§ p-adic L-functions for P-ordinary Hida families on unitary groups

History:
Given $k \geq 4$ $\rightsquigarrow E_k(z) \rightsquigarrow$ constant term $\zeta(1-k)/2$

<u>Fact</u> After "p-depletion" $E_k \rightsquigarrow E_k^{(p)}$, <u>all</u> q-expansion coefficients satisfy certain "mod p congruences"

Conclusion: $\exists$ "p-adic measure" $d\tilde{\mu}$ on $\mathbb{Z}_p^\times$ s.t. $\int_{\mathbb{Z}_p^\times} x^k \, d\tilde{\mu} = E_k^{(p)}$

$\Rightarrow$ compose w/ taking constant term $\rightsquigarrow d\mu$ s.t.

$$\int_{\mathbb{Z}_p^\times} x^k \, d\mu = (1-p^{k-1})\,\zeta(1-k)$$

Note: $d\mu$ can be $\int$ against <u>any</u> C^0-fct on $\mathbb{Z}_p^\times$, i.e. $d\mu \in \Lambda = \mathbb{Z}_p[\![\mathbb{Z}_p^\times]\!]$ & "$x \mapsto x^k$" is really a homomorphism "$\Lambda \to \mathbb{Z}_p$" $\in \operatorname{Spec}\Lambda(\mathbb{Z}_p)$

In fact: $\forall$ finite-order char χ on $\mathbb{Z}_p^\times$,

$$\int_{\mathbb{Z}_p^\times} x^k \chi(x) \, d\mu = (1-\chi(p)p^{k-1})\,L(1-k;\chi)$$

This led to a long story of finding "mod p congruences" for L-fctz as the "automorphic datum" varies p-adically

Our setup: K = quadratic imaginary $/\mathbb{Q}$

V = Herm. K-v.sp of dim n $\rightsquigarrow$ $G = GU(V)$ "$= GU(a,b)$"

Fix $p = \mathfrak{p}\bar{\mathfrak{p}}$ in K (Very convenient for structure of $G(\mathbb{Q}_p)$)

Goal: Study $L(\frac{k+1}{2}; \pi, \chi_u)$ as $\underbrace{\text{1) } \pi = \text{cusp AR on } G(\mathbb{A}) \text{ varies } p\text{-adically}}_{\text{To be defined}}$

2) $\chi = |\cdot|^{-k/2} \chi_u$ = Hecke char. of K of cond. p^r varies p-adically (Same as before $\chi \in \operatorname{Spec} \Lambda(\mathcal{O}_{\mathbb{C}_p})$)

How? Doubling method:

One can construct carefully an Eisenstein series $\mathcal{E}_{\chi,\pi}$ & choose vectors $\phi \in \pi$, $\phi^\vee \in \pi^\vee$ s.t.

$$\int \mathcal{E}_{\chi,\pi}(g_1, g_2)\, \phi(g_1)\, \phi^\vee(g_2)\, dg_1\, dg_2 \approx L(\tfrac{k+1}{2}; \pi, \chi)$$

This $\int$ generalizes "taking constant term" above.

Q1: Given π, which ϕ (& $\phi^\vee$) to choose to make this $\approx$ rigorous?

Q2: How to construct $\mathcal{E}_{\chi,\pi}$?

Q3: How to vary this p-adically, i.e. get a "p-adic measure" from this family $\mathcal{E}_{\bullet,\bullet}$?

