Hilbert Modular Varieties.

F totally real field $F \neq \mathbb{Q}$, discriminant $d(F)$

p unramified in F. $\mathcal{O} = \mathcal{O}_F$

$G = \mathrm{Res}_{\mathcal{O}/\mathbb{Z}}\, GL_2$, $G_1 = \mathrm{Res}_{\mathcal{O}/\mathbb{Z}}\, SL_2$, $PG = G^{ad} = \mathrm{Res}_{\mathcal{O}/\mathbb{Z}}\, PGL_2$

T torus of diagonals in G_1

$I = \mathrm{Hom}(F, \bar{\mathbb{Q}})$

$X(T) \simeq \mathbb{Z}[I]$

$G(\mathbb{Q})_+ = G(\mathbb{Q})^+ = \{ a \in G(\mathbb{Q}) \mid \det(\sigma a) > 0,\ \forall \sigma \in I \}$

$\mathfrak{a} \subset F$ lattice, $\mathfrak{a}^* = \{ x \in F \mid \mathrm{tr}_{F/\mathbb{Q}}\, \mathfrak{a}x \subset \mathbb{Z} \}$

$\delta^{-1} = \mathcal{O}^*$

$g = [F : \mathbb{Q}]$

AVRM $/S$ $\quad (A, \lambda, \iota)$

- A/S abelian scheme
- $\iota : \mathcal{O} \hookrightarrow \mathrm{End}(A)$
- λ $\mathcal{O}$-linear polarization of A s.t. Rosati involution acts trivially on $\mathcal{O}$
- $\mathrm{Lie}\, A \simeq \mathcal{O} \otimes_{\mathbb{Z}} \mathcal{O}_S$ as $\mathcal{O} \otimes_{\mathbb{Z}} \mathcal{O}_S$-mod

$\Leftrightarrow$ $\forall \alpha \in \mathcal{O}$, α acts on $\mathrm{Lie}\, A$ over $\mathcal{O}_S$ has char. poly.

$$\prod_{\sigma} (t - \sigma(\alpha))$$

$\lambda : A \longrightarrow {}^t A \Rightarrow \ker \lambda = A[\mathfrak{c}^{-1}]$, $\mathfrak{c}^{-1}$ integral ideal

λ is called $\mathfrak{c}$-polarization.

Prop.

If N invertible on S, $A[N] \simeq (\mathcal{O}/N\mathcal{O})^2$ etale locally

If S^{red} of char p, A/S ordinary then etale locally

$$A[p^n]^{\circ} \simeq \mathcal{O}^* \otimes_{\mathbb{Z}} \mu_{p^n}, \quad A[p^n]^{et} \simeq \mathcal{O}/p^n\mathcal{O}$$

Level structure.

$\mathfrak{c}$ integral ideal.

$\mathcal{M}(\mathfrak{c}, \Gamma(N)) : \mathrm{Sch}/\mathbb{Z}[\frac{1}{N}] \longrightarrow \mathrm{Sets}$

$S \longmapsto \{(A, \lambda, \iota, \phi_N)\}/\simeq$

- $(A, \lambda, \iota)/S$ AVRM, λ $\mathfrak{c}$-polar.
- $\phi_N : (\mathcal{O}^* \otimes_{\mathbb{Z}} \mu_n) \oplus (\mathcal{O} \otimes_{\mathbb{Z}} N^{-1}\mathbb{Z}/\mathbb{Z}) \xrightarrow{\sim} A[N]$

$\mathcal{O}$ - linear s.t. the Weil pairing

$e_N : A[N] \times {}^tA[N] \longrightarrow \mu_n$

compose with λ identifies ϕ_N with

$\langle \ \rangle_N : (\mathcal{O}^* \otimes \mu_n) \oplus (\mathcal{O} \otimes N^{-1}\mathbb{Z}/\mathbb{Z}) \longrightarrow \mu_N$

$(a \otimes \zeta) \oplus (b \otimes m) \longmapsto e^{2\pi i \, tr_{F/\mathbb{Q}} ab} \zeta^m$

Assume $N \geq 3$, $\mathcal{M}(\mathcal{C}, \Gamma(N))$ is rep'le by quasi-proj. scheme $/ \mathbb{Z}[\frac{1}{N}]$ and smooth over $\mathbb{Z}[\frac{1}{N d(F) c}]$, $c = \mathbb{Q} \cap \mathcal{C}$.

