists \pi: P_1^* \Sigma \rightarrow P_2^* \Sigma$$

let  $\omega_i = \omega_{\Sigma/X}$  normal sheaf

$$\forall k \in \mathbb{Z}, \quad \pi_k: P_2^* \omega^k \dashrightarrow P_1^* \omega^k$$

$$T_{P, k}^{\text{naive}} = P_2^* \omega^k \xrightarrow{\pi_k} P_1^* \omega^k \xrightarrow{\text{Tr}_{P_1}} P_1^! \omega^k$$

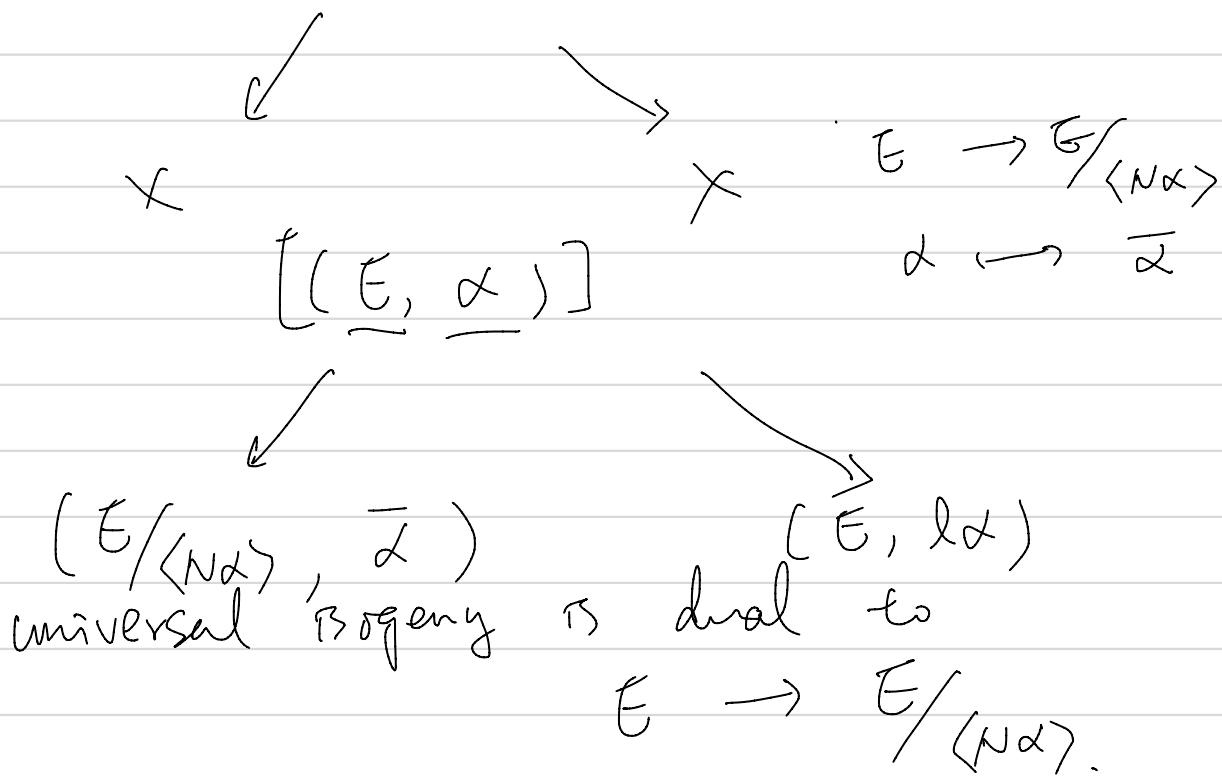
$$T_{P, k} := P^{\text{int}}(1, k) T_{P, k}^{\text{naive}}$$

We can also define  $T_\ell$  for  $\ell \nmid N$  in the same way.

