Recall, we have

$\Gamma(N) \rightsquigarrow$ Classifies basis of $A[N]$
$\Gamma_1^1(\mathfrak{n}) \rightsquigarrow$ Classifies point of $A[N]$ of order $\mathfrak{n}$
$\Gamma_1(\mathfrak{n}) \rightsquigarrow$ Classifies point of $A[N]/\mathbb{A}_F^{\hat{\mathcal{O}}}$ (or order $\mathfrak{n}$)
$\Gamma_0(\mathfrak{n}) \rightsquigarrow$ " subgp of $A[N]$ of order $\mathfrak{n}$

$\rightsquigarrow \widetilde{\Gamma}(\mathfrak{n}), \widetilde{\Gamma}_1^1(\mathfrak{n}), \widetilde{\Gamma}_1(\mathfrak{n}), \widetilde{\Gamma}_0(N) \subset G(\mathbb{A}^{(\infty)})$

$\rightsquigarrow S_?^?(-) = \begin{pmatrix} d & \\ & 1\end{pmatrix}^{-1} \widetilde{\Gamma}_?^?(-) \begin{pmatrix} d & \\ & 1\end{pmatrix}, \quad d\hat{\mathcal{O}} \cap F = \partial$

$$\Rightarrow G(\mathbb{Q})\backslash G(\mathbb{A})/S(N)Z(\mathbb{R})C_0 = \coprod_{\mathfrak{c} \in Cl_F^+(N)} \mathfrak{M}(\mathfrak{c}, \Gamma(N))(\mathbb{C})$$

In fact, $Sh_{S(N)}(G,X)_{/\mathbb{Q}(\mu_N)} = \coprod_{\mathfrak{c} \in Cl_F^+(N)} \mathfrak{M}(\mathfrak{c}, \Gamma(N))$

Picture:

$$\begin{tikzcd} \mathfrak{M}(\mathfrak{c}, \Gamma(N)) \subset Sh_{S(N)}(G,X) \arrow[d] & /\mathbb{Q}(\mu_N) \\ Sh^{(p)}_{S(N)}(G,X) & /\mathbb{Z}_{(p)}(\mu_N) \end{tikzcd}$$

Goal: Construct $\omega_\varepsilon^\kappa$ (& $\omega_{\kappa,\varepsilon}$) on $Sh^{(p)}$ w/

1) Compatibility $Sh = \varprojlim_K Sh_K$

$S_0(\mathfrak{n})\begin{pmatrix} a^{-1} & \\ & d\end{pmatrix} S_0(\mathfrak{n}) \cap GL_2(F_{\mathfrak{n}})$
$= M_2(\hat{\mathcal{O}}) \cap GL_2(\mathbb{A}_F^{(\mathfrak{n}\infty)})$

2) $$\begin{tikzcd} \omega_\varepsilon^\kappa \arrow[d] & \omega_\varepsilon^\kappa \arrow[l, "g^*"'] \arrow[d] \\ Sh \arrow[r, "g^{-1}"] & Sh \end{tikzcd}$$, $\forall g \in \Delta_0(\mathfrak{n}) = \Delta_{\mathfrak{n}} \times \Delta^{(\mathfrak{n})}$

3) $Tr_{K'/K} : H^\circ(Sh_K, \omega_{K,\varepsilon}^\kappa) \to H^\circ(Sh_{K'}, \omega_{K',\varepsilon}^\kappa)$

$\mathcal{W}$ = pullback via $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ of
$\mathcal{O}_K \subset K$ w/ $K/\mathbb{Q}_p$ max unram'd ↓

PART 1

(We work $/ \mathcal{W}$, $W = \varprojlim \mathcal{W}/p^n\mathcal{W}$)
Fix $\mathcal{W} \hookrightarrow \mathbb{C}$

Step 1: $\underline{\omega}^k$ on Sh

Easy:

$$\begin{tikzcd}
\underline{\omega}^k \arrow[d] \arrow[r, phantom, "\overset{\forall \mathbb{C}}{\rightsquigarrow}"] & \underline{\omega}^k_{S(N)} \arrow[d] \\
\mathfrak{M}_1(\mathbb{C}, \Gamma(N)) & Sh_{S(N)}(X, G)
\end{tikzcd}$$

By pullback:
$$\begin{tikzcd}
\underline{\omega}^k \arrow[d] \\
Sh(X, G)
\end{tikzcd}$$

Step 2: $\underline{\omega}^k$ on $Sh^{(p)}$ (much harder)