$\mathfrak{n}$ nonzero ideal of $\mathcal{O}$, $N = \mathfrak{n} \cap \mathbb{Q}$.

$\mu_{\mathfrak{n}}$ locally free group scheme

$$\mu_{\mathfrak{n}}(R) = \{ x \in \mathbb{G}_m(R) \otimes_{\mathbb{Z}} \mathcal{O}^* \mid \mathfrak{n} x = 0 \}$$

e.g. $\mathfrak{n} = (N)$, $\mu_{\mathfrak{n}} = \mu_N \otimes_{\mathbb{Z}} \mathcal{O}^*$

$$\mathcal{M}(\mathcal{C}, \Gamma_1^1(\mathfrak{n})) : \mathrm{Sch}/\mathbb{Z}[\tfrac{1}{N}] \longrightarrow \mathrm{Sets}$$

$$S \longmapsto \{(A, i_{\mathfrak{n}})\}/\simeq$$

$i_{\mathfrak{n}} : \mu_{\mathfrak{n}} \hookrightarrow A[\mathfrak{n}]$

$\mathcal{M}(\mathcal{C}, \Gamma_1^1(\mathfrak{n}))$ rep'le if $\mathfrak{n}$ deep enough.

$F_+^\times \subset F$ the group of totally positive elements,

$O_+^\times = F_+^\times \cap O^\times$

$$\mathcal{E}_{\Gamma_1(n), c} : Sch / \mathbb{Z}[\tfrac{1}{N}] \longrightarrow Sets$$

$$S \longmapsto \{(A, \iota, O_+^\times \lambda, i_n)\} / \simeq$$

We can always find $\epsilon \in O_+^\times$, $\epsilon \equiv 1 \mod n$

$\Rightarrow$ ϵ gives nontrivial auto.

$\mathcal{E}_{\Gamma_1(n), c}$ is not rep'l by schemes.

For Γ above, the coarse moduli scheme $M(c, \Gamma)$ always exists.

Complex analytic Hilbert modular forms.

Over $\mathbb{C}$, Riemann - Poincare - Lefschetz

$$(A, \lambda, \iota, \phi(i)) \sim (\mathcal{L}, \lambda, \phi(i))$$

$\mathcal{L}$ $\mathcal{O}$- lattice in $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{C} \simeq \mathbb{C}^I$

$\lambda : \mathcal{L} \wedge_{\mathcal{O}} \mathcal{L} \simeq \mathfrak{c}^*$

$\phi : \mathfrak{n}^* / \mathcal{O}^* \times \mathcal{O}/\mathfrak{n} \hookrightarrow F\mathcal{L}/\mathcal{L}$

$i : \mathfrak{n}^* / \mathcal{O}^* \hookrightarrow F\mathcal{L}/\mathcal{L}$

$$(\mathcal{L}, \lambda, \phi(i)) \longmapsto \mathbb{C}^2/\mathcal{L}$$

$$(A, \cdots) \longmapsto \mathcal{L}_A = \left\{ \int_\gamma \omega \in \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{C},\ \gamma \in H_1(A, \mathbb{Z}) \right\}$$

$\mathcal{Z} = \mathcal{H}^1 \subset \mathbb{C}^1$

identify $\mu_N \simeq N^{-1}\mathbb{Z}/\mathbb{Z} \implies \mu_{(N)} \simeq (N)^*/\mathcal{O}^*$

$$\mu_{\mathfrak{n}} \simeq \mathfrak{n}^*/\mathcal{O}^*$$

Choose $\mathfrak{a}, \mathfrak{b}$ prime to $\mathfrak{n}$ s.t. $\mathfrak{a}\mathfrak{b}^{-1} = \mathfrak{c}$