For  $\ell \mid N$ , we can define  $U_\ell$ :

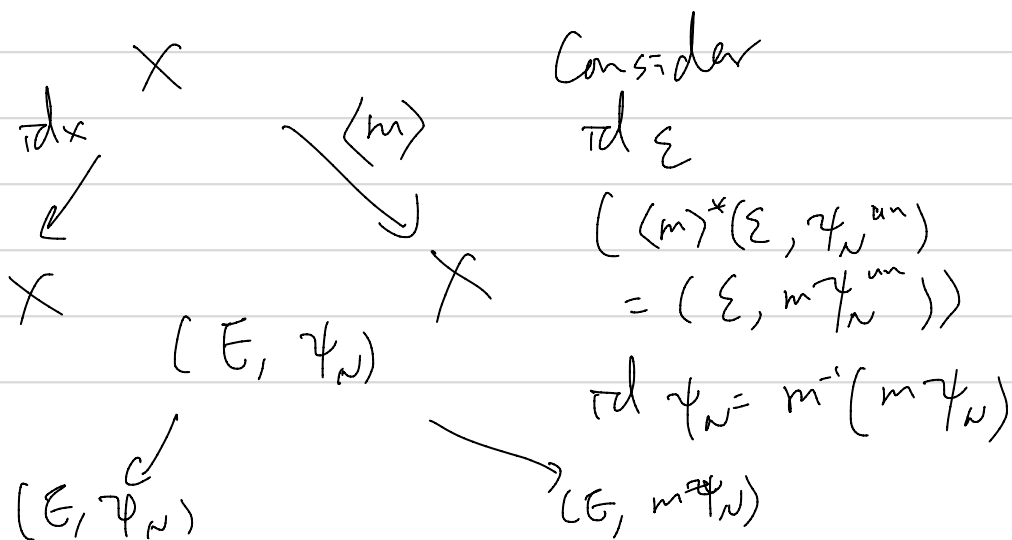
$$X_0(\mathbb{C}) = X(\Gamma_1(N) \cap \Gamma_0(N\ell))$$

$$= X(\Gamma_1(N\ell)) / \left( \frac{1+N\mathbb{Z}}{1+N\ell\mathbb{Z}} \right)$$



$(\mathbb{Z}/N\mathbb{Z})^\times$  acts on  $X$   $m \in (\mathbb{Z}/N\mathbb{Z})^\times$

$$\langle m \rangle: (E, \psi_N) \mapsto (E, m\psi_N)$$



$$\rightsquigarrow \langle m \rangle : \mathcal{R}\Gamma(X, \omega^k) \rightarrow \mathcal{R}\Gamma(X, \omega^k)$$

$$(\langle m \rangle f)(E, \mathcal{Y}_N, \omega) := f(E, m\mathcal{Y}_N, \omega)$$

- $T_p, T_x, U_x, \langle m \rangle$  commutes with one another.

Duality theory of stacks project Ch 48

Let  $S$  Noetherian scheme

$\text{FTS}_S =$  category of finite type

Let  $f: X \rightarrow Y$  on  $\text{FTS}_S$  separated schemes /  $S$

$$\exists f^! : D_{\text{Coh}}^+(\mathcal{O}_Y) \rightarrow D_{\text{Coh}}^+(\mathcal{O}_X)$$

If  $W_Y$  is a dualizing complex on  $Y$

$W_X := f^! W_Y$  is a dualizing complex on  $X$ .

$$f^! m = D_X(Lf^*(D_Y(m))) \quad \forall m \in D_{\text{Coh}}^+(\mathcal{O}_Y)$$

If  $f$  is proper,  $f^!$  is right adjoint to  $Rf_*$  and

$$\bullet \quad Rf_* R\text{Hom}_{\mathcal{O}_X}(K', f^! M) \cong R\text{Hom}_{\mathcal{O}_Y}(Rf_* K', M)$$

$$\forall K' \in D_{\text{Coh}}^-(\mathcal{O}_X), \quad M \in D_{\text{Coh}}^+(\mathcal{O}_Y)$$

$\Rightarrow$  Serre duality

$$Rf_* D_X(K') \cong D_Y(Rf_* K')$$

In our case,  $\zeta = \text{Spec}(\bar{\mathcal{O}}_P)$

$$f: X \rightarrow \text{Spec}(\bar{\mathcal{O}}_P)$$

$$g: X_0(P) \rightarrow \text{Spec}(\bar{\mathcal{O}}_P)$$

$$\omega_X = f^! \mathcal{O}_{\bar{\mathcal{O}}_P} = \Omega^1_{X/\bar{\mathcal{O}}_P} [1]$$

$$\underline{\omega}_{X_0(P)} = g^! \mathcal{O}_{\bar{\mathcal{O}}_P} \cong p_1^! \Omega^1_{X/\bar{\mathcal{O}}_P} [1]$$

