Hecke and diamond operators

$N \geq 3$, $p \nmid N$, $X = X_1(N)/\mathbb{Z}_p$, $\mathcal{E} \longrightarrow X$ semi-abelian scheme, extending universal curve

$$\begin{tikzcd} & X_0(p) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & X \end{tikzcd}$$

$$\begin{tikzcd} & (E, \eta_N, \varphi : E \longrightarrow F) \arrow[dl] \arrow[dr] & \\ (E, \eta_N) & & \underline{(F, \varphi_* \eta_N)} \end{tikzcd}$$

$\exists\ \tau : p_1^* \mathcal{E} \longrightarrow p_2^* \mathcal{E}$

Let $\omega := \omega_{\mathcal{E}/X}$ conormal sheaf

$\forall\ k \in \mathbb{Z}$, $\tau_k : p_2^* \omega^k \dashrightarrow p_1^* \omega^k$

$$T_{p,k}^{\text{naive}} = p_2^* \omega^k \overset{\tau_k}{\dashrightarrow} p_1^* \omega^k \xrightarrow{Tr_{p_1}} p_1^! \omega^k$$

$$T_{p,k} := p^{-\inf(1,k)} T_{p,k}^{\text{naive}}$$

We can also define T_ℓ for $\ell \nmid N$ in the same way.

For $\ell \mid N$, we can define U_ℓ :

$$X_0(\ell) = X(\Gamma_1(N)\cap \Gamma_0(N\ell))$$
$$= X(\Gamma_1(N\ell)) \Big/ \left(1+N\mathbb{Z} \big/ 1+N\ell\mathbb{Z} \right)$$

$$\begin{tikzcd}
 & X_0(\ell) \arrow[dl] \arrow[dr] & \\
X & & X
\end{tikzcd}
\qquad
\begin{aligned}
E &\longrightarrow E/\langle N\alpha\rangle \\
\alpha &\longmapsto \bar{\alpha}
\end{aligned}$$

$$\begin{tikzcd}
 & {[(\underline{E}, \underline{\alpha})]} \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
(E/\langle N\alpha\rangle, \bar{\alpha}) & & (E, \ell\alpha)
\end{tikzcd}$$

universal isogeny is dual to $E \longrightarrow E/\langle N\alpha\rangle$.

$(\mathbb{Z}/N\mathbb{Z})^\times$ acts on X $\quad m \in (\mathbb{Z}/N\mathbb{Z})^\times$

$$\langle m\rangle : (E, \psi_N) \longmapsto (E, m\psi_N)$$

$$\begin{tikzcd}
 & X \arrow[dl, "\mathrm{id}_X"'] \arrow[dr, "\langle m\rangle"] & \\
X & & X
\end{tikzcd}$$

$$\begin{tikzcd}
 & (E, \psi_N) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
(E, \psi_N) & & (E, m\psi_N)
\end{tikzcd}$$

Consider $\mathrm{id}_{\mathcal{E}}$

$(\langle m\rangle^*(\mathcal{E}, \psi_N^{un}) = (\mathcal{E}, m\psi_N^{un}))$

$\mathrm{id}\ \psi_N = m^{-1}(m\psi_N)$

$\rightsquigarrow \langle m \rangle : R\Gamma(X, \omega^k) \longrightarrow R\Gamma(X, \omega^k)$

$$(\langle m \rangle f)(E, \psi_N, \omega) := f(E, m\psi_N, \omega)$$

- T_p, T_ℓ, U_ℓ, $\langle m \rangle$ commutes with one another.

Duality theory cf Stacks project Ch 48

Let S Noetherian scheme

FTS_S = category of finite type separated schemes $/S$

Let $f : X \to Y$ in FTS_S

$\exists\ f^! : D^+_{QCoh}(\mathcal{O}_Y) \longrightarrow D^+_{QCoh}(\mathcal{O}_X)$.

If $\omega_Y^\cdot$ is a dualizing complex on Y

$\omega_X^\cdot := f^! \omega_Y^\cdot$ is a dualizing complex on X.

$\underline{f^! M = D_X(Lf^*(D_Y(M)))}$ $\forall M \in D^+_{QCoh}(\mathcal{O}_Y)$.