$z \in \mathcal{Z}$, $\quad L_z = 2\pi i(\mathfrak{b}z - \mathfrak{a}^*)$

$$\lambda_z(2\pi i(az-b),\ 2\pi i(cz-d)) = ad - bc \in \mathbb{C}^*$$

$$i_z : \mu_{\mathfrak{n}} \simeq \mathfrak{n}^*/\mathcal{O}^* \hookrightarrow \mathbb{C}^1/L_z$$

$$[a] \longmapsto -2\pi i a + L_z$$

$$\phi_{\mathfrak{n},z} : \mathfrak{n}^*/\mathcal{O}^* \times \mathcal{O}/\mathfrak{n} \longrightarrow \mathbb{C}^1/L_z$$

$$a \quad b \longmapsto [2\pi i(-a+bz)]$$

Define congruence subgps of $G(\mathbb{Q})_+$

$$\Gamma_1(\mathfrak{n}; \mathfrak{a}, \mathfrak{b}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \begin{pmatrix} \mathcal{O} & (\mathfrak{a}\mathfrak{b})^* \\ \mathfrak{n}\mathfrak{a}\mathfrak{b}\delta & \mathcal{O} \end{pmatrix} \middle| ad - bc \in \mathcal{O}_+^\times,\ a-1 \in \mathfrak{n} \right\}$$

$$\Gamma_1'(\mathfrak{n}; \mathfrak{a}, \mathfrak{b}) = \Gamma_1(\mathfrak{n}; \mathfrak{a}, \mathfrak{b}) \cap SL_2(F)$$

$$\Gamma(\mathfrak{n}; \mathfrak{a}, \mathfrak{b}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1'(\mathfrak{n}; \mathfrak{a}, \mathfrak{b}) \middle| b \in \mathfrak{n}(\mathfrak{a}\mathfrak{b})^* \right\}$$

$$(A_z, \lambda_z, i_z) \simeq (A_w, \lambda_w, i_w) \Leftrightarrow w \in \Gamma_1'(\mathfrak{n}; \mathfrak{a}, \mathfrak{b}) z$$

$$\Rightarrow M(c, \Gamma_1'(\mathfrak{n}))(\mathbb{C}) = \Gamma_1'(\mathfrak{n}; \mathfrak{a}, \mathfrak{b}) \backslash \mathfrak{Z}$$

Fact. $(\mathfrak{a}, \mathfrak{b})$ s.t. $\mathfrak{a}\mathfrak{b}^{-1} = \mathfrak{c}$ are bijective to cusps of $\Gamma(\mathcal{O}; \mathcal{O}, \mathfrak{c}^{-1})$, i.e. $\Gamma \backslash \mathbb{P}^1 F$

$$\gamma \in SL_2(F). \quad \gamma \begin{pmatrix} \mathfrak{a}^* \\ \mathfrak{b} \end{pmatrix} = \begin{pmatrix} \mathcal{O}^* \\ \mathfrak{c}^{-1} \end{pmatrix}$$

$$\Gamma_1'(\mathfrak{n}; \mathcal{O}, \mathfrak{c}^{-1}) \backslash \mathfrak{Z} \xrightarrow[\gamma]{\sim} \Gamma_1'(\mathfrak{n}; \mathfrak{a}, \mathfrak{b}) \backslash \mathfrak{Z}$$

$X(T) \simeq \mathbb{Z}[I]$

$\underline{\omega}^{\kappa}$ over $\mathcal{M}$ be the auto. v.b. of wt $\kappa \in X(T)$

($\pi : A^{univ} \longrightarrow \mathcal{M}$, $\underline{\omega} = \pi_* \Omega$ is $T = Res_{\mathcal{O}/\mathbb{Z}} \mathbb{G}_m$ - mod

$\Rightarrow \underline{\omega} = \bigoplus_{\sigma \in I} \underline{\omega}^{\sigma}$.

$\kappa \in \mathbb{Z}[I]$. $\underline{\omega}^{\kappa} = \otimes (\underline{\omega}^{\sigma})^{\otimes \kappa_\sigma}$)

$G_\kappa(\mathfrak{c}, \mathfrak{n}; R) = H^0(\mathcal{M}(\mathfrak{c}, \mathfrak{n})_{/R}, \underline{\omega}^{\kappa})$

$R / \mathbb{Z}[\frac{1}{N d(F)}]$

$G_\kappa(\mathfrak{c}, \Gamma_1 \mathfrak{n}; R) = H^0(\mathcal{M}(\mathfrak{c}, \Gamma_1 \mathfrak{n})_{/R}, \underline{\omega}^{\kappa})$

$G_\kappa(\mathfrak{c}, \Gamma(N); R) = H^0(\mathcal{M}(\mathfrak{c}, \Gamma(N))_{/R}, \underline{\omega}^{\kappa})$

$f \in G_\kappa(\mathfrak{c}, \mathfrak{n}; \mathbb{C})$, $f : \mathfrak{Z} \longrightarrow \mathbb{C}$ holo.

