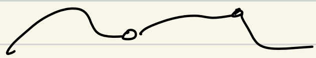
Summary

$$T_{m,\alpha} = \{(E, \phi_N, \mu_{p^\alpha} \hookrightarrow E[p^\alpha])\}$$

$$\downarrow \ \mathrm{Aut}(\mu_{p^\alpha}) = GL(1, \mathbb{Z}/p^\alpha\mathbb{Z})$$

Part 1 (Geometry)

1-dim'l

$$X \supset S := X[\tfrac{1}{\Sigma}]$$  $$\rightsquigarrow S_m = S \times \mathbb{Z}/p^m\mathbb{Z}$$

Affine for $GL(2)$

$$V_{m,\alpha} = H^0(T_{m,\alpha}, \mathcal{O}_{T_{m,\alpha}})^{N(\mathbb{Z}/p^\alpha\mathbb{Z})} \rightsquigarrow V_{m,\infty} = \varinjlim_{\alpha} V_{m,\alpha}$$

$\Rightarrow$ Study properties of both

$$\mathcal{V} = \varinjlim_{m} V_{m,\alpha} \quad \& \quad V = \varprojlim_{m} V_{m,\alpha}$$

$$(\mathbb{Z}/p^m\mathbb{Z} \hookrightarrow \mathbb{Z}/p^{m+1}\mathbb{Z}) \qquad (\mathbb{Z}/p^{m+1}\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^m\mathbb{Z})$$

Part 2 (Hida Theory)

as $T(\mathbb{Z}_p) = \mathbb{G}_m(\mathbb{Z}_p) = \mathbb{Z}_p^\times$-modules & Hecke modules

Part 1

Recall $f: E \to S$ is an ell. curve $/S$ if

(E1) $\exists$ section $0_E \in E(S)$

(E2) $\dim_S E = 1$ & E is geom. conn'd

(E3) $f_* \Omega_{E/S} \simeq R^1 f_* \mathcal{O}_E$ is loc. free of rk 1 $/S$

$\Rightarrow$ E/S is a g'p scheme & $\exists\, \omega \in \Omega_{E/S}$ is nowhere vanishing inv. diff'l

$(\frac{1}{6} \in R)$

By R.R., given $(E, \omega)/R$, $\exists!\, g_2(E,\omega), g_3(E,\omega) \in R$ s.t.

$$y^2 = 4x^3 - g_2 x - g_3 \quad \& \quad \omega = \frac{dx}{y}$$

$(\Delta = g_2^3 - 27 g_3^2)$

In other words, over $\mathbb{Z}[\frac{1}{6}]$

$$\operatorname{Spec}(R) = \operatorname{Spec}\left(\mathbb{Z}\left[\tfrac{1}{6}, g_2, g_3, \tfrac{1}{\Delta}\right]\right) = \mathcal{M}_1 \quad \leftarrow Y(1)$$

is the moduli space for $\mathcal{P}(R) = [(E,\omega)_{/R}]$ $\leftarrow$ $\cong$-class

It is equipped w/ univ. $(\mathbb{E}, \underline{\omega}) \longrightarrow \mathcal{M}_1$

Q: Adding $\Gamma_1(N)$-level?

We have $\mathcal{P}'_{\Gamma_1(N)}(R) = [(E, \underset{\text{order } N}{P}, \omega)_{/R})]$

$$\wr\wr$$

$$\mathcal{P}_{\Gamma_1(N)}(R) = [(E, \mu_N \hookrightarrow E[N], \omega)_{/R}]$$

and $P \in E[N]$ is equiv to

$$\begin{tikzcd} \operatorname{Spec} R \arrow[r, "P"] \arrow[rr, bend right, "\varphi_P"'] & E \arrow[r, "\varphi_E"] \arrow[d] & \mathbb{E} \arrow[d] \\ & \operatorname{Spec} R \arrow[r] & \mathcal{M}_1 \end{tikzcd}$$

$$\Rightarrow \mathcal{P}'_{\Gamma_1(N)}(R) \cong \Big(\mathbb{E}[N] - \bigcup_{N > d \mid N} \mathbb{E}[d]\Big)(R)$$

<u>THEOREM</u> $\exists$ affine scheme $\mathcal{M}_{\Gamma_1(N)} = \operatorname{Spec}(R_{\Gamma_1(N)})_{/\mathbb{Z}[1/6N]}$

