p-adic L-functions for P-ordinary Hida families on unitary groups

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SETUP

- K = quadratic imaginary field, $V_{/K}$ = Hermitian vector space.
- $G = GU(V)$ of signature $(a, b) = (a, n - a)$.
- $p = \mathfrak{p}_w \mathfrak{p}_{\overline{w}}$ = prime that splits in K.
- Our goal is to study special values of standard L-function

$$L\left(\frac{k+1}{2}; \pi, \chi_u\right) \text{ , } k \geq 0 \text{ critical,}$$

  for $\pi =$ cuspidal *P-ordinary* (at p) automorphic repr'n for G, $\chi_u =$ unitary Hecke character of K of conductor dividing p^∞.
- BASIC CASE : π is a analogue of a Hecke eigencusp form f on GL(2) of level $\Gamma_1(p^r)$. Also, (P-)ordinary $\iff$ $a_p(f)$ = p-adic unit.
- Remark : We can add a tame level $p \nmid N$ but let's omit this here.

Outline of the talk

- Step 0 : Setup some notation.
- Step 1 : What is a P-ordinary automorphic representation? A P-ordinary family?

  Local property about π_p. Can be ”glued” using Hecke algebras.
- Step 2 : Statement of the main result

  Function over a family with some interpolation property
- Step 3 : Outline of the proof/construction

  Eisenstein series and then more Eisenstein series
- Step 4 : Quick conclusion

  Our p-adic L-function = Eisenstein series + how to pair them with π

Step 0) Notation

- Because p splits $\Rightarrow$ nice basis of V over $K \otimes \mathbb{Q}_p$, i.e.

$$G(\mathbb{Q}_p) = \mathbb{Q}_p^\times \times \mathrm{GL}_n(K_w)$$

- It contains the "Hodge parabolic"

$$\mathbb{Q}_p^\times \times P_{a,b}(\mathbb{Q}_p) := \mathbb{Q}_p^\times \times \begin{pmatrix} \mathrm{GL}_a(K_w) & * \\ 0 & \mathrm{GL}_b(K_w) \end{pmatrix}$$

- Let P be *any* standard parabolic in $P_{a,b} \Rightarrow P$ = (Levi) · (unipotent radical) = $L \cdot P^u$.

  Here $L = \prod_{i=1}^{d} \mathrm{GL}(n_i)$ is "d blocks".

Step 0) Notation (cont.)

- Let π be an automorphic representation (AR) of G. Write

$$\pi = \otimes'_q \pi_q = \pi_\infty \otimes \pi_p \otimes \pi^{p,\infty}$$

- For this talk : mostly ignore $\pi^{p,\infty}$.
- For $\pi_\infty \Rightarrow$ only care about its weight

$$\kappa = (\kappa_1, \ldots, \kappa_a; \kappa_{a+1}, \ldots, \kappa_n) \in X^*(T)\,,$$

  where T is the maximal torus of $\mathrm{GL}(n)$. Note that $T \subset L$.
- DEFINITION : We say a weight ρ of T is *P-parallel* if it extends to a character of L, i.e. $X^*(L) \subset X^*(T)$ is the set of P-parallel weights.
- Let $[\kappa] = \kappa + X^*(L) = \{\kappa + \rho : \rho \text{ is } P\text{-parallel}\}$.

Step 1) p-adic families

- Recall : We want to study $L\left(\frac{k+1}{2}; \pi, \chi\right)$.
- IDEA : vary π and $\chi = |\cdot|^{-k/2}\chi_u$ inside *p-adic families* $\Rightarrow$ p-adic L-function $\mathcal{L}_p$.
- Question : How to vary χ?
- Answer : χ is essentially an $\mathcal{O}_{\mathbb{C}_p}$-point of an Iwasawa algebra Λ.
- Question : What about π?
- Answer : more complicated
- BASIC CASE : Replace ordinary modular form f with a system of Hecke eigenvalues $\lambda : \mathbb{T} \to R$ (some p-adic ring R). Then, f is an R point of $\mathbb{T}$.
- $\mathfrak{m}_\pi = \ker(\mathbb{T} \to R \to R/p) \Rightarrow$ Hida family is completed ring $\mathbb{T}_{\mathfrak{m}_\pi}$.
- "classical" points of $\mathbb{T}_{\mathfrak{m}_\pi} \leftrightarrow$ ordinary Hecke eigencusp forms g of level $\Gamma_1(p^\infty)$ such that $a_q(g) \equiv a_q(f) \bmod p$, for all primes q.