If f is proper, $f^!$ is right adjoint to Rf_* and

- $Rf_* R\mathcal{H}om_{\mathcal{O}_X}(K^\cdot, f^! M^\cdot) \cong R\mathcal{H}om_{\mathcal{O}_Y}(Rf_* K^\cdot, M)$

$\forall\ K^\cdot \in D^-_{Coh}(\mathcal{O}_X)$, $M^\cdot \in D^+_{Coh}(\mathcal{O}_Y)$.

$\Rightarrow$ Serre duality

$$Rf_* D_X(K^\cdot) \cong D_Y(Rf_* K^\cdot).$$

In our case, $S = \mathrm{Spec}(\mathbb{Z}_p)$

$f : X \longrightarrow \mathrm{Spec}(\mathbb{Z}_p)$

$g : X_0(p) \longrightarrow \mathrm{Spec}(\mathbb{Z}_p)$

$\omega^{\bullet}_X = f^! \mathcal{O}_{\mathbb{Z}_p} = \Omega^1_{X/\mathbb{Z}_p}[1]$

$\underline{\omega^{\bullet}_{X_0(p)}} = g^! \mathcal{O}_{\mathbb{Z}_p} \cong p_1^! \Omega^1_{X/\mathbb{Z}_p}[1]$

$\cong p_2^! \Omega^1_{X/\mathbb{Z}_p}[1]$.

Want to study

$D(T_p) : D(R\Gamma(X, \omega^k)) \longrightarrow D(R\Gamma(X, \omega^k))$

$T_p : R\Gamma(X, \omega^k) \xrightarrow{p_2^*} R\Gamma(X_0(p), p_2^*\omega^k) \xrightarrow{T_p}$
$R\Gamma(X_0(p), p_1^! \omega^k) \longrightarrow$
$R\Gamma(X, \omega^k)$

$D(T_p) : R\Gamma(X, D(\omega^k)) \xrightarrow{p_1^*} R\Gamma(X_0(p), p_1^* D(\omega^k))$

$\xrightarrow{D(T_p)} R\Gamma(X_0(p), p_2^! D(\omega^k))$

$\longrightarrow R\Gamma(X, D(\omega^k))$

$D(\omega^k) = \omega^{-k} \otimes_{\mathcal{O}_X} \Omega^1_{X/\mathbb{Z}_p}[1]$

shift by -1 and we study the map

$p_1^*(\omega^{-k} \otimes \Omega^1_{X/\mathbb{Z}_p}) \longrightarrow p_2^!(\omega^{-k} \otimes \Omega^1_{X/\mathbb{Z}_p})$.

$T^{naive}_{p,k}$: $p_2^* \omega^k \xrightarrow{\pi_k} p_1^* \omega^k \xrightarrow{Tr_{p_1}} p_1^! \omega^k$

dual : $p_1^*(\omega^{-k} \otimes \Omega^1_{X/\mathbb{Z}_p}) \xrightarrow{Tr_{p_1}} p_1^!(\omega^{-k} \otimes \Omega^1_{X/\mathbb{Z}_p})$

$\cong p_1^* \omega^k \otimes \omega_{X_0(p)}^{\circ}$

$\xrightarrow{\pi_k^{-1} \otimes id} p_2^* \omega^k \otimes \omega_{X_0(p)}^{\circ}$

$\cong p_2^!(\omega^{-k} \otimes \Omega^1_{X/\mathbb{Z}_p})$.

Consider K-S isom :

$\omega^2(-C) \cong \Omega^1_{X/\mathbb{Z}_p}$ where C is the boundary divisor on X

$\Rightarrow$ we get a map

$D(T^{naive}_{p,k})$: $p_1^*(\omega^{2-k}(-C))$

$\longrightarrow p_2^!(\omega^{2-k}(-C))$

We would like to stud the map

$p_1^* \omega^2(-C) \xrightarrow{\cong} p_1^* \Omega^1_{X/\mathbb{Z}_p}$

$\xrightarrow{Tr_{p_1}} p_1^! \Omega^1_{X/\mathbb{Z}_p} \xrightarrow{\cong} \omega_{X_0(p)}^{\circ} \cong p_2^! \Omega^1_{X/\mathbb{Z}_p}$

