Higher Hida Theory on the modular curve

**Aim of (classical) Hida Theory**
Construct $p$-adic analytic families of Hecke eigenforms.

**Some great applications: progress towards**

- Langlands' program
  e.g. (Wiles) attached Galois rep. to Hilbert eigenforms of wts.

- BSD conjecture
  e.g. Bertolini-Darmon, Skinner-Urban

- Algebracicity of Stark-Heegner points
  e.g. Bertolini-Darmon

- Explicit class field theory
  Darmon-Pollack-Vowk, Dasgupta-tekde
Aim of higher Hida theory
Construct p-analytic families of
Hecke eigen-classes in the coherent co-
of Shimura varieties.

Some great applications

- Potential modularity of abelian surfaces over CM fields
  (Boxer-Calegari-Gee-Pilloni)

- Bloch-Kato conjecture beyond GL_2
  (Jeffries-Zerbes, L-Pilloni-Skinner-Z.)
Goal of the seminar

Understand basic ideas of higher Hida theory in the simplest case of modular curves.

Set up $N \geq 3$ prime to $p$

$X/\mathbb{Z}_p$ compactified modular curve of level $\Gamma_0(N)$

$\xi \quad \text{universal semi-abelian scheme}$

$\omega := e^* \Omega^1_{\xi/\mathbb{X}}$ line bundle cotangent space set origin.

Then

$M_k(\Gamma_0(N)) = H^0(X, \omega^k)$

$S_k(\Gamma_0(N)) = H^0(X, \omega^k(-D))$

where $D$ is the boundary divisor.
Let $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ 
$\forall k \in \mathbb{Z}$ 
$k: \Lambda \to \mathbb{Z}_p$ 
$[u] \mapsto u^k$

**Main Theorem**

$\exists$ finite, projective $\Lambda$-modules $M, N$ 
with prime to $p$ Hecke action 
endowed with canonical Hecke equivariant isomorphisms $\forall k \geq 3$

- $M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p = e_p H^0(X, \omega^k)$
- $N \otimes_{\mathbb{Z}_p} \mathbb{Z}_p = e_p H^1(X, \omega^{2-k}(D))$

Moreover,

$\exists M \times N \to \Lambda$ perfect pairing interpolating classical Sen's duality.
Comments

1) $e_p = \lim T^n_p$ ordinary projector

2) $M, N$ have geometric construction

$N$ isn't simply defined as $N^*$. 

Question

Let $H^{nd}$ be the Hecke algebra

Given $d: H^{nd} \rightarrow A$ a $A$-algebra homomorphism,

how do we find $\eta_\tau \in N$ s.t.

$T \cdot \eta_\tau = d(T) \cdot \eta_\tau \quad \forall T \in H^{nd}$?
Strategy of the proof

1) Igusa tower construction

Let $X$ be the $p$-adic completion of $X/\mathbb{Z}_p$.

Concretely,

$$X = \lim_{\to} X_n$$

and

$$X_n := X \times \mathbb{Z}_p^2/\mathbb{Z}_p$$

$X$ and its ordinary locus

(c.f. talk on Hasse invariant)

(profinite stalk over

$$\mathbb{Z}_p$$

(cf. talk on $p$-adic theory)
and \( E \in \Omega \subset (\mathcal{O}_{\hat{M}} \hat{\otimes} \mathbb{Z}_p \wedge) \) is an invertible \( \mathcal{O}_{\hat{M}} \hat{\otimes} \mathbb{Z}_p \wedge \)-module.

s.t.

\[ \Omega \otimes \mathbb{Z}_p \cong \omega^k \mathbb{Z}_p. \]

**Definition**

\[
M := e(U_p) H^0(\mathcal{X}^{\text{ord}}, \Omega)
\]

\[
N := e(F) H^1_c(\mathcal{X}, \Omega(-D))
\]

Now, it is easy to see that

- \( N \otimes \mathbb{Z}_p \cong e(U_p) H^0(\mathcal{X}^{\text{ord}}, \omega^k) \)

- \( N \otimes \mathbb{Z}_p \cong e(F) H^1_c(\mathcal{X}, \omega^{2-k}(-D)) \)
2) **Classicality**

Need to prove that \( \chi_k \geq 3 \)

- \( \epsilon_p H^0(X, \omega^k) = e(C_p) H^0(X^0, \omega^k) \)
- \( \epsilon_p H^1(X, \omega^k(-D)) = e(F) H^1_c(X^0, \omega^k(-D)) \)

Enough to prove it mod \( p \) (i.e. on special fiber)

Then use Nakayama.

Classicality mod \( p \) relies on careful study of \( T_p \)-correspondence.

\[
\begin{array}{c}
X_0(p) \\
\pi_1 \\
X_1 \\
\pi_2 \\
X_2 \\
E_1 \\
(E_1 \rightarrow E_2) \\
\end{array}
\]
Here the basic idea is that

- $H^0(X_{\text{odd}}, \omega^k) = \varprojlim_n H^0(X_1, \omega^{k(n-SS)})$
- $H^1_{\text{c}}(X_{\text{odd}}, \omega^{-k}(-D)) = \varprojlim_n H^1(X_1, \omega^{-k}(-D-nSS))$

and after applying ordinary projector

all transitions become isomorphisms.
Talks for the seminar

1) Geometry of $X_0(p)/\mathbb{Z}_p$ (Katz-Nazm)
   David

2) Hasse invariant (Katz-Nazm, Katz)
   Sam

3) $T^p$ correspondence (ch. 3 Boxer-Pilloni)
   Hung

4) Classicality mod $p$ (ch. 4.1 BP)
   Haodong

5) Igusa tower and p-adic Hodge (ch. 4.2 BP)
   Avi

6) Main result and possibly duality (ch. 4.2.5 BP)
   Jiaxi