$f(z) = f(\mathcal{L}_z, \mathcal{O}_+^{\times} \lambda_z, i_z)$

$f(\gamma z) = (cz+d)^{\kappa} f(z)$, $\gamma \in \Gamma_1(\mathfrak{c}, \mathfrak{n})$

At cusp $(\mathfrak{a}, \mathfrak{b})$ $f(z) = \sum_{\xi \in \mathfrak{a}\mathfrak{b}} a(\xi) e^{2\pi i \xi z}$

Toroidal compactification.

Let (a, b) be a cusp for $\mathcal{M}(C, \Gamma(1))$

it is fixed by $\Gamma_\infty(0; a, b) = \Gamma(0; a, b) \cap B(\mathbb{Q})$

B upper triangular Borel of G

cusps of $\mathcal{M}(C, \Gamma(N))$ are $(a, b, \phi_N) / \Gamma_\infty(0; a, b)$

Study toroidal compactification of $\mathcal{M}(C, \Gamma(N))$ at cusp (a, b, ϕ_N).

$C = \{ \xi \in F_\infty \mid \sigma\xi > 0 \quad \forall \sigma \in I \}$ cone

a simplicial cone σ in C of dim m is an open span $\sigma = \langle v_1, \dots, v_m \rangle_{\mathbb{R}_+}$ for $\{v_i\}$ linear independent cone decomposition $\mathcal{C}$: $C = \coprod_{\sigma} \sigma$

Choose $\mathcal{C} = \mathcal{C}(\mathfrak{a}, \mathfrak{b}, \phi_N)$ s.t.

PC 1 σ open simplicial cone

PC 2 cones in $\mathcal{C}$ are permuted under multiplication by ϵ^2, $\epsilon \in T(\mathbb{Z})(N) = \{\epsilon \in \mathcal{O}^\times \mid \epsilon \equiv 1 \mod N\}$

There are finitely many orbits and

$\epsilon(\sigma) \cap \sigma \neq \phi \Rightarrow \epsilon = 1$

PC 3 σ smooth i.e. generated by part of $\mathbb{Z}$-basis of $(\mathfrak{a}\mathfrak{b})^*$

PC 4 $\mathcal{C}$ good enough to make toroidal compact. projective.

minimal compactification $M^*(\mathbb{C}, \Gamma(N))$ is characterized by it is covered by any smooth compactification of $M(\mathbb{C}, \Gamma(N))$

The formal stalk of M^* at cusp $(\mathfrak{a}, \mathfrak{b}, \phi_N)$ is $H^0(T(\mathbb{Z})(N), \hat{R}_S(N))$

$$\hat{R}_S(N) = \left\{ a_0 + \sum_{\xi \in N^{-1}\mathfrak{a}\mathfrak{b} \cap C} a_\xi q^\xi \;\middle|\; a_\xi \in \mathbb{Z}[\tfrac{1}{N}, \mu_N] \right\}$$

$\epsilon \in T(\mathbb{Z})(N)$ acts by $q^\xi \longmapsto q^{\epsilon^2 \xi}$

Thm. Let $\mathcal{C} = \{\mathcal{C}_S\}_S$ collection of cone decompositions with PC 1-4 indexed by equiv. classes of cusps of $M(\mathbb{C}, \Gamma(N))$ then there is a smooth proj. toroidal compactification $M_{\mathcal{C}}(\mathbb{C}, \Gamma(N))$ s.t.

(1) the semi-abelian scheme $\mathcal{G}_{\mathcal{C}}$ extends A^{univ} over $M(\mathbb{C}, \Gamma(N))$ and

degenerates into $\mathbb{G}_m \otimes \mathfrak{a}^*$ over cuspidal divisor D_s for $s = (\mathfrak{a}, \mathfrak{b}, \phi_N)$ and coincide with $\mathrm{Tate}_s(q)$ on the formal completion $\widehat{\mathcal{M}}_s(c, \Gamma(N))$ of $\mathcal{M}_e$ along D_s.

$\mathcal{M} = \mathcal{M}(c, \Gamma(N)), \mathcal{M}(c, \Gamma_1(n)), \mathcal{M}(c, n) \quad / \mathbb{Z}[\frac{1}{N}]$

$e \Rightarrow M \quad / \mathbb{Z}[\frac{1}{N}, \mu_N]$, $\mathcal{G}/M$ semi-AV with level struc. canonically determined by e.