K-S map

Let E an elliptic curve

$$\begin{array}{c} E \\ \downarrow f \\ S \end{array}$$

$$\overset{0\text{th}}{} \qquad \overset{1\text{st}}{}$$

$$\Omega^\bullet_{E/S} = \quad 0 \to \mathcal{O}_E \to \Omega^1_{E/S} \to 0$$

Hodge filtration:

$$F^0\Omega^\bullet_{E/S} = \quad 0 \to \mathcal{O}_E \to \Omega^1_{E/S} \to 0$$
$$F^1\Omega^\bullet_{E/S} = \quad 0 \to 0 \to \Omega^1_{E/S} \to 0$$
$$F^2\Omega^\bullet_{E/S} = \quad 0$$

$$\exists\ E_1^{p,q} = Rf_*^{p+q}(gr^p\,\Omega^\bullet_{E/S}) = Rf_*^q\,\Omega^p_{E/S} \Rightarrow Rf_*^{p+q}\,\Omega^\bullet_{E/S}$$

Thus splits at E_1

$\to$ short exact sequence

$$0 \to \underset{\underset{\omega_{E/S}}{\parallel}}{f_*\Omega^1_{E/S}} \to \underline{H^1(E/S)} \to \underset{\underset{\omega_{E/S}^{-1}}{\parallel}}{Rf_*^1\mathcal{O}_E} \to 0$$

If $f': E' \to S$ another curve

$\varphi: E \to E'$ S-isogeny

$$\Rightarrow \varphi^*: H^1(E'/S) \to H^1(E/S)$$

$f'_*\Omega^1_{E'/S} \to f_*\Omega^1_{E/S}$ is pull back of global differentials

$$Rf_*^1\mathcal{O}_E = \mathrm{Lie}(E^\vee/S) \cong \mathrm{Lie}(E/S)$$

$\mathrm{Lie}(E'/S) \to \mathrm{Lie}(E/S)$ is the tangent map of $\varphi^\vee$. (Look at Picard functor).

If S is affine, we can express $H^1(E/S)$ in terms of Čech hypercocycles.

Let $\{U_i\}$ affine cover of E

a class of $\mathbf{H}^1(E/S)$ is represented by

$\{(U_i), (\eta_{ij})\}$, $\omega_i \in \Omega^1_{E/S}(U_i)$

$\eta_{ij} \in \mathcal{O}_E(U_i \cap U_j)$, $\omega_i - \omega_j = d\eta_{ij}$

$\eta_{jk} = \eta_{ij} - \eta_{ik}$.

Suppose S/k smooth curve (stacks project Ch 50)

first exact sequence:

$$0 \to f^* \Omega^1_{S/k} \to \Omega^1_{E/k} \to \Omega^1_{E/S} \to 0,$$

splits.

$F^0 \Omega^1_{E/k} := \Omega^1_{E/k}$, $F^1 \Omega^1_{E/k} := f^* \Omega^1_{S/k}$,

$F^2 \Omega^1_{E/k} = 0$

$\rightsquigarrow$ filtration on $\Omega^\bullet_{E/k}$, s.t

$$\mathrm{gr}^p \Omega^\bullet_{E/k} = f^* \Omega^p_{S/k}[-p] \otimes_{\mathcal{O}_E} \Omega^\bullet_{E/S}$$

$$\Rightarrow E_1^{p,q} = Rf_*^{p+q}(f^* \Omega^p_{S/k}[-p] \otimes_{\mathcal{O}_E} \Omega^\bullet_{E/S})$$

$$= \Omega^p_{S/k} \otimes H^q(E/S).$$

$$\nabla = d_1^{0,q} : H^q(E/S) \to \Omega^1_{S/k} \otimes_{\mathcal{O}_S} H^q(E/S)$$

K-S map:

$$0 \to \omega_{E/S} \to H^1(E/S) \to \omega^{-1}_{E/S} \to 0$$

$$\omega_{E/S} \to H^1(E/S) \xrightarrow{\nabla} \Omega^1_{S/k} \otimes_{\mathcal{O}_S} H^1(E/S)$$
$$\to \Omega^1_{S/k} \otimes_{\mathcal{O}_S} \omega^{-1}_{E/S}.$$

This map is $\mathcal{O}_S$-linear.