$=$ to moduli space of $\mathcal{P}_{\Gamma_1(N)} \cong \mathcal{P}'_{\Gamma_1(N)}$

The map $\mathcal{M}_{\Gamma_1(N)} \longrightarrow \mathcal{M}_1$ is finite étale of degree $\varphi(N)$

Really,

$$\begin{array}{c} \mathcal{M}_{\Gamma_1(N)} \\ \big\downarrow \\ Y_1(N) \end{array} \qquad \begin{array}{l} \lambda \cdot [(E,P,\omega)] = [(E,P,\lambda\omega)] \\ \text{is a } \mathbb{G}_m\text{-torsor} \end{array}$$

Thus $\mathbb{G}_m \circlearrowright \mathcal{M}_{\Gamma_1(N)} \rightsquigarrow \mathbb{G}_m \circlearrowright \mathcal{R}_{\Gamma_1(N)}$, so

$$\mathcal{R}_{\Gamma_1(N)/R} = \bigoplus_{k\in\mathbb{Z}} R_k(\Gamma_1(N);R)^{\circlearrowleft \mathbb{G}_m \text{ acts by } z\mapsto z^{-k}}$$

$$\Rightarrow Y_1(N)_{/R} = \mathrm{Spec}(R_0(\Gamma_1(N);R)) = \mathrm{Proj}(\mathcal{R}(\Gamma_1(N))_{/R}) = \mathbb{G}_m \backslash \mathcal{M}_1(\Gamma_1(N))$$

A (nearly) modular form $f \in R_k(\Gamma_1(N);R)$ is

$$f : (E, \phi_N, \omega)_{/A} \to A \qquad (A = R\text{-alg})$$

(G0) $f(E, \phi_N, \lambda\omega) = \lambda^{-k} f(E, \phi_N, \omega)$, $\forall \lambda \in \mathbb{G}_m(A)$

(G1) Dep. only on $[(E, \phi_N, \omega)_{/A}]$

(G2) Given $\rho: A \to A'$, $f(E_{A'}, \phi_{N,A'}, \omega_{A'}) = \rho(f(E, \phi_N, \omega))$

<u>THEOREM</u> $Y_1(N) = \mathbb{G}_m \backslash \mathcal{M}_{\Gamma_1(N)}$ is a curve over $\mathbb{Z}[1/6N]$ s.t.

1) loc. free of finite rk $/ \mathcal{M}_1 = \mathbb{P}^1(j) - \{\infty\}$

2) (coarse if $N<4$) moduli space of ell. curves

To get $X_1(N)$ & modular forms $\rightsquigarrow$ Compactify.

$$\begin{tikzcd} \mathcal{M}_{\Gamma_1(N)} \arrow[d] & \subset & \overline{\mathcal{M}}_{\Gamma_1(N)} \arrow[d] \\ \mathcal{M}_1 & \subset & \mathrm{Spec}(\mathbb{Z}[1/6, g_2, g_3]) \arrow[ull, dashed] \end{tikzcd}
\qquad
\begin{tikzcd} \mathbb{Z}[\frac{1}{6}, g_2, g_3] \arrow[r] \arrow[dr, dashed, "\text{int. closure}"] & \mathcal{R}_{\Gamma_1(N)}/\mathbb{Z}[1/6] \\ & G_{\Gamma_1(N)}(\mathbb{Z}[1/6N]) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

We still have

$$G_{\Gamma_1(N)}(R) = G_{\Gamma_1(N)}(\mathbb{Z}[1/6N]) \otimes R = \bigoplus_{k=0} G_k(\Gamma_1(N); R)$$

$$\Rightarrow X_1(N)_{/R} = \mathrm{Proj}(G_{\Gamma_1(N)}(R)) = \mathrm{Spec}(G_0(\Gamma_1(N); R)) = \mathbb{G}_m \backslash \mathcal{H}_{\Gamma_1(N)}$$

Really,

$$\begin{tikzcd} Y_1(N) \arrow[r, phantom, "\subset"] \arrow[d] & X_1(N) \arrow[d] \\ \mathbb{A}^1(j) \arrow[r, phantom, "\subset"] & \mathbb{P}^1(j) \end{tikzcd}$$

$\left(j = \frac{(12g_2)^3}{\Delta}\right)$

$\left(\begin{array}{l} X_1(N) \text{ is normalization} \\ \text{of } X(1) = \mathbb{P}^1(j) \text{ in } Y_1(N) \end{array}\right)$