$\pi : \mathcal{G} \to \mathcal{M}_e$, $\underline{\omega} = \pi_* \Omega$ locally free over $\mathcal{M}_e$

Koecher principle: $H^0(\mathcal{M}_e, \det(\underline{\omega})^{\otimes j}) = H^0(\mathcal{M}, \det(\underline{\omega})^{\otimes j})$
$j \in \mathbb{Z}$

$\mathcal{O}_M = \bigoplus_{k \geq 0} H^0(\mathcal{M}_e, \det(\underline{\omega})^{\otimes k})$, $\quad M^* = \mathrm{Proj}(\mathcal{O}_M)$

$= \mathcal{M} \cup \{\text{cusps}\}$

Fix $\overline{\mathbb{Q}} \longrightarrow \overline{\mathbb{Q}}_p$, W pullback of $\check{\mathbb{Z}}_p$

$W \otimes \mathcal{O} \simeq W^I$, $W = \widehat{W}^p$

Assume $[$ prime to p.

$\mathcal{G}/M$ over $\mathbb{Z}[\frac{1}{\mathrm{Nd}(F)}, \mu_N]$

$\pi: \mathcal{G} \to M$, $\underline{\omega} = \pi_* \Omega$ has $T = \mathrm{Res}_{\mathcal{O}/\mathbb{Z}}\, \mathbb{G}_m$ action

coming from action of $\mathcal{O}$

$\Rightarrow \underline{\omega} = \bigoplus_{\sigma \in I} \underline{\omega}^\sigma$

$\forall \kappa \in \mathbb{Z}[I]$, $\underline{\omega}^\kappa = \bigotimes (\underline{\omega}^\sigma)^{\otimes \kappa_\sigma}$ line bundle $/M$

R W-alg. of char p

(A, ω) semi-abelian scheme $/R$ of rel. dim g

$A \simeq \mathcal{G} \times_W R$

ω base of $\Omega_{A/R}$ over R

$F_{abs}: A_{/R} \longrightarrow A_{/R}$

η dual base of ω for $T_{A/R}$

D derivation of $\mathcal{O}_{A,0}$, $D^p(xy) = xD^p y + yD^p x$

$\Rightarrow$ $F^*: D \longrightarrow D^p$ is F_{abs} - linear endo. of $T_{A/R}$

Define $H(A,\omega) \in R$ by $F^*(\wedge^g \eta) = H(A,\omega)(\wedge^g \eta)$

$H(A,\omega) = 0 \Leftrightarrow A$ not ordinary

$H \in H^0\left(\left(\mathrm{Proj} \bigoplus_{k\in\mathbb{Z}[2]} \underline{\omega}^k\right) \otimes_W \mathbb{F}, \underline{\omega}^{(p-1)2}\right)$

as $\bigoplus_{k\in\mathbb{Z}[2]} \underline{\omega}^k$ rep's the functor $R \longmapsto (A,\omega)/\simeq$

and for $a \gg 0$

$$H^0(M^*, \underline{\omega}^{a(p-1)2}) \otimes_W \mathbb{F} = H^0(M^*_{\mathbb{F}}, \underline{\omega}^{a(p-1)2})$$

So we have a lift of H^a

$$E \in H^0(M^*, \underline{\omega}^{a(p-1)1}) = H^0(\mathcal{M}, \underline{\omega}^{a(p-1)1})$$
$$= H^0(M, \underline{\omega}^{a(p-1)1})$$

$S^* = M^*[\frac{1}{E}]$ is affine and irr. as E is a section of an ample line bundle.