To vary π p-adically, we restrict our attention:

1) π_∞ is holo. disc. series of wgt κ
2) π_p is $(P-)$ ordinary

For a moment, forget "P". Then, "ordinary" generalizes the notion

"a modular form f is ordinary (at p) if $a_p(f)$ is a p-adic unit"

Work of Eischen-Harris-Li-Skinner:

1) Built on the work of Hida to construct & study ordinary Hida families

Answer: $\exists$ (localized) ordinary Hecke algebra

$$\mathbb{T}^{ord} = \left(\varprojlim_{r} \mathbb{T}^{ord}_{I_r, K^p, \kappa} \right)_{m_\pi}$$

s.t. π has an associated $\lambda_\pi : \mathbb{T}^{ord} \to R$ (some p-adic ring), i.e. $\pi \in \operatorname{Spec} \mathbb{T}(R)$

$\Rightarrow$ The other "classical" points of $\mathbb{T}^{ord}$ are all other cusp AR of $G(\mathbb{A})$ that are

1) ordinary
2) congruent to π mod p, i.e.

$$\lambda_\pi : \mathbb{T}^{ord} \to R \to R/p$$
$$\lambda_{\pi'} : \mathbb{T}^{ord} \to R \to R/p$$

are $=$

(This mimics the notion of 2 cusp forms f & f' whose q-exp coeff. are congruent mod p)

2) For all $\pi \in \mathrm{Spec}\, \mathbb{T}^{ord}(R)$,
singled-out the "correct" $\phi \in \pi$ (& $\phi^\vee \in \pi^\vee$)

Idea: $\pi = \pi_\infty \otimes \pi_p \otimes \pi^{p,\infty}$

- π_∞ has wgt $\kappa \rightsquigarrow \phi_\infty =$ holo. vector of wgt κ
- π_p is ordinary $\rightsquigarrow$ its "ordinary" part $\pi_p^{ord} \subset \pi_p$ is 1-dim'l. In fact, π_p^{ord} is a character ψ of $T(\mathbb{Z}_p)$ = max'l torus of $G(\mathbb{Q}_p)$
- $\pi^{p,\infty} \rightsquigarrow$ "standard" choice of $\phi^{p,\infty}$

$\rightsquigarrow \phi = \phi_\infty \otimes \phi_p \otimes \phi^{p,\infty}$

3) Construct $\mathcal{E}_{\chi,\pi} = \mathcal{E}_{\chi,\kappa,\psi}$
+ compute Fourier coeff to verify mod p congruences.
($\Rightarrow \exists$ p-adic measure $dEis$)

THM (EHLS, 2020) Assuming certain technical hypothesis,
$\exists\, \mathcal{L}_p \in \mathbb{T}^{ord} \hat{\otimes} \Lambda$ s.t. $\forall (\pi, \chi) = (\pi, |\cdot|^{-\frac{k}{2}} \chi_u)$ "critical"
$$\mathcal{L}_p(\pi,\chi) \text{"}=\text{"} \langle \int (\chi, \kappa\cdot\psi)\, dEis,\ \phi_\pi \otimes \phi_{\pi^\vee} \rangle$$
$$= (*)\, E_p\left(\tfrac{k+1}{2}; \pi, \chi\right) E_\infty\left(\tfrac{k+1}{2}; \pi, \chi\right) L^{p,\infty}\left(\tfrac{k+1}{2}; \pi, \chi\right)$$
algebraic term + some minor details

Fact: "ordinary" is hard to satisfy
Q: Is there a way to extend this result to more AR's?
A: Yes by going from "ordinary" $\rightsquigarrow$ "P-ordinary"
Note: The "P-ord" setting was done by Liu-Rosso for GSp instead of GU

Main difficulty: $\pi_p^{P\text{-ord}} \subset \pi_p$ is not 1-dim'l

Q: How to replace φ?

§ P-ordinary AR's & families

Fact $G(\mathbb{Q}_p) = \mathbb{Q}_p^\times \times GL_n(\mathbb{Q}_p)$

Consider $P_{a,b} = \left(\begin{array}{c|c} GL(a) & * \\ \hline 0 & GL(b) \end{array}\right) \subset GL_n(\mathbb{Q}_p)$

"Hodge parabolic"

We choose P to be any (standard) parabolic subgroup of $G(\mathbb{Q}_p)$ contained in $P_{a,b}$

E.g. 1) $P = P_{a,b} = \left(\begin{array}{c|c} GL(a) & \\ \hline & GL(b) \end{array}\right)\left(\begin{array}{c|c} 1_a & Mat(a,b) \\ \hline & 1_b \end{array}\right)$

2) $B =$ (upper-triangular) Borel
$= T \cdot B^u$

max'l torus; unipotent upper-Δ

In general, $P = L P^u$ where $L = \prod GL(n_i)$
"block diagonal"

Q: What to do with P?