Step 1) P-ordinary automorphic repr'ns

- Discussed $\pi_\infty \rightsquigarrow \kappa$ and $\pi^{p,\infty}$ briefly earlier. What about π_p?
- Choice of $P \rightsquigarrow$ Hecke operator $U_p = U_{P,p}$.
- BASIC CASE : For GL(2), we have

$$U_p = \Gamma_1(p^r) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma_1(p^r)$$

- OUR CASE : $U_p := I_{P,r} t_p I_{P,r}$, where

$$I_{P,r} = \{g \bmod p^r \in P^u(\mathbb{Z}_p/p^r\mathbb{Z}_p)\}$$

  and $t_p \in Z(L)$ = center of L is a *regular* element (i.e. "decreasing powers of p"). In reality, need to normalize $u_P = \kappa(t_p)^{-1} U_P$.

Definition

π is P-ordinary (at p) $\leftrightarrow \exists \varphi \in \pi_p^{I_r}$ such that $u_p\varphi = c\varphi$ with c = p-adic integral unit.

- $\pi^{(P-\mathrm{ord},r)}$:= $\{$all such φ's$\}$ and $\pi^{P-\mathrm{ord}} := \varinjlim_r \pi^{(P-\mathrm{ord},r)}$.

Step 1) Structure theorem

Lemma (Jacquet)

If π_p is P-ordinary, then $\pi_p \hookrightarrow \mathrm{Ind}_P^{G(\mathbb{Q}_p)} \sigma$ for some admissible representation σ of $L(\mathbb{Q}_p)$.

Theorem (M., 2024)

If π_p is P-ordinary, σ is supercuspidal, and κ is "very regular", then

$$\pi_p^{P-\mathrm{ord}} \hookrightarrow \pi_p \hookrightarrow \mathrm{Ind}_P^{G(\mathbb{Q}_p)} \sigma \xrightarrow{f \mapsto f(1)} \sigma$$