$S^*_m = S^* \times_W W_m$, $S^*_\infty = \varprojlim S^*_m$ affine formal scheme

$S = M[\frac{1}{E}]$. S_m, S_∞

M_∞ formal completion of M along $M_1 = M \times_W W_1$

then $S_\infty \subset M_\infty$ is ordinary locus

Let $T_{m,n}/W_m = \mathrm{Isom}_{\mathcal{O}}(\mathcal{O}\otimes\mu_{p^n}, \mathcal{G}[p^n]^\circ)$

$$\simeq \mathrm{Isom}_{\mathcal{O}}(\widehat{\mathcal{G}[p^n]^\circ}, \mathcal{O}/p^n\mathcal{O})$$

$$\simeq \widehat{\mathcal{G}[p^n]^\circ} - \bigcup_{p^n \subsetneq \mathfrak{n}} \widehat{\mathcal{G}[p^n]^\circ[\mathfrak{n}]}$$

then $T_{m,n}/S_m$ etale covering. Gal $= T(\mathbb{Z}/p^n\mathbb{Z}) = (\mathcal{O}/p^n)^\times$

called Hilbert modular Igusa tower over S_m

K. A. Ribet: $T_{m,n}$ is irr.

$\Rightarrow$ q-exp. principle.

Define $\underline{\omega}_\kappa \subset \underline{\omega}^\kappa$ invertible subsheaf vanishing over cuspidal divisors, called sheaf of cusp forms of wt κ.

p-adic Hilbert modular forms of level $\Gamma(N)$

fix N prime to p

the proof of VCT extends to our case except

$$(*) \quad H^0(S, \underline{\omega}_\kappa) \otimes_W W_m \simeq H^0(S_m, \underline{\omega}_\kappa \otimes_W W_m)$$

where in elliptic modular form case S_m is affine.

Write $\pi : S \longrightarrow S^*$. $H^0(S, \mathcal{L}) = H^0(S^*, \pi_* \mathcal{L})$ as S^* affine.

ETS $\quad \pi_*(\underline{\omega}_\kappa) \otimes_W W_m \simeq \pi_*(\underline{\omega}_\kappa \otimes_W W_m)$

as then

$$\begin{aligned} H^0(S, \underline{\omega}_\kappa) \otimes_W W_m &= H^0(S^*, \pi_* \underline{\omega}_\kappa) \otimes W_m \\ &= H^0(S^*_m, (\pi_* \underline{\omega}_\kappa) \otimes W_m) \qquad S^* \text{ affine} \\ &= H^0(S^*_m, \pi_*(\underline{\omega}_\kappa \otimes W_m)) \\ &\simeq H^0(S_m, \underline{\omega}_\kappa \otimes W_m) \end{aligned}$$

Check (*) stalk by stalk.

Outside cusps true.

At cusp $x = (\mathfrak{a}, \mathfrak{b})$

$$\widehat{\pi_*(\underline{\omega}^k/R)_x} \simeq H^0(T(\mathbb{Z})(N), R[\![\tfrac{1}{N}(\mathfrak{a}\mathfrak{b})_{\geq 0}]\!])$$

$$= \left\{ \sum_{\xi \in \frac{1}{N}(\mathfrak{a}\mathfrak{b})_{\geq 0}} a(\xi) q^{\xi} \;\middle|\; a(\epsilon^2 \xi) = \epsilon^k a(\xi) \right\}$$

$(\mathfrak{a}\mathfrak{b})_{\geq 0} = \{ \xi \in \mathfrak{a}\mathfrak{b} \mid \xi \text{ totally positive} \} \cup \{0\}$

$\epsilon \in T(\mathbb{Z})(N)$ acts on $R[\![\tfrac{1}{N}(\mathfrak{a}\mathfrak{b})_{\geq 0}]\!]$ by

$$\epsilon \cdot \left(\sum_{\xi} a(\xi) q^{\xi} \right) = \sum_{\xi} \epsilon^{-k} a(\epsilon^2 \xi) q^{\xi}$$

$H^0(T(\mathbb{Z})(N), R)$ commutes with $- \otimes_W W_m$ iff

$H^1(T(\mathbb{Z})(N), R) = 0$

For ω_k, the q- expansion has no constant term so

R is isom. to product of copies of $\mathrm{Hom}(\mathbb{Z}[U], R) = R^U$

so H^1 vanishes.