1) Level subgroup: $I_{P,r} = \{ g \bmod p^r \in P^u(\mathbb{Z}/p^r\mathbb{Z}) \}$
(compare w/ $\Gamma_1(p^r) \subset GL_2(\mathbb{Z}_p)$)

2) Hecke operator $u_p = I_{P,r}\, t_p\, I_{P,r}$, where
$t_p \in Z(L(\mathbb{Q}_p))$ is "regular"
(compare w/ $\Gamma_1(p^r)\, t_p\, \Gamma_1(p^r)$)

Note: Need to normalize u_p (depends on wgt κ) but let's ignore this

Fact: $u_p \curvearrowright \pi_p^{I_r}$

Better: $u_p \curvearrowright \pi_p^{P^u(\mathbb{Z}_p)}$ $\left(\varinjlim_r\right)$

Let's break $\pi_p^{P^u(\mathbb{Z}_p)}$ into (generalized) eigenspaces

$\Rightarrow \pi_p^{P\text{-ord}} = \bigoplus \left\{ \begin{array}{l} u_p\text{-eigenspaces} \\ \text{w/ } p\text{-adic} \\ \text{unit eigenvalues} \end{array} \right\} \subset \pi_p^{P^u(\mathbb{Z}_p)}$

<u>DEF</u> An AR π is <u>P-ord</u> if $\pi_p^{P\text{-ord}} \neq 0$

<u>Q</u> What does $\pi_p^{P\text{-ord}}$ look like?

<u>THM</u> (M., 2024) If π is P-ord, then

1) $\pi_p \hookrightarrow \mathrm{Ind}_P^G \sigma$
2) $\pi_p^{P\text{-ord}} \to \mathrm{Ind}_P^G \sigma \xrightarrow{f \mapsto f(1)} \sigma$ is an $\cong$ of $L(\mathbb{Z}_p)$-repr'n
3) $\exists$ some smooth irred. rep'n τ of $L(\mathbb{Z}_p)$ s.t. $\mathrm{Hom}_{L(\mathbb{Z}_p)}(\tau, \pi_p^{P\text{-ord}}) = \mathbb{C}.\langle \iota_\tau \rangle$, i.e. τ has multiplicity 1

(1) is "easy", 2) is hard, 3) is an easy consequence of 2) using "Schneider-Zink types")

Conclusion: τ replaces ψ (even though τ is not unique so just fix one)

In my thesis:

- Given $\pi = \pi_\infty \otimes \pi_p \otimes \pi^{p,\infty}$ of weight κ & P-ord

$\rightsquigarrow \phi_v = \phi_\kappa \otimes \phi_{p,v}^{P\text{-ord}} \otimes \phi^{p,\infty} \in \pi = \pi_\infty \otimes \pi_p \otimes \pi^{p,\infty}$

To vary this p-adically $\Rightarrow$ view π as a point of some Hecke algebra.

Start w/ $\mathbb{T}_{\kappa,\tau}$ "$= \langle T_\ell \ (\ell \nmid pN), u_{p,r} \rangle$" of level K_r

acting on $S_\kappa(K_r, \tau; \mathbb{C}) = H^0(Sh_{K_r}; \omega_{\kappa,\tau}^{cusp})$

Then, π contributes, i.e.

some "big enough" p-adic ring

$$\pi_\infty^\kappa \otimes \mathrm{Hom}_{L(\mathbb{Z}_p)}(\tau, \pi_p^{P\text{-ord}}) \otimes (\pi^{p,\infty})^{K^{p,\infty}} \underset{\text{mult. 1 hyp}}{\cong} S_\kappa(K_r, \tau; E)[\lambda_\pi] \otimes_E \mathbb{C}$$