Around $j = \infty \in \mathbb{P}^1(j)$, we use $j^{-1} \in q\mathbb{Z}[q] \rightsquigarrow \mathbb{P}^1(q)$

$$\begin{tikzcd} \mathrm{Tate}(q): ZY^2 = 4X^3 - g_2(q)XZ^2 + g_3(q)Z^3 \arrow[r] \arrow[d] & \mathbb{E} \arrow[r] \arrow[d] & \overline{\mathbb{E}} \arrow[d] \\ \mathrm{Spec}\, \mathbb{Z}[1/6]((q)) \arrow[r, "\mathrm{fpqc}"'] & \mathbb{P}^1(j) - \{\infty\} \arrow[r] & \mathbb{P}^1(j) \end{tikzcd}$$

$$= \mathbb{P}^1(q) - \{0\}$$

Since $\mathrm{Tate}(q)$ is basically $\hat{\mathbb{G}}_{m/\mathbb{Z}[[q]]}/q^{\mathbb{Z}}$, it has a canonical $\Gamma_1(N)$-level

$$\phi_N^{can} : \mu_N \hookrightarrow \hat{\mathbb{G}}_m \twoheadrightarrow \mathrm{Tate}(q)$$

$\Rightarrow$ We also have

$$\begin{tikzcd} (\mathrm{Tate}(q), \phi_N^{can}) \arrow[r] \arrow[d] & (\mathbb{E}, \underline{\phi_N}) \arrow[d] \\ \iota_\infty : \mathrm{Spec}(\mathbb{Z}[\frac{1}{6N}]((q))) \arrow[r] & Y_1(N) \end{tikzcd}$$

$\Rightarrow$ Formal completion along ∞ of $X_1(N)_{/R}$ is $\underset{\hat{\iota} \text{ by } \iota_\infty}{\simeq} R[[q]]$

The same can be done around all cusps but we need to add μ_N & $q^{1/N}$, so completed local rings look like

$$R[\mu_N][[q^{1/N}]]$$

$\Rightarrow f \in G_k(\Gamma_1; R)$ is $(E, \phi_N, \omega)_{/A} \mapsto A$ s.t.

$$(G0\text{-}2) \ \& \ (G3) \quad f(\mathrm{Tate}(q), \phi_N, \omega) \in R[\zeta_N][[q^{1/N}]]$$

—— o ——

The Hasse Invariant H is a modular form of weight $p-1$ & level 1 $/\mathbb{F}_p$:

$$H \in G_{p-1}(\Gamma_1(1); \mathbb{F}_p)$$

Important Properties:

1) $H(\mathrm{Tate}(q), \underset{\text{pull back of } \frac{dt}{t}}{\omega_{can}}) = 1$

2) $H((E, \omega)_{/\bar{\mathbb{F}}_p}) = 0 \iff E$ is s.s.

3) H can be lifted (non-canonically) to $W = \mathbb{Z}_p$.

Let $\mathcal{E}$ be a lift of H to $W = \mathbb{Z}_p$, fix $p \nmid N$.

Let $(\underline{E}, \underline{\phi_N}, \underline{\omega})$ be the semi-abelian variety extending the universal ell. curve, so

$$\begin{array}{c}(\mathbb{E}, \underline{\phi_N}) \\ \downarrow \\ M = X_1(N)_{/W}\end{array}$$

Let $\mathcal{M}_m = X_1(N)_{/W_m} = X_1(N) \times_W W_m$, where $W_m = \mathbb{Z}/p^m\mathbb{Z}$.

We consider the ordinary locus

$$S_m = \mathcal{M}_m[\tfrac{1}{\mathcal{E}}], \quad S_0 = \mathcal{M}[\tfrac{1}{\mathcal{E}}]$$

Indep of $\mathcal{E}$ Dep of $\mathcal{E}$

We also take $S_\infty = \varprojlim_m S_m$ (formal completion of S_0 along S_1)

In case of $GL(2)$: $S_m = \mathrm{Spec}(V_{m,0})$ is affine for $V_{m,0}$ = flat W_m-alg. !