Problem is that we can't quite relate

$$Sh^{(p)}_K/_{\mathcal{W}} \quad (\text{some } K \overset{\text{compact, open}}{\subset} G(\mathbb{A}^{(\infty)}))$$

to (integral) moduli problems $\mathfrak{M}(\mathbb{C}, \Gamma)_{/\mathcal{W}}$

$\Rightarrow$ We have to work with $(A, \bar{\lambda}, \bar{\eta}^{(p)}) \in Sh^{(p)}_K$,

where $A_{/S}$ is an ab. var. w/ RM by F as usual & $\bar{\lambda}$ is an isogeny class of polarization & $\bar{\eta}^{(p)}$ is a $K^{(p)}$-orbit of isom.

$$\eta^{(p)} : V(\mathbb{A}^{(p\infty)}) \xrightarrow{\sim} V^{(p)}(A)$$

$\Rightarrow$ Constructing $\underline{\omega}^k_K$ over $Sh^{(p)}_K$ amounts to choosing (functorially) a section ω^k of $\underline{\omega}^k$, for any $\omega \in H^0(A, \Omega_{A/S})$, that only depends on the image of $(A, \lambda, \eta^{(p)})$ in $Sh^{(p)}_K$.

The trick is to "normalize" the polarization $\bar{\lambda}$ & the prime-to-p level K structure $\bar{\eta}^{(p)}$.

In fact, in the construction of $Sh_K^{(p)}$, i.e. the proof of representability of the moduli problem $\mathcal{E}_K^{(p)}$, Hida proves that

Given $(A, \lambda, \eta^{(p)}) \in [(A, \bar{\lambda}, \bar{\eta}^{(p)})]$, λ determines a polarization ideal $\mathfrak{c}$ (unique up to some terms), i.e.

$$\mathfrak{c} \in Cl(K) = (\mathbb{A}_F^{(\infty)})^\times / F_+^\times \det(K),$$

and there exists uniques

1) $A' \xrightarrow{isog} A$ (w/ $V(A) = V(A')$)

2) $\mathfrak{c}$-pol. λ' on A'

3) $\eta^{(p)}(\hat{L}) = T^{(p)}(A')$ $\quad (L = \partial^{-1} \oplus \mathfrak{c}^{-1} \subset V)$

Therefore, using this "normalized" $(A', \lambda', \eta^{(p)})$, Hida constructs the desired $\underline{\omega}^k$ over $Sh_K^{(p)}$.

Sketch: Given $\omega \in H^0(A, \Omega_{A/S})$, take $((\alpha^*)^{-1}\omega)^{\otimes k}$, where $\alpha : A \to A'$ comes from above

HOWEVER, he needs some assumptions:

1) $K_p \subset G(\mathbb{Z}_p)$ & $K^{(p)} \subset G_1(\mathbb{A}^{(p\infty)}) \Rightarrow \det(K^{(p)}) = 1$

2) $K^{(p)}$ is small (i.e. $K^{(p)} \subset S(N)^{(p)}$, $p \nmid N \geq 3$)

Step $2\frac{1}{2}$ It will be convenient to work w/

$$K \subset G(\mathbb{A})$$

instead of just $K \subset G(\mathbb{A}^{(\infty)})$.

Let $G(\mathbb{Z}_p \times \mathbb{A}^{(p\infty)})^+ := G(\mathbb{Z}_p) \times G(\mathbb{A}^{(p\infty)}) \times G(\mathbb{R})^+$

Then, Step 2 gave us a well-defined $\underline{\omega}^k_K$ on

$$Sh^{(p)}_K := Sh^{(p)}/K^{(\infty)} \quad \left(\text{i.e. } G(\mathbb{R})^+ \text{ acts trivially}\right)$$

for 1) Any $k \geq 1$

2) $K \subset G(\mathbb{Z}_p) \times G_1(\mathbb{A}^{(p\infty)}) \times G(\mathbb{R})^+$ small enough

Note that by pullback we finally have $\underline{\omega}^k$ on $Sh^{(p)}$

Step 3 Obviously, we want $\underline{\omega}^k_K$ over $Sh^{(p)}_K$ for any

$$K \subset G(\mathbb{Z}_p \times \mathbb{A}^{(p\infty)})^+$$

b/c its to restrictive to only have $\det(K^{(p)}) = 1$ when trying to define Hecke operators

Idea: Let K as above & $K_1 = (K \cap G_1(\mathbb{A}^{(p\infty)})) \times G(\mathbb{R})$

We have $\underline{\omega}^k_{K_p \times K_1} \Rightarrow$ Define an action by $G(\mathbb{Z}_p \times \mathbb{A}^{(p)})^+$

& let ω^k_K be the K-invariant sections

Step 3.1 First guess for an action of $G(\mathbb{Z}_p \times \mathbb{A}^{(p\infty)})^{f}$

Really, we can only do it for $g \in \Delta_0(\mathfrak{n})$ but this will suffice

Take $(A, \bar{\lambda}, \bar{\eta}^{(p)}) \rightsquigarrow (A, \lambda, \eta^{(p)}) \xrightarrow{\alpha} (A', \lambda', \eta^{(p)})$ normalized as above.