π_∞^κ — $(\mathfrak{g}, K)$-module

$$\phi_\kappa \otimes \iota_\tau \otimes \phi^{p,\infty} \longleftrightarrow \Phi_\pi^{P\text{-ord}}$$

for some Hecke character $\quad (E = R[1/p])$

$$\lambda_\pi : \mathbb{T}_{\kappa,\tau} \twoheadrightarrow \mathbb{T}_{\kappa,\tau}^{P\text{-ord}} \xrightarrow{\lambda_\pi} R$$

$\rightsquigarrow \pi \longleftrightarrow \lambda_\pi \in \mathrm{Spec}\, \mathbb{T}_{\kappa,\tau}^{P\text{-ord}}(R)$

But $\mathbb{T}_{\kappa,\tau}$ <u>too</u> small

Idea: Look at $\left(\varprojlim_r \mathbb{T}_{K_r,\kappa,\tau}^{P\text{-ord}}\right)_{\mathfrak{m}_\pi} =: \mathbb{T}_{K^p,\kappa,\tau,\pi}^{P\text{-ord}}$

where $\mathfrak{m}_\pi = \ker\left(\mathbb{T} \xrightarrow{\lambda_\pi} R \to R/p\right)$

Conj Given $x, x' \in X^*(T)$, if $x - x' \in X^*(L)$, then
$$\mathbb{T}^{P\text{-ord}}_{K^p, x, \tau, \pi} \simeq \mathbb{T}^{P\text{-ord}}_{K^p_r, x', \tau, \pi}$$

FACT This is known in the $P = B$ case.

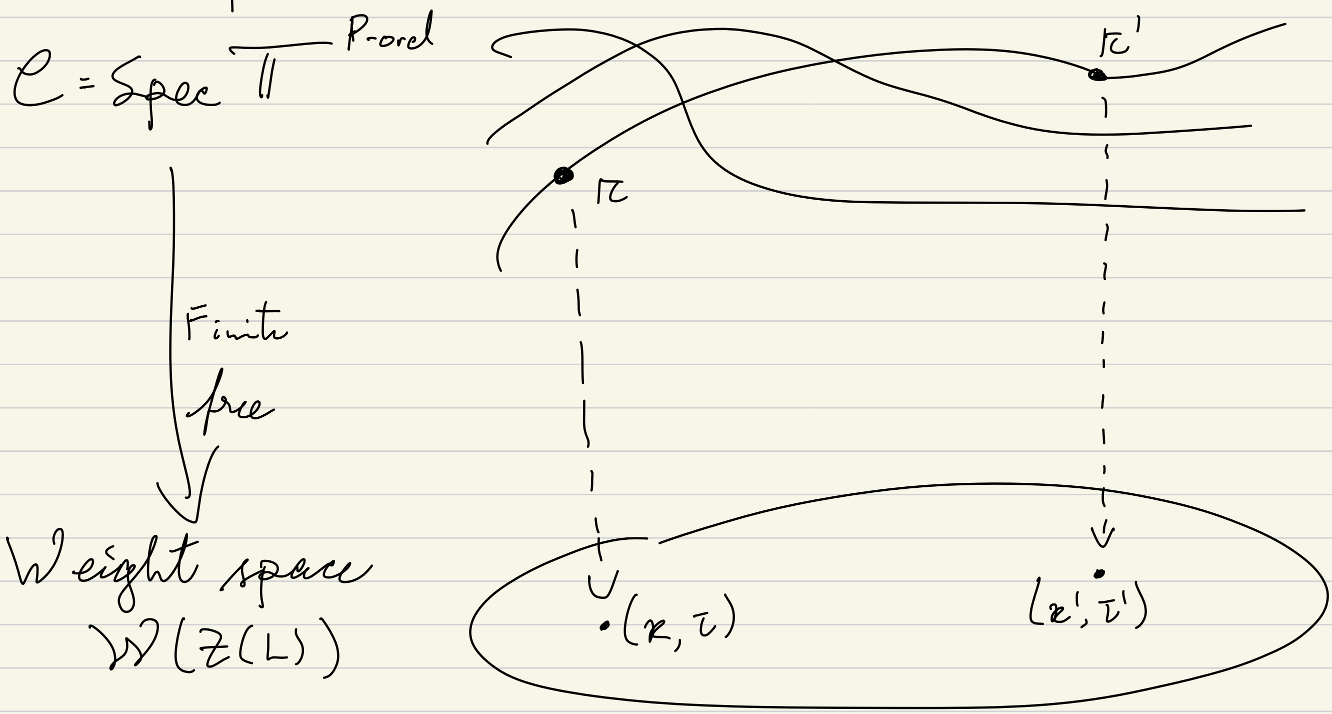
$\Rightarrow$ Let $[x] = x + X^*(L)$ $\rightsquigarrow$ $\mathbb{T}^{P\text{-ord}}_{K^p, [x], \tau, \pi}$