$$\cong p_2^! \Omega^1_{X/\bar{\mathcal{O}}_P} [1].$$

Want to study

$$D(T_P): D(R\Gamma(X, \omega^k)) \rightarrow D(R\Gamma(X, \omega^k))$$

$$T_P: R\Gamma(X, \omega^k) \xrightarrow{p_2^*} R\Gamma(X_0(P), p_2^* \omega^k) \xrightarrow{T_P} R\Gamma(X_0(P), p_1^! \omega^k) \rightarrow R\Gamma(X, \omega^k)$$

$$D(T_P): R\Gamma(X, D(\omega^k)) \xrightarrow{p_1^*} R\Gamma(X_0(P), p_1^* D(\omega^k))$$

$$\xrightarrow{D(T_P)} R\Gamma(X_0(P), p_2^! D(\omega^k)) \rightarrow R\Gamma(X, D(\omega^k))$$

$$D(\omega^k) = \omega^{lc} \otimes_{\mathcal{O}_X} \Omega^1_{X/\bar{\mathcal{O}}_P} [1]$$

shift by  $-1$  and we study the map

$$p_1^* (\omega^{lc} \otimes \Omega^1_{X/\bar{\mathcal{O}}_P}) \rightarrow p_2^! (\omega^{lc} \otimes \Omega^1_{X/\bar{\mathcal{O}}_P}).$$

$$T_{P, k}^{\text{naive}} : P_2^* W^k \xrightarrow{\pi_{1k}} P_1^* W^k \xrightarrow{\text{Tr}_{P_1}} P_1! W^k$$

$$\begin{aligned} \text{dual: } P_1^* (W^{-k} \otimes \Omega_{X/\mathbb{Z}_p}^1) &\xrightarrow{\text{Tr}_{P_1}} P_1! (W^{-k} \otimes \Omega_{X/\mathbb{Z}_p}^1) \\ &\cong P_1^* W^k \otimes W_{X_0}(P) \\ &\xrightarrow{\pi_{1k}^{-1} \otimes \text{id}} P_2^* W^k \otimes W_{X_0}(P) \\ &\cong P_2! (W^{-k} \otimes \Omega_{X/\mathbb{Z}_p}^1). \end{aligned}$$

Consider  $K$ - $S$  isom:

$$W^2(-C) \cong \Omega_{X/\mathbb{Z}_p}^1 \text{ where } C \text{ is the boundary divisor on } X$$

$\Rightarrow$  We get a map

$$D(T_{P, k}^{\text{naive}}) : P_1^* (W^{2-k}(-C)) \rightarrow P_2! (W^{2-k}(-C))$$

We would like to study the map

$$\begin{aligned} P_1^* W^2(-C) &\xrightarrow{\cong} P_1^* \Omega_{X/\mathbb{Z}_p}^1 \\ \xrightarrow{\text{Tr}_{P_1}} P_1! \Omega_{X/\mathbb{Z}_p}^1 &\xrightarrow{\cong} W_{X_0}(P) \cong P_2! \Omega_{X/\mathbb{Z}_p}^1 \end{aligned}$$

$K$ - $S$  map

Let  $E$  an elliptic curve

$$\begin{array}{c} \downarrow f \\ S \end{array}$$

$$\Omega^1 E/S = 0 \rightarrow \mathcal{O}_E \xrightarrow{\begin{smallmatrix} 0 \\ \text{in} \end{smallmatrix}} \Omega^1 E/S \rightarrow 0$$

Hodge filtration:

$$F^0 \Omega^1 E/S = 0 \rightarrow \mathcal{O}_E \rightarrow \Omega^1 E/S \rightarrow 0$$

$$F^1 \Omega^1 E/S = 0 \rightarrow 0 \rightarrow \Omega^1 E/S \rightarrow 0$$

$$F^2 \Omega^1 E/S = 0$$

$$\exists E_i^{p,q} = Rf_{*}^{p+q} (\text{gr}^p \Omega^1 E/S) = Rf_{*}^{p+q} \Omega^p E/S \Rightarrow Rf_{*}^{p+q} \Omega^1 E/S$$

This splits at  $E_i$

$$0 \rightarrow f_{*} \Omega^1 E/S \rightarrow H^1(E/S) \rightarrow Rf_{*}^1 \mathcal{O}_E \rightarrow 0$$

