Igusa tower: \( \sim N K^m = (Q_\mathbf{g} \otimes A)^M \)

\( N \geq 3, \ (N, p) = 1. \)

\( I_{g_n} : R \in \text{Nil}_p \mathcal{L} \mathcal{P}_p \rightarrow (E/R, \varphi_n, \varphi_p^n) \)

\( \varphi_p^n : E \mathcal{P}^n \rightarrow \mathcal{M} \mathcal{P}^n. \)

\( I_{g_n} \) is representable by a formal affine scheme.

Let \( n \rightarrow \infty. \)

\( I_g : R \in \text{Nil}_p \mathcal{L} \mathcal{P}_p \rightarrow (E/R, \varphi_n, \varphi_p^n) \)

\( \varphi_p^n : E \mathcal{P}^n \rightarrow \mathcal{M} \mathcal{P}^n \simeq \hat{\mathbb{G}}_m \)
\[ M_{\text{max}}(R) = (\mathbb{1} + \text{Nil}_R(R), x) = \hat{G}_m(R), \]

\[ x \quad \text{s.t.} \quad (x-1)^n = 0 \quad \text{for some } n, \]

\[ P^m = 0 \text{ in } R \quad (x-1)^m = 0 \quad \text{for some } m. \]

\( I_g \) is also represented by an affine formal scheme, pro-étale \( \mathbb{Z}_p \)-torsor over \( X(\text{ord}(N)) \).

\[ \mathbb{Z}_p^\times = \text{Aut}(M_{\text{max}}) = \text{Aut}(\mathbb{Z}_p) \]

acts on \( \Phi_p \).

\[ f \in H^0(X_{\text{ord}}(N), \omega^k) \quad p : Ig \to X_{\text{ord}}(N). \]
$\rightarrow H^0(\mathcal{I}_g, p^*\omega^k) \Rightarrow p^*f$

Upon induces $\Omega_{\mathcal{I}_g}(\frac{dt}{t}) \Rightarrow f$

a trivialization of invariant differentials by pulling back for $(\hat{G}_m, \frac{dt}{t})$.

Weights translates into a character of $\mathbb{Z}_p^\times$.

$\alpha \in \mathbb{Z}_p^\times \subset \hat{G}_m$

$t \rightarrow t^\alpha \mathrm{Lie}$

$a : \mathrm{Lie}\hat{G}_m \rightarrow \mathrm{Lie}\hat{G}_m$
dualizing $a: \frac{dt}{t} \rightarrow a\frac{dt}{t}$

$$a \cdot \hat{f} = a^{-k} \hat{f}$$

$\chi: \mathbb{Z}_p^x \rightarrow \mathbb{Z}_p^x$ a character

$$S^\text{ord}_\chi(N) := \{ f \in \mathcal{O}_\mathcal{F}_g \mid a^\cdot f = \chi(a) f \}.$$ 

Serve's duality:

$$\chi/\mathbb{Z}_p \cong \mathbb{H}^{1}(X, \Omega^{1}/\mathbb{Z}_p) \rightarrow \mathbb{Z}_p.$$ 

Want to construct

$$\langle , \rangle: H^0(X^{\text{ord}}, \omega_{\mathbb{C}^{\text{un}}}^k) \times H^{1}_{c}(X^{\text{ord}}, \omega_{\mathbb{C}^{\text{un}}}^{2-k}) \rightarrow \mathbb{C}.$$
\[ \to \quad \mathbb{H}^1_c(\mathcal{X}^\text{ord}, w^2(-D) \otimes \mathbb{Z}_p \Lambda) \to \Lambda. \]

\[ \to \quad \mathbb{H}^1_c(\mathcal{X}^\text{ord}, w^2(-0) \otimes \mathbb{Z}_p \Lambda) \]

\[ \to \quad \mathbb{H}^1_c(\mathcal{X}, w^2(-0) \otimes \Lambda) \to \mathbb{H}^1_c(\mathcal{X}, w^2(-D)) \otimes \Omega \to \Omega \]

\[ K \otimes 1 \to \mathbb{H}^1_c(\mathcal{X}, \mathcal{S}_2 \mathcal{X}/\mathcal{Z}_p) \otimes \Lambda \to \Lambda. \]

\[ f \in \mathbb{H}^0(\mathcal{X}^\text{ord}, \mathcal{W}^\text{num}) \Leftrightarrow \Omega_{\mathcal{X}^\text{ord}} \to \mathcal{W}^\text{num}. \]

\[ w^2 - \mathcal{W}^\text{num} (-D) := w^2(-D) \otimes \text{Hom}(\mathcal{W}^\text{num}, \Lambda \otimes \Omega_{\mathcal{X}^\text{ord}}) \]