Actually, consider $(A, \lambda, \eta^{(p)} \circ g^{-1}) \xrightarrow{\alpha_g} (A'_g, \lambda'_g, \eta^{(p)}_g)$

Given $g \in \Delta_0(\mathfrak{n})$ and $\omega \in H^0(A, \Omega_{A/S})$, define

$$g^{-1} \cdot \omega^{\otimes k} := (g^{-1}\omega)^{\otimes k} = \left((\alpha_g^{-1})^* \omega \right)^{\otimes k}$$

(Again, this "somehow" only depends on isogeny classes, i.e. images in $Sh^{(p)}$)

We have:

$$\begin{tikzcd} \underline{\omega}^k_{K_p \times K_1} \arrow[r, "g^{-1}"] \arrow[d] & \underline{\omega}^k_{g(K_p \times K_1)} \arrow[d] \\ Sh^{(p)}_{K_p \times K_1} \arrow[r, "g^{-1}"] & \underline{\omega}^k_{g(K_p \times K_1)} \end{tikzcd}$$

$^gK = gKg^{-1}$

Now, we want to do descent

$$\begin{tikzcd} \underline{\omega}^k_{K_p \times K_1} \arrow[r, dashed] \arrow[d] & \underline{\omega}^k_{K} \arrow[d] \\ Sh^{(p)}_{K_p \times K_1} \arrow[r] & Sh^{(p)}_{K} \end{tikzcd}$$

by taking $K/(K_p \times K_1) \cong K^{(p)}/K_1$ invariants

(???)

Apparently, the issue is that the center $Z(\mathbb{Q})$ doesn't act trivially.

So far, we have descent to $\underline{\omega}_K^k$ if

$$K \subset G(\mathbb{Z}_p) \times G_1(\mathbb{A}^{(p\infty)}) \times G(\mathbb{R})^+$$

but now, we want it for all $K \subset G(\mathbb{Z}_p \times \mathbb{A}^{(p\infty)})^+$, i.e. all $K^{(p)} \subset G(\mathbb{A}^{(p)})^+ := G(\mathbb{A}^{(p\infty)}) \times G(\mathbb{R})^+$.

(I think the issue is that we know

$$\begin{tikzcd} \underline{\omega}^k \arrow[r, bend left, "\text{Invariants}"] \arrow[d] & \underline{\omega}^k_{K_p \times K_1} \arrow[l, bend left, "\text{pullback}"] \arrow[d] \\ Sh^{(p)} \arrow[r] & Sh^{(p)}_{K_p \times K_1} \end{tikzcd}$$

but as of now

$$\begin{tikzcd} \underline{\omega}^k \arrow[r, bend left, "\times" description] \arrow[d] & \underline{\omega}^k_K \arrow[l, bend left, "\times" description] \arrow[d] \\ Sh^{(p)} \arrow[r] & Sh^{(p)}_K \end{tikzcd}$$

)

Step 3.2 Define two "normalized" actions that fix all of our problems.

There are many ways to do that.

Choose $\omega \in \mathbb{Z}[I] = X(T)$ = character group of torus $T = \mathrm{Res}^{\mathbb{Z}}_{\mathcal{O}} \mathbb{G}_m$ inside G_1

Choose $\varepsilon : T_G(\hat{\mathbb{Z}}) \to W^\times$, continuous char. of $T_G \cong T^2$

$\Rightarrow$ Induces a "central" $\varepsilon_+ : Z(\hat{\mathbb{Z}}) \to W^\times$ as

$$\varepsilon_+(z) = \varepsilon_1(z)\varepsilon_2(z)$$

$$\left(\varepsilon\begin{pmatrix} a & \\ & d \end{pmatrix} = \varepsilon_1(a)\varepsilon_2(d)\right)$$

Define $\varepsilon^-: T(\widehat{\mathbb{Z}}) \to \mathcal{W}^\times$ by $\varepsilon^-(z) = \varepsilon_2^{-1}(z)\varepsilon_1(z)$