Variation of τ w/ $[\tau] := \left\{ \tau \otimes \psi \;\middle|\; \begin{array}{l} \psi : L(\mathbb{Z}_p) \to \mathbb{Q}_p^\times \\ \text{smooth char.} \end{array} \right\}$ is easier:

$\rightsquigarrow$ $\mathbb{T}^{P\text{-ord}} := \mathbb{T}^{P\text{-ord}}_{[x,\tau],\pi}$ def'd as above but replace

$S_x(K_r, \tau; \mathbb{C})$ w/ $S_x(K_r, [\tau]; \mathbb{C}) = \bigoplus_{\tau' \in [\tau]} S_x(K_r, \tau'; \mathbb{C})$

- Construct & study "P-ordinary families":
$\exists$ (localized) P-ordinary Hecke algebra $\mathbb{T}^{P\text{-ord}}$ whose spectrum "looks like":



i.e. "classical" points of $\mathbb{T}^{P\text{-ord}}$ are

1) π' = cusp P-ord AR "congruent to π mod p"
2) $x' \in [x]$ and $\tau' \in [\tau]$

(condition 2 was not "apparent" in the B-ord case)

- Construct family of Eisenstein series

$$\mathcal{E}_{\chi,\pi} = \mathcal{E}_{\chi,\tau,\kappa}$$

that is "compatible" with $\mathcal{C}$ (i.e. "good" for doubling method $\int$)

+ Fourier coeff's satisfy nice "mod p congruences"

$\Rightarrow \exists$ p-adic measure $dEis^{[\kappa,\tau]}$ s.t.

$$\int \Big(\chi, \underbrace{(\kappa+\rho)}_{\kappa'}\cdot\underbrace{(\tau\otimes\psi)}_{\tau'}\Big)\, dEis^{[\kappa,\tau]} = \mathcal{E}_{\chi,\tau',\kappa'}$$

<u>THM</u> (M., 2024) Assuming tech. hyp.,

$\exists\, \mathcal{L}_p^{p\text{-ord}} \in \mathbb{T}^{p\text{-ord}} \otimes \Lambda$ s.t. $\forall\, (\pi', \chi)$ "critical",

$$\mathcal{L}_p^{p\text{-ord}}(\pi',\chi) \text{``}=\text{''} \Big\langle \int(\chi, \tau'\cdot\kappa')\, dEis^{[\kappa,\tau]},\ \Phi_\pi^{p\text{-ord}} \otimes \Phi_{\pi^\vee}^{p\text{-ord}} \Big\rangle_{Ser}\text{''}$$

$$= (*)\, E_p(\dots)\, E_\infty(\dots)\, L^{p,\infty}\Big(\tfrac{k+1}{2}; \pi, \chi\Big)$$