The Igusa tower consists of

$$T_{m,\alpha}, \quad T_{m,\alpha}(R) = \{[(E, P \in E[p^\alpha], \phi_N)_{/R}]\}$$

$$\mathrm{Aut}(\mu_{p^\alpha}) = (\mathbb{Z}/p^\alpha\mathbb{Z})^\times \curvearrowright \big\downarrow \text{étale, Galois} \qquad \cong \{[(E, \phi_p : \mu_{p^\alpha} \xrightarrow{\sim} E[p^\alpha]^\circ, \phi_N)_{/R}]\}$$

$$S_m$$

We see $\underbrace{T_{m,\alpha/S_m}}_{\mathrm{Spec}(V_{m,\alpha})} = (E[p^\alpha]^{\text{ét}} - E[p^{\alpha-1}]^{\text{ét}})_{/S_m}$ is affine.

The curve $T_{m,\alpha}/W_m$ is called the Igusa curve of level Np^α.

We also have $T_{m,\infty/W_m} = \varprojlim_\alpha T_{m,\alpha}$, which is a $\mathbb{Z}_p^\times$-cover of S_m

<u>THEOREM</u> The Igusa curve $T_{m,\alpha/\mathbb{F}_p}$ is irred. for all $m, \alpha \geq 1$.

We have $V_{m,0} \subset V_{m,1} \subset \ldots$ & $T_{m,\infty} = \mathrm{Spec}(V_{m,\infty})$, where $V_{m,\infty} = \bigcup_\alpha V_{m,\alpha}$

We define

$\mathbb{Q}_p/\mathbb{Z}_p$-duals?

$$\mathcal{V} = \mathcal{V}_{\Gamma_1(N)} := \varinjlim_m V_{m,\infty} \quad \& \quad V = V_{\Gamma_1(N)} := \varprojlim_m V_{m,\infty}$$

The space $V_{\Gamma_1(N)}$ is the space of p-adic mod. forms of level $\Gamma_1(N)$ (no weight yet!)

We can see $f \in V_{m,\alpha}$ as $(E, \phi_p : \mu_{p^\alpha} \hookrightarrow E[p^\alpha]^\circ, \phi_N)_{/R} \mapsto R$ s.t.

(Gp1) $f(E, \phi_p, \phi_N) = f(E', \phi_p', \phi_N')$

(Gp2) $f((E, \phi_p, \phi_N) \times_{R,\rho} R') = \rho(f(E, \phi_p, \phi_N))$

(Gp3) $f(\mathrm{Tate}(q), \phi_p^{can}, \phi_N) \in W[[q^{1/N}]]$

where $\phi_p^{can} : \mu_{p^\alpha} \hookrightarrow \mathbb{G}_m(W[[q^{1/p^\alpha}]])/q^{\mathbb{Z}}$
$= \mathrm{Tate}(q)(W[[q^{1/p^\alpha}]])$

As a consequence of the irreducibility of $T_{m,\alpha}$, we get

<u>COROLLARY</u> If q-expansion of $f \in V_{\Gamma_1(N)} \hat{\otimes}_W B$ vanishes, for B = p-adically complete W-alg, then $f = 0$.

Part 2

Goal: Study the action of $T(\mathbb{Z}_p) = \mathbb{G}_m(\mathbb{Z}_p) = \mathbb{Z}_p^\times$ on $\mathcal{V}_{\Gamma_1(N)}$

And construct ordinary projector e out of Hecke operators.

Let's first approach this axiomatically: $(k \geq 3)$

It's easy to see that $H^0(X_1(N)_{/R}, \underline{\omega}^k) \overset{(*)}{\cong} G_k(\Gamma_1(N); R)$ for any $\mathbb{Z}[1/6N]$-alg R.

$$\begin{tikzcd} (E,\phi_N,\omega) \arrow[r, mapsto] \arrow[d] & (E,\phi_N,\omega) \arrow[d] \\ \operatorname{Spec} R \arrow[r] & X_1(N) \end{tikzcd}$$

Similarly, $H^0(S_m, \underline{\omega}^k) = V_{m,\infty}[k] = \{(z.f) = z^k f,\ \forall z \in \mathbb{Z}_p^\times\}$

$f \in H^0(S_m, \underline{\omega}^k)$, $\quad$ Given $\phi_p : \mu_{p^\infty} \to E$