\[ f \text{ induces } \quad \mathcal{W}^\text{num} (-D) \to \mathcal{W}^2(-0) \otimes \text{Hom}(\mathcal{W}^\text{num}, \Lambda \otimes \Omega_{\mathcal{X}^\text{ord}}) \]

\[ \mathbb{H}^1_c(\mathcal{X}^\text{ord}, w^2(-D)) \to \mathbb{H}^1_c(\mathcal{X}^\text{ord}, w^2(-0) \otimes \Lambda) \]
Prop. 4.17. \( (f, g) \in H^0(\mathcal{X}_{\text{ord}}, \mathcal{W}^{k\text{mun}}) \times H^1_c(\mathcal{X}_{\text{ord}}, \mathcal{W}^{2-k\text{mun}}(-D)) \), we have

\[
\langle Up f, g \rangle = \langle f, Fg \rangle.
\]

Proof. Step 1.

\[
H^0(\mathcal{X}_{\text{ord}}, \mathcal{W}^{k\text{mun}}) \times H^1_c(\mathcal{X}_{\text{ord}}, \mathcal{W}^{2-k\text{mun}}(-D)) \rightarrow \Delta
\]

\[
\uparrow
\]

\[
\prod_{k \in \mathbb{Z}} H^0(\mathcal{X}_{\text{ord}}, \mathcal{W}^k) \times H^1_c(\mathcal{X}_{\text{ord}}, \mathcal{W}^{2-k}(-D)) \rightarrow \prod_{k \in \mathbb{Z}} 2
\]

Step 2: Prove \( \langle Up f, g \rangle = \langle f, Fg \rangle \) for every \( k \in \mathbb{Z} \).
Work with $X_n^{\text{ord}}$. By defn of

$$H_c^1 (X_n^{\text{ord}}, \mathfrak{f}H) := \lim H_c^1 (X_n, \mathcal{I}_n^\infty \mathfrak{f}H)$$

On $H_c^1$, $F$ acts as

$$F: p_2^! \mathcal{I}\text{m}^k w^{-L} \rightarrow p_1^! \mathcal{I}^{-L+k} w^{-k}$$

Apply $R\text{Hom}(\quad, \mathcal{W}_{X_{\text{log}}}(2p/2p)) =: D$

$$D(F): p_2^! \mathcal{I}^{-L+k} w^k \rightarrow p_1^! \mathcal{I}^{-k} w^k$$

Lemma 4.14? $D(F) = \text{Up}$.

$\supset\quad \uparrow$

prop 3.6? $\Omega^{\text{fl}}_\bullet (\log (S, D))$. 
Prop 4.18. Restrict $\langle \cdot, \cdot \rangle$ to
\[ e(U_F) H^0(\mathcal{X}^{ord}, \mathbb{W}^{km}) \times e(F) H^1_c(\mathcal{X}^{ord}, \mathbb{W}^{km}), \]
\[(1) \langle \cdot, \cdot \rangle \text{ becomes a perfect pairing.}\]
\[(2) \langle U_F f, g \rangle = \langle f, F g \rangle. \]
\[(3) \text{ Have commutative diagram} \]
\[ e(U_p) H^0(\mathcal{X}_{\text{ord}}, \omega_{k}) \times e(F) H^1_c(\mathcal{X},\omega_{k}\langle i \rangle) \to \mathbb{Z}_p \]

\[ \begin{array}{ccc}
& i \uparrow & \\
\downarrow & & \downarrow j \\
e(T_p) H^0(X, \omega_{k}) & \times & e(T_p) H^1(X, \omega_{k}\langle i \rangle) \\
\end{array} \]

Serve's duality.

**Proof.** (2) follows from 4.17.

(1) follows from (3): i & j are isom for \( k \geq 3 \) & the same trick as step 1 in 4.17.

Prove (3) for \( k \geq 3 \).
Observe that,

\[
H^0(\mathcal{X}^{\text{ord}}, \omega^k) \times H^1_c(\mathcal{X}^{\text{ord}}, \omega^{2-k}(-D)) \to \mathbb{P}^n
\]

\[
i \uparrow \quad \downarrow \quad \bigstar
\]

\[
H^0(X, \omega^k) \times H^1(X, \omega^{2-k}(-D))
\]

commutes.

For any \( f \in e(T_p) \) and \( g \in e(F) \),

\[
\langle e(L_p) i(f), g \rangle = \langle i(f), e(F) g \rangle
\]

\[
= \langle i(f), g \rangle.
\]

\[
\langle f, e(T_p) j(g) \rangle = \langle e(T_p) f, j(g) \rangle
\]
\[ \langle f, j \rangle(g) \] 
\[ = \langle f, 1 \rangle(g) \] 
\[ = \langle f, 1, g \rangle. \]