Suppose S is affine, $\Omega^1_{S/k} = \mathcal{O}_S\, dt$

$\omega \in \Gamma(E, \Omega^1_{E/S})$, $\omega_i := \omega|_{U_i}$,

$\tilde{\omega}_i \in \Omega^1_{E/k}(U_i)$ a lifting

$\tilde{\omega}_i - \tilde{\omega}_j \in f^*\Omega^1_{S/k}(U_i \cap U_j)$, say $\eta_{ij} f^* dt$

$\eta := (\eta_{ij}) \in H^1(E, \mathcal{O}_E)$

$KS(\omega) = dt \otimes \eta.$

If $\varphi: E \to E'$ isogeny

$$\begin{tikzcd} \omega_{E'/S} \arrow[r, "\varphi^*"] \arrow[d, "KS"] \arrow[dr, phantom, "\circlearrowright"] & \omega_{E/S} \arrow[d, "KS"] \\ \Omega^1_{S/k} \otimes \omega^{-1}_{E'/S} \arrow[r, "id \otimes \varphi^\vee_*"] & \Omega^1_{S/k} \otimes \omega^{-1}_{E/S} \end{tikzcd}$$

We only have to determine the map $p_1^* D(\omega^k)$
$\to p_2^! D(\omega^k)$ at the generic point of $X_0(p)$.
We can look at the generic fiber and look at the affine part.
$\omega_{X_0(p)/\mathbb{Q}_p} = \Omega^1_{X_0(p)/\mathbb{Q}_p}[1]$. With $\pi : p_1^* \mathcal{E} \to p_2^* \mathcal{E}$
We have, on the affine part,

$$\begin{tikzcd}
p_2^* \omega \arrow[r, "\pi_1"] \arrow[d, "KS"'] & p_1 \omega \arrow[d, "KS"] \\
\Omega^1_{Y_0(p)/\mathbb{Q}_p} \otimes p_2 \omega^{-1} \arrow[r, "id \otimes (\pi_1^\vee)^{-1}"] & p_1 \omega^{-1}
\end{tikzcd}$$

$$\Rightarrow \begin{tikzcd}
p_2^* \omega^2 \arrow[r, "\pi_1 \otimes (\pi_1^\vee)^{-1}"] \arrow[dr, "(tr_{p_1})"'] & p_2^* \omega^2 \arrow[d, "(tr_{p_2})"] \\
 & \Omega^1_{Y_0(p)/\mathbb{Q}_p}
\end{tikzcd}$$

$p_i^* \Omega^1_{Y/\mathbb{Q}_p} \xrightarrow{tr_{p_i}} p_i^! \Omega^1_{Y/\mathbb{Q}_p} \cong \omega_{X_0(p)/\mathbb{Q}_p}$
$\cong \underline{\Omega^1_{X_0(p)/\mathbb{Q}_p}}$
is the natural map.

Now the map

$$p_1^*(\omega^{-k} \otimes \omega^2(-C)) \longrightarrow p_2^* \omega^{-k} \otimes p_2^!(\omega^2(-C))$$

is the map $(\pi_{1-k}^{-1} \otimes tr_{p_1})(1 \otimes tr_{p_2} \circ tr_{p_1}^{-1} \otimes \pi_1^\vee \pi_1^{-1})$

$$= \pi_{1-k}^{-1} \otimes \pi_1^\vee \otimes tr_{p_2} \qquad \pi_{1-k}\pi_{1-k}^\vee = p^{1-k}$$

$$= \underline{p^{k-1} \pi_{2-k}^\vee \circ tr_{p_2}}.$$

Let $T_{p,k}^{\vee,\text{naive}}$ be the cohomological correspondence given by

$$\begin{tikzcd} & X_0(p) \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\ X & & X \end{tikzcd}$$

and $\pi^\vee : p_2^*\mathcal{E} \to p_1^*\mathcal{E}$

$$\Rightarrow D(T_{p,k}^{\text{naive}}) = p^{k-1} \underline{T_{p,2-k}^{\vee,\text{naive}}}$$

$$\begin{tikzcd} & (E, \psi_N, \varphi : E \to F) \arrow[dl] \arrow[dr] & \\ (F, \varphi\psi_N) & & (E, \psi_N) \end{tikzcd}$$

$$\varphi^\vee(\varphi\psi_N) = p\psi_N$$

$$\Rightarrow T_{p,2-k}^{\vee,\text{naive}} = \langle p\rangle^{-1} T_{p,2-k}^{\text{naive}}$$

$$\Rightarrow \underline{D(T_{p,k}) = \langle p\rangle^{-1} T_{p,2-k}}.$$

Also we have $D(\langle m\rangle) = \langle m\rangle^{-1}$

$$\Rightarrow D^2(T_{p,k}) = T_{p,k} \quad \text{(San check)}.$$