$\parallel$   $\parallel$   
 $\mathcal{W}_{E/S}$   $\mathcal{W}_{E/S}^{-1}$

If  $f': E' \rightarrow S$  another curve  
 $\varphi: E \rightarrow E'$   $S$ -isomorphism

$$\Rightarrow \varphi^*: H^1(E'/S) \rightarrow H^1(E/S)$$

$f'_* \Omega^1 E'/S \rightarrow f_* \Omega^1 E/S$  is pull back of global differentials

$$Rf_{*}^1 \mathcal{O}_E = \text{Lie}(E^v/S) \subseteq \text{Lie}(E/S)$$

$\text{Lie}(E^v/S) \rightarrow \text{Lie}(E/S)$  is the tangent map of  $\varphi^v$ . (Look at Picard functor).

If  $S$  is affine, we can express  $H^1(E/S)$  in terms of Čech hypercocycles.

Let  $\{U_i\}$  affine cover of  $E$   
 a class of  $H^1(E/S)$  is represented by

$$\{ (U_i), (\eta_{ij}) \}, \quad w_i \in \Omega^1_{E/S}(U_i)$$

$$\eta_{ij} \in \mathcal{O}_E(U_i \cap U_j), \quad w_i - w_j = d\eta_{ij}$$

$$\eta_{ik} = \eta_{ij} - \eta_{jk}.$$

Suppose  $S/k$  smooth curve (stacks project Ch 50)  
 first exact sequence:

$$0 \rightarrow f^* \Omega^1_{S/k} \rightarrow \Omega^1_{E/k} \rightarrow \Omega^1_{E/S} \rightarrow 0,$$

splits.

$$F^0 \Omega^1_{E/k} := \Omega^1_{E/k}, \quad F^1 \Omega^1_{E/k} := f^* \Omega^1_{S/k},$$

$$F^2 \Omega^1_{E/k} := 0$$

$\leadsto$  filtration on  $\Omega^i_{E/k}$ , s.t

$$\text{gr}^p \Omega^i_{E/k} = f^* \Omega^p_{S/k} \otimes_{\mathcal{O}_E} \Omega^i_{E/S}$$

$$\exists \quad E_{i, h}^p = R\Gamma_x^{p+h} (f^* \Omega^p_{S/k} \otimes_{\mathcal{O}_E} \Omega^i_{E/S})$$

$$= \Omega^p_{S/k} \otimes H^h(E/S).$$

$$\mathcal{D} = d_{i, h}^0 : H^h(E/S) \rightarrow \Omega^1_{S/k} \otimes_{\mathcal{O}_S} H^h(E/S)$$

$K-S$  map:

$$0 \rightarrow W_{E/S} \rightarrow H^1(E/S) \rightarrow W_{E/S}^* \rightarrow 0$$

$$\begin{aligned} W_{E/S} \rightarrow H^1(E/S) &\xrightarrow{\Delta} \Omega_{S/k}^1 \otimes_{\mathcal{O}_S} H^1(E/S) \\ &\rightarrow \Omega_{S/k}^1 \otimes_{\mathcal{O}_S} W_{E/S}^* \end{aligned}$$

This map is  $\mathcal{O}_S$ -linear.

Suppose  $S$  is affine,  $\Omega_{S/k}^1 = \mathcal{O}_S dt$

$W \in \Gamma(E, \Omega_{E/S}^1)$ ,  $W_i := W|_{U_i}$ ,

$\tilde{W}_i \in \Omega_{E/k}^1(U_i)$  a lifting

$\tilde{W}_i - \tilde{W}_j \in \Gamma^*(\Omega_{E/k}^1(U_i \cap U_j))$ , say  $\eta_{ij} t^* dt$

$\eta := (\eta_{ij}) \in H^1(E, \mathcal{O}_E)$

$KS(W) = dt \otimes \eta$ .

If  $\varphi: E \rightarrow E'$  is a map

$$W_{E'/S} \xrightarrow{\varphi^*} W_{E/S}$$

$$\downarrow KS \quad \supseteq \quad \downarrow KS$$

$$\Omega_{S/k}^1 \otimes W_{E'/S}^{-1} \xrightarrow{id \otimes \varphi_*} \Omega_{S/k}^1 \otimes W_{E/S}^{-1}$$



We only have to determine the map  $P_i^* D(W^k)$

$\rightarrow P_i^! D(W^k)$  at the generic point of  $X_0(p)$ .