$\Big\}$ using $(*)$ $\qquad \omega_{can} = \frac{dt}{t} \mapsto (\phi_p)_*(\omega_{can}) =: \omega$

f is a fct on $(E, \phi_N, \omega)_{/R}$

$$f(E, \phi_p, \phi_N) := f(E, \phi_N, (\phi_p)_*(\omega_{can}))$$

$$\& \quad (z.f)(E, \phi_p, \phi_N) = f(E, z^{-1}\phi_p, \phi_N) = f(E, \phi_N, z^{-k}\omega)$$
$$= z^k f(E, \phi_N, \omega) = z^k f(E, \phi_p, \phi_N)$$

By taking limits, $\underbrace{\mathcal{V}[k]}_{\text{p-divisible}} = H^0(S_{\infty/W}, \underline{\omega}^k \otimes \mathbb{Q}_p/\mathbb{Z}_p)$

$$\cong H^0(S_{\infty/W}, \underline{\omega}^k) \otimes \mathbb{Q}_p/\mathbb{Z}_p$$

This $\cong$ is because S_0 is affine $\rightsquigarrow$ base change is easy.
In general, this is hard.

One axiom needed is

(F_k) $\operatorname{corank}_W e\mathcal{V}[k] = \operatorname{rk}_W \operatorname{Hom}(e\mathcal{V}[k], \mathbb{Q}_p/\mathbb{Z}_p)$
is finite for some k.

In practice, we show this by proving

$$(*)\quad \underbrace{eH^0(\mathfrak{I}_{\infty/W}, \underline{\omega}^k \otimes \mathbb{Q}_p/\mathbb{Z}_p)}_{p\text{-divisible}} \cong \underbrace{eH^0(X_1(N)_{/W}, \underline{\omega}^k \otimes \mathbb{Q}_p/\mathbb{Z}_p)}_{\text{finite corank b/c } X_1(N) \text{ is projective}}$$

Assume $p>2$, $\mathbb{Z}_p^\times = \underbrace{(1+p\mathbb{Z}_p)}_{=\Gamma_T} \times \underbrace{(\mathbb{Z}/p\mathbb{Z})^\times}_{=\Delta}$. $\gamma \in \Gamma_T$ generator

p-profinite group $\to \Gamma_T$ $\qquad$ Δ prime-to-p finite g'p

Let $\Lambda = W[[X]] \simeq W[[\Gamma_T]]$ by $1+X \mapsto \gamma$ $\quad ((1+X)^s \mapsto \gamma^s)$

We have $W[[\mathbb{Z}_p^\times]] = \Lambda[\Delta]$

Let $\mathcal{V}^{ord} = e\mathcal{V}$ & $V^{ord} = \operatorname{Hom}_{cont}(\mathcal{V}^{ord}, \mathbb{Q}_p/\mathbb{Z}_p)$ be its Pontryagin dual.

<u>THEOREM</u> Suppose that (F_k) holds for one $k \in \mathbb{Z}$.

Then, V^{ord} is finite-free over Λ.

Moreover, if $(*)$ holds for $\kappa \in X(T) = X(\mathbb{G}_m) = \mathbb{Z}$ such that $\kappa|_\Delta = k|_\Delta$ (i.e. $\kappa \equiv k \ (p-1)$), then

$$\star = V^{ord} \otimes_{W[[T(\mathbb{Z}_p)]]} W \cong \operatorname{Hom}_W(G_\kappa^{ord}(\Gamma_1(N); W), W)$$

$$\begin{tikzcd} \operatorname{Spec} \star \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \operatorname{Spec} V^{ord} \arrow[d] \\ \operatorname{Spec} W \arrow[r, "\kappa"] & \operatorname{Spec} W[[T(\mathbb{Z}_p)]] \end{tikzcd}$$

Proof Let's do the 2^{nd} part with $\kappa = k$ first.

Let $\chi = k|_\Delta \in \hat{\Delta} = \mathrm{Hom}(\Delta, W)$. Then,

$$\begin{aligned}
\star &= V^{ord}[\chi] \otimes_{\Lambda, k} W \\
&= V^{ord}[\chi] / (X+1-\gamma^k) V^{ord}[\chi] \\
&= \mathrm{Hom}_{cont}(\underbrace{\mathcal{V}^{ord}[\chi, (X+1-\gamma^k)]}_{= e\mathcal{V}[k]}, \mathbb{Q}_p/\mathbb{Z}_p) \\
&\cong \mathrm{Hom}_{\mathbb{Z}_p}(\varinjlim_m H^0_{ord}(X_1(N)_{/W_m}, \underline{\omega}^k_{/W_m}), \varinjlim_m p^{-m}\mathbb{Z}_p/\mathbb{Z}_p) \\
&\cong \varprojlim_m \mathrm{Hom}_W(H^0_{ord}(X_1(N)_{/W_m}, \underline{\omega}^k_{/W_m}), W_m) \\
&= \mathrm{Hom}_W(H^0_{ord}(X_1(N), \underline{\omega}^k), W) \\
&\cong \mathrm{Hom}_W(G_k(\Gamma_1(N); W), W)
\end{aligned}$$