We assume that

1. $\varepsilon_+(\epsilon) = \epsilon^{k-2w}$, $\forall$ units ϵ in $\mathcal{O}^\times$ (w/ $\mathcal{W}^\times \to \mathbb{C}^\times$ previously Fixed)
2. ε_+ extends to $Z(\mathbb{A})/Z(\mathbb{Q}) \to \mathbb{C}^\times$

It is possible to extend the "Neben character"

$$\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_+)$$

to

1. $\varepsilon_\Delta : \Delta_0((p)\cap\mathfrak{n}) \to \overset{\text{completion of } \mathcal{W}}{W^\times}$
2. $\varepsilon_S : \mathcal{Z}(\mathbb{A})S_0((p)\cap\mathfrak{n}) \to W^\times$
3. Such that ε_Δ & ε_S agree on $S_0((p)\cap\mathfrak{n})$

(Not worth the details, see (ex0) - (ex3) on p.169-170)

$\Rightarrow$ We define

$$g_{w,\varepsilon}\, \omega^{\otimes k} := \varepsilon_S(\overset{g\in\mathcal{Z}(\mathbb{A})S_0(\mathfrak{n})}{g})^{-1} \det(g)_p^{-w} (g\cdot\omega^{\otimes k})$$

$$g^*(\omega^{\otimes k}) := \varepsilon_\Delta(\underset{g\in\Delta_0(\mathfrak{n})}{g}) \det(g)_p^{w} (g^{-1}\cdot\omega^{\otimes k})$$

FINALLY:

$$H^0(Sh_K^{(p)}, \underline{\omega}^k_{w,\varepsilon,K}) := H^0(\overset{K^{(p)}\text{-action is } g_{w,\varepsilon} \text{ here}}{K^{(p)}}, H^0(Sh^{(p)}_{K_1\times K_p}, \underline{\omega}^k_{K_1\times K_p}))$$

Step 3.3 Combine the info from ω & k to a single weight κ of T_G.

Let $\kappa = (\kappa_1, \kappa_2) \in \mathbb{Z}[I]^2 = X(T_G)$ s.t.

1) $\kappa_2 - \kappa_1 + I = k$

2) $\kappa_1 + \kappa_2 = k - 2\omega + \overset{= (1, \dots, 1)}{I}$

Denote $\underline{\omega}^{k}_{\omega, \varepsilon, K}$ by $\underline{\omega}^{\kappa}_{\varepsilon, K}$ from now on.

THEOREM

(1) If K is maximal at p and $Sh_K^{(p)}$ is smooth, then $\underline{\omega}^{\kappa}_{\varepsilon, K}$ (& $\underline{\omega}_{\kappa, \varepsilon, K}$) is well defined over $\mathcal{W}$

(2) Analogue $/\mathbb{C}$

(3) Analogue for $Ig_{K/B}$ $\left(\begin{array}{l} B = \text{ring of char. } p \\ \text{or } B = \mathcal{W} \end{array} \right)$

We now have

$$\underline{\omega}^{\kappa}_{\varepsilon/W} = \underline{\omega}^{\kappa}_{\varepsilon} \otimes_{\mathcal{W}} W \quad \& \quad \underline{\omega}^{\kappa}_{\varepsilon/\mathbb{C}} = \underline{\omega}^{\kappa}_{\varepsilon} \otimes_{\mathcal{W}} \mathbb{C}$$

at the same time.

PART 2

We finally have our Hecke operators

$$[K g K'] : H^{\bullet}(Sh_K, \underline{\omega}^{\kappa}_{\varepsilon,K}) \to H^{\bullet}(Sh_{K'}, \underline{\omega}^{\kappa}_{\varepsilon,K'})$$

when $K, K' \subset \Delta_0(\mathfrak{n})$ as

$$|\det(g)|_{\mathbb{A}} \cdot Tr_{K'/(K^g \cap K)} \circ \overset{\text{induced from } g^*}{[g]} \circ Res_{K/(K \cap gK')}$$

We write $T_p(1,y)$ for $[K\begin{pmatrix}1 & \\ & y\end{pmatrix}K]$

NOTE: Our construction of g^* & $g_{\omega,\varepsilon}$ implicitly depend on the fact that we worked $/\mathbb{R}$