Set $V_{m,n} = H^0(T_{m,n}, \mathcal{O})$

$\mathcal{V} = \varinjlim_m V_{m,\infty}$, $V = \varprojlim_m V_{m,\infty}$

$R_\ell = \bigoplus_{k \geq \ell} H^0(M, \underline{\omega}^k)$

$V_{m,n}^{cusp} = H^0(T_{m,n}, \mathcal{O}(-D_{cusp}))$

$\mathcal{V}_{cusp} = \varinjlim V_{m,\infty}^{cusp}$, $V_{cusp} = \varprojlim V_{m,\infty}^{cusp}$

$R_\ell^{cusp} = \bigoplus_{k \geq \ell} H^0(M, \underline{\omega}_k)$

$D_\ell = \beta(R_\ell)[\frac{1}{p}] \cap V$, $D_\ell^{cusp} = \beta(R_\ell^{cusp})[\frac{1}{p}] \cap V_{cusp}$

where $D_{cusp} = (M - \mathcal{M}) \times_M T_{m,n}$

$\beta(\sum_k f_k) = \sum_k \beta_k f_k$

$(\beta_k f_k)(A, \lambda, \phi_p) = f_k(A, \lambda, \phi_p, \omega_{can})$

β_K sends classical modular forms to p-adic ones

where W_{can} induced by $\frac{dt}{t} \otimes 1$ on $\hat{\mathbb{G}}_m \otimes \mathcal{O}^*$.

We shall define Hecke operators

$U(p)$, $T(1,p)$ acting on cusp forms of level divisible by p / prime to p

$$e = \lim_n U(p)^{n!} \quad , \quad e^\circ = \lim_n T(p)^{n!}$$

$$(-)_{ord} = e - \quad , \quad (-)^{ord} = e^\circ - .$$

Thm.

$\mathcal{M} = M(C, \Gamma(N))$. $S = \mathcal{M}[\frac{1}{E}]$

Suppose $N \geq 3$, p prime to $N d(F) C$. Then

(1) D_e^{cusp} dense in V_{cusp}

(2) $V_{cusp}^{ord,*} = \mathrm{Hom}_W(V_{cusp}^{ord}, W)$ is proj. $W[\![T(\mathbb{Z}_p)]\!]$-mod of f.t.

(3) $V_{cusp}^{ord,*} \otimes_{W[\![T(\mathbb{Z}_p)]\!], \kappa} W \simeq \mathrm{Hom}_W(H^0_{ord}(S, \underline{\omega}_\kappa), W)$

if $\kappa \geq 3I$. $\kappa : W[\![T(\mathbb{Z}_p)]\!] \longrightarrow W$

$x \longmapsto \prod_\sigma \sigma(x)^{\kappa_\sigma}$

(4) If $\kappa \geq 3I$, e induces

$$\begin{array}{ccc} H^0_{ord}(S, \underline{\omega}_\kappa) & \simeq & H^0_{ord}(\mathcal{M}, \underline{\omega}_\kappa) \\ \| & & \| \\ e\, H^0(S, \underline{\omega}_\kappa) & & e^0\, H^0(\mathcal{M}, \underline{\omega}_\kappa) \end{array}$$

Hecke operators.

R is called p-adic if it is p-adic complete.

$\mathfrak{q}$ prime ideal of F, we shall define $U(\mathfrak{q}^r)$ and

$T(\mathfrak{q}) = T(1, \mathfrak{q})$.

$U(p) = \prod_{\mathfrak{p}|p} U(\mathfrak{p})$, $\quad T(p) = \prod_{\mathfrak{p}|p} T(\mathfrak{p})$.

A subgroup C of A/S is cyclic of order $\mathfrak{q}^r$

if $C \simeq \mathcal{O}/\mathfrak{q}^r$ or $C \simeq \mu_{\mathfrak{q}^r}$ etale locally.

(etale cyclic)

Consider $\Gamma_1^1(\mathfrak{n})$-level.

- Classical modular forms.

  1) Assume $N(\mathfrak{q})$ invertible in base R.

Let C' etale cyclic subgrp of $A/_S$ of order q^r

$\pi: A \longrightarrow A' = A/C$ is etale.

c-polar. λ induces cq^r-polar. $\pi_* \lambda$.

If $q \mid n$, $\mu_q \subset \mu_n$, $C = i(\mu_q)$.

C' is disjoint from C if $C \cap C'$ reduced to $\{0\}$

$\Rightarrow$ $\pi \circ i$ gives $\Gamma_1'(n)$-level on A'.

For $f \in G_k(cq^r, \Gamma_1'(n); R)$

$$(U(q^r) f)(A, \lambda, i, \omega) = \frac{1}{N(q^r)} \sum_{C'} f(A/C', \pi_* \lambda, \pi \circ i, (\pi^*)^{-1} \omega)$$

where C' runs over all etale cyclic subgrps of order q^r that disjoint from C.