$(X[\alpha])^* = X^*/\alpha X^*$

finite rank by (F_k)

by $(\dagger)$

The same can be done for any $\kappa|_\Delta = k|_\Delta$.

For the first part, we know by (F_k) & above that

$$V^{ord}[\chi]/(X+1-\gamma^k)V^{ord}[\chi] \text{ is finite free } /W$$

$$\overset{\text{NAKAYAMA}}{\Rightarrow} V^{ord}[\chi] \text{ is finite } / \Lambda$$

$\Rightarrow \exists\ \pi : \Lambda^s \twoheadrightarrow V^{ord}[\chi]$, where $s = s(\chi) = \mathrm{corank}_W\, e\mathcal{V}[k]$

& π is $\cong$ modulo $(1+X-\gamma^k) \Rightarrow \mathrm{Ker}(\pi) \subset \bigcap_k \mathrm{Ker}(\pi \ (1+X-\gamma^k))$
$= 0$

The same is true for all $\kappa|_\Delta = \chi$, so $\Rightarrow V^{ord}[\chi]$ is finite free & $V^{ord} = \bigoplus_{\chi \in \hat{\Delta}} V^{ord}[\chi]$ $\square$

We can show that as long as our operator e satisfies
$$e(\mathcal{E}f) = \mathcal{E}(ef), \quad \forall f \in H^0(S_1, \underline{\omega}^k),$$
then $\dim_{\mathbb{Q}_p} G_k^{ord}(\Gamma_1(N), \mathbb{Q}_p)$ is unif bounded in k

$\Rightarrow$ (F_k) holds for all k

(In fact, if $k \geq 3$, this $\dim^n$ only depends on $k \in \mathbb{Z}/(p-1)\mathbb{Z}$)

In our case, $e = \varinjlim_n U(p)^{n!}$, where
$$f|_{U(p)}(E, \phi_p, \phi_N) = \frac{1}{p}\sum_C f(E/C, \phi_p, \phi_N),$$
where C goes over all order p subg'p outside $\mathrm{im}(\phi_p)$.

So final result is

<u>THEOREM</u> [VCT] If $k \geq 3$,
$$V^{ord} \otimes_{W[\![T(\mathbb{Z}_p)]\!],k} W \cong \mathrm{Hom}_W(G_k^{ord}(\Gamma_1(N); W), W)$$
And a similar result holds for cuspidal subspace.

We now consider $G(\mathcal{X}, \Lambda) = \mathrm{Hom}_\Lambda(V^{ord}[\mathcal{X}], \Lambda)$

Note that $a(n) : V^{ord} \to \Pi_p$, $a(n) \in V^{ord}$ $(\in V^{ord}[\mathcal{X}]?)$

Given $\Phi \in G(\mathcal{X}, \Lambda)$, let $\Phi(q) = a(n, \underset{\substack{\parallel \\ \Phi(a(n))}}{\Phi}) q^n \in \Lambda[\![q]\!]$

<u>THEOREM</u> For each $k \geq 2$, we have

(1) $G(\chi;\Lambda)$ is finite free $/\Lambda$

(2) $G(\chi;\Lambda)\otimes_{\Lambda,k} W \cong G_k^{ord}(\Gamma_1(N)\cap\Gamma_0(p), \chi\omega^{-k}; W)$,

where ω = Teichmüller character mod p

(3) The above $\cong$ is $\Phi \mapsto \sum_n a(n,\Phi)(\gamma^k-1)q^n \in W[[q]]$

The same can be done with tame level $\Gamma_0(N)$ & cusps.

We say $\Phi \in G(\chi,\Lambda)$ is a p-ordinary Λ-adic form & $\{\Phi(\gamma^k-1)\}_{k\geq 2}$ is a p-adic analytic family of modular forms