Technically, if we were working $/\mathbb{C}$, we would have a slightly different action. We write

$$T_p(1,y) := [K\begin{pmatrix}1 & \\ & y\end{pmatrix}K]_p \quad \& \quad T_\infty(1,y) := [K\begin{pmatrix}1 & \\ & y\end{pmatrix}K]_\infty$$

to differentiate the two. In general, we have

$$T_p(1,y) = y_p^{-\kappa} T_\infty(1,y)$$

If $S_1'((p)\cap\mathfrak{n}) \subset K \subset S_0((p)\cap\mathfrak{n})$ & $y^{(p\mathfrak{n})} = 1$, we write

$$U_\gamma(y) := T_\gamma(1,y) \quad (\gamma = p \text{ or } \infty)$$

$$\Rightarrow e = \prod_{\mathfrak{p} | p} e_{\mathfrak{p}} \text{ w/ } e_{\mathfrak{p}} := \lim_{n\to+\infty} U_p(\varpi_{\mathfrak{p}}')^{n!} = \lim_{n\to+\infty} U_p(\varpi_{\mathfrak{p}})^{n!}$$

$$= \lim_{n\to+\infty} U_p(\mathfrak{p})^{n!}$$

THEOREM Under certain technical conditions:

If $y_p = 1$, $T_p(1, y) = T_\infty(1, y)$ is $\mathcal{O}$-integral.

We can also say something about $\mathcal{O}$-integrality of $T_p(1, \infty)$ & $U_p(y)$ ($y_p \neq 1$) in a lot of cases

(See THM 4.28 on p. 178 $\leftsquigarrow$ Compute effect on q-exp.)

Define $G_\kappa(K, \varepsilon; R) := H^0(Sh_{\bar{K}/R}, \underline{\omega}^{\kappa}_{\varepsilon, K/R})$

$\mathcal{O}$-algebra

$S_\kappa(K, \varepsilon; R) := H^0(Sh_{\bar{K}/R}, \underset{\sim}{\omega}_{\kappa, \varepsilon, K/R})$,

where $K \subset G(\mathbb{A}^{(\infty)})$, max'l at p, & $\bar{K} = Z(\mathbb{A}^{(\infty)})K$.

For $z \in Z(\mathbb{A}^{(\infty)})$, let $\langle z \rangle_v := |z|_{\mathbb{A}}^{-2} [KzK]_v$ ($v = p$ or ∞).
If $z \in Z(\mathbb{A}^{(p\infty)})$, i.e. $z_p = 1$, then

$$\langle z \rangle := \langle z \rangle_p = \langle z \rangle_\infty \text{ acts via } \varepsilon_+(z)$$
on both.

THM If κ is dominant ($\kappa_2 \geq \kappa_1$), then

$$G_\kappa(K, \varepsilon; R) \ \& \ S_\kappa(K, \varepsilon; R)$$

are both finite-type R-module ($K_p = G(\mathbb{Z}_p)$)

PART 3

$\leftarrow$ p unram'd in $F/\mathbb{Q}$

Let $h_k(K,\varepsilon,R)$ be the Hecke algebra gen'd by the image of $T_\infty(1,\varpi_q)$ $(q\nmid\mathfrak{N})$ & $U_\infty(\varpi_q)$ $(q\mid\mathfrak{N})$ in

$$\mathrm{End}(S_k(K,\varepsilon;R))$$

Let $S_k^{n.ord}(K,\varepsilon;R) = \begin{cases} eS_k(K,\varepsilon;R), & \text{if } p\mid\mathfrak{N} \\ e_0S_k(K,\varepsilon;R), & \text{if } p\nmid\mathfrak{N}\end{cases}$

Here $S_1'(\mathfrak{N})\subset K\subset S_0(\mathfrak{N})$, $e=\lim U_p(p)^{n!}$, $e_0=\lim T_\infty(p)^{n!}$

Similarly, $h_k^{n.ord}(K,\varepsilon;R) = \begin{cases} eh_k(K,\varepsilon;R), & p\mid\mathfrak{N} \\ e_0h_k(K,\varepsilon;R), & p\nmid\mathfrak{N}\end{cases}$

<u>THM</u> $S_1(p^r)\cap S_1'(\mathfrak{N})\subset K\subset S_0(p^r\mathfrak{N})$, $p\nmid\mathfrak{N}$.

The pairing $\langle t,f\rangle := \underset{\sim}{\underline{a}}_p(1,f|t)^{\in R}$, we get perfect duality b/w

$\llcorner$ "Adelic q-exp."

1) $h_k(K,\varepsilon;R)$ & $S_k(K,\varepsilon;R)$

2) $h_k^{n.ord}(K,\varepsilon;R)$ & $S_k^{n.ord}(K,\varepsilon;R)$

3) p-adic analogues,

each under their own set of conditions.