If $q \nmid n$, for $f \in G_k(C_{q^r}, \Gamma_1'(n); R)$

$$(T(1,q^r)f)(A,\lambda,\iota,\omega) = \frac{1}{N(q^r)} \sum_{C'} f(A/C', \pi_*\lambda, \pi\circ\iota, (\pi^*)^{-1}\omega)$$

where C' runs over all etale cyclic subgps of order q^r.

Check $U(q^r)f$, $T(1,q^r)f \in G_k(C, \Gamma_1'(n); R)$.

2) If $N(q)$ may not be invertible in R.

Consider $R[\frac{1}{N(q)}]$ flat over R, by flat base change

$$G_k(C, \Gamma; R[\tfrac{1}{N(q)}])$$

$$= H^0(\mathcal{M}(C,\Gamma), \underline{\omega}^{k2}/R[\tfrac{1}{N(q)}])$$

$$\simeq H^0(\mathcal{M}(C,\Gamma), \underline{\omega}^{k2}/R) \otimes_R R[\tfrac{1}{N(q)}]$$

$$= G_k(C, \Gamma; R) \otimes_R R[\tfrac{1}{N(q)}]$$

The operators are defined over $R[\frac{1}{Nq}]$ and check they preserve R-integral struture.

Serre-Tate deformation theory / q-expansion principle

$\Rightarrow$ $U(q^r)$ always integral

$T(1, q^r)$ integral if $k \gg 0$.

- p-adic modular forms

Over p-adic base ring R, consider

$(A, \lambda, i_p : \mu_{p^\infty} \otimes \mathcal{O}^* \hookrightarrow A[p^\infty], \phi)/S$

where ϕ is prime to p struture of level Γ.

If q prime to p, $N(q)$ invertible, define

$U(q^r)$ $(q \mid n)$, $T(1, q^r)$ $(q \nmid n)$

replacing $(\pi^*)^{-1}\omega$ by $\pi \circ i_p$.

Let $\mathfrak{P} | p$.

S_∞ / W formal completion of $M(C, \Gamma)[\frac{1}{E}]$ along p-fibre.

$(X^{univ}, \lambda^{univ}, \phi^{univ})_{/M(C,\Gamma)}$ universal object.

$\mathcal{C}' \subset X^{univ}[\mathfrak{P}]/_{S_\infty}$ étale cyclic of order $\mathfrak{P}$

$\Rightarrow$ $\mathcal{C}'$ can be defined only over S'_∞ / S_∞

locally free covering of rank $N(\mathfrak{P})$.

Given $(A, \lambda, \phi)_{/R}$, $\exists !$ φ: Spec $R \longrightarrow S_\infty$

$\varphi^*(\text{univ.}) = (A, \lambda, \phi)$.

$\varphi^* S'_\infty = \operatorname{Spec} R'$, R' locally free R-alg. of rank $N(P)$. $tr: R' \to R$.

$\varphi^* C' = C'$ étale cyclic subgp of $A_{/R'}$ of order P.

The operator

$$\big((N(P)U(P))f\big)(A, \lambda, i_p, \phi) = tr(f(A/C', \pi_* \lambda, \pi \circ i_p, \pi \circ \phi))$$

$$= \sum_{C'} f(A/C', \pi_* \lambda, \pi \circ i_p, \pi \circ \phi)$$

where C' cyclic subgp of order P - generically different from $A[P]^\circ_{/R}$.

Again use q-expansion or Serre - Tate to check divisibility by $N(P)$.

$U(q^r) = U(q)^r$.

$\lambda \longmapsto \xi^{-1}\lambda$

If $\xi \in F_{+}^{\times}$ then $G_K(C, r_1' n; R) \xrightarrow{\sim} G_K(C(\xi), r_1' n; R)$

$\Rightarrow$ $U(p)$, $T(p)$ acting on $G_K(C, r_1' n; R)$.

In summary

$U(q^r)$, $T(1, q^r)$ $f(A, \cdots)$

$$= \sum_{C'} f(A/C', \pi \circ (\cdots))$$

where C' runs over all etale cyclic subgrps of order q^r s.t.

1) $(U(q^r))$ disjoint from C, $C = A[p]^\circ$ $p \mid P$, $i(\mu_q)$ $q \mid n$

2) $(T(1, q^r))$ no condition