We can look at the generic fiber and look at the affine part.

$$W_{X_0(p)}/\mathcal{O}_p = \Omega^1_{X_0(p)}/\mathcal{O}_p [1]. \text{ With } \pi_i: P_i^* \Sigma$$

We have, on the affine part,  $P_i^* \Sigma$

$$\begin{array}{ccc} P_2^* W & \xrightarrow{\pi_1} & P_1 W \\ \downarrow \text{KS} & & \downarrow \text{KS} \\ \Omega^1_{X_0(p)}/\mathcal{O}_p \otimes P_2 W^1 & \xrightarrow{\text{id} \otimes (\pi_1^\vee)^{-1}} & P_1 W^1 \end{array}$$

$$\Rightarrow \begin{array}{ccc} P_2^* W^2 & \xrightarrow{\pi_1 \otimes (\pi_1^\vee)^{-1}} & P_2^* W^2 \\ \searrow (\text{tr}_{P_1}) & & \swarrow (\text{tr}_{P_2}) \\ & \Omega^1_{X_0(p)}/\mathcal{O}_p & \end{array}$$

$$P_i^* \Omega^1_{X/\mathcal{O}_p} \xrightarrow{\text{tr}_{P_i}} P_i^! \Omega^1_{X/\mathcal{O}_p} \cong W_{X_0(p)}/\mathcal{O}_p \cong \Omega^1_{X_0(p)}/\mathcal{O}_p$$

is the natural map.

Now the map

$$p_1^*(W^k \otimes W^2(-c)) \rightarrow p_2^* W^k \otimes p_2^*(W^2(-c))$$

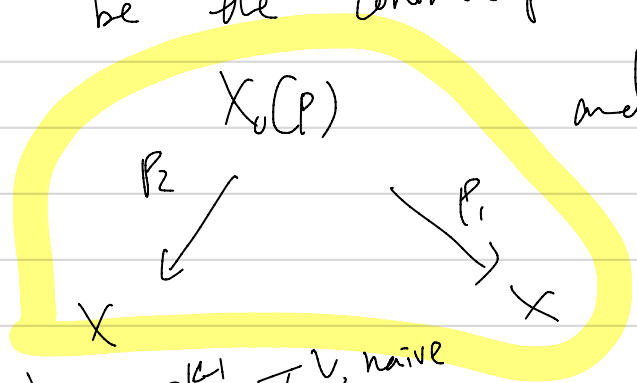
is the map  $(\pi_1^{-1} \otimes \text{tr}_{p_1}) (1 \otimes \text{tr}_{p_2} \circ \text{tr}_{p_1}^{-1} \otimes \pi_1^\vee \pi_1^{-1})$

$$= \pi_{1-k}^{-1} \otimes \pi_1^\vee \otimes \text{tr}_{p_2} \quad \pi_{1-k} \pi_{1-k}^\vee = p^{1-k}$$

$$= p^{k-1} \pi_{2-k}^\vee \circ \text{tr}_{p_2}$$

Let  $\overline{T}_{p,k}^{\vee, \text{naive}}$  be the cohomological correspondence

given by



$$\text{and } \pi^\vee: p_2^* \mathcal{E} \rightarrow p_1^* \mathcal{E}$$

$$\Rightarrow \mathcal{D}(\overline{T}_{p,k}^{\vee, \text{naive}}) = p^{k-1} \overline{T}_{p,2-k}^{\vee, \text{naive}}$$

$$(\mathcal{E}, \psi_\omega, \varphi: \mathcal{E} \rightarrow \mathcal{F})$$

$$(\mathcal{F}, \varphi \psi_\omega)$$

$$(\mathcal{E}, \psi_\omega)$$

$$\varphi^\vee(\varphi \psi_\omega) = p \psi_\omega$$

$$\Rightarrow \overline{T}_{p,2-k}^{\vee, \text{naive}} = \langle p \rangle^{-1} \overline{T}_{p,2-k}^{\vee, \text{naive}}$$

$$\Rightarrow \mathcal{D}(\overline{T}_{p,k}) = \langle p \rangle^{-1} \overline{T}_{p,2-k}$$

$$\text{Also we have } \mathcal{D}(\langle m \rangle) = \langle m \rangle^{-1}$$

$$\Rightarrow \mathcal{D}^2(\overline{T}_{p,k}) = \overline{T}_{p,k} \text{ (same check)}$$