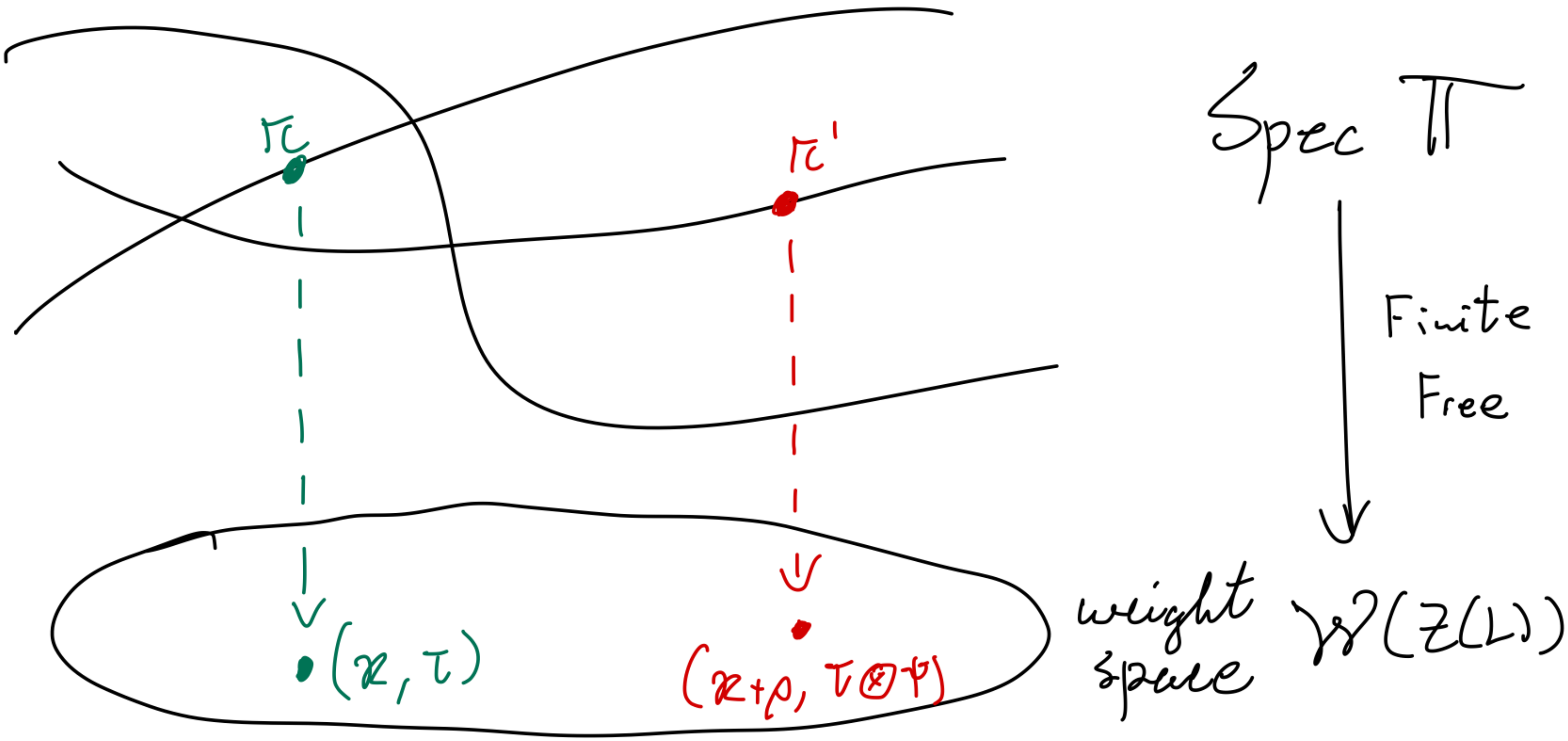
is an isomorphism of $L(\mathbb{Z}_p)$-representations.

- Fix *Schneider-Zink* type $\iota_\tau : \tau \hookrightarrow \sigma|_{L(\mathbb{Z}_p)}$ (multiplicity 1).
- BASIC CASE : τ replaces the role of nebentypus of f.
- Takeaway : $\mathrm{Hom}_{L(\mathbb{Z}_p)}(\tau, \pi_p^{P-\mathrm{ord}}) = \mathbb{C}.\langle \iota_\tau \rangle$

Step 1) P-ordinary families

- $[\kappa] = \kappa + X^*(L)$. Let $[\tau] = \{\tau \otimes \psi \mid \psi : L(\mathbb{Z}_p) \to \overline{\mathbb{Q}}_p^\times \text{ smooth}\}$
- To this $\rightsquigarrow$ some Hecke algebra $\mathbb{T}_{[\kappa,\tau]}^{P-\mathrm{ord}}$ and a character $\lambda_\pi : \mathbb{T}_{[\kappa,\tau]}^{P-\mathrm{ord}} \to R$ (some p-adic ring R) associated to π.
- Let $\mathfrak{m}_\pi := \ker(\mathbb{T}_{[\kappa,\tau]}^{P-\mathrm{ord}} \to R \to R/p)$ (a non-Eisenstein ideal) and $\mathbb{T} := (\mathbb{T}_{[\kappa,\tau]}^{P-\mathrm{ord}})_{\mathfrak{m}_\pi}$ = a P-ordinary Hida family containing π.
- Note : $\mathbb{T}$ implicitly depends on our choice of $\tau \Rightarrow$ could be referred to as a (P, τ)-ordinary Hida family containing π.
- Idea : “classical” points of $\mathbb{T} \leftrightarrow$ cuspidal P-ordinary automorphic representations π' on G, of some weight $\kappa' = \kappa + \rho \in [\kappa]$ and some type $\tau' = \tau \otimes \psi \in [\tau]$, that are “congruent to π modulo p”

Step 1) Geometry (P-ordinary Hida theory)



Step 2) Main result

Theorem (M., 2024)

Assuming certain technical hypothesis, given a P-ordinary family $\mathcal{C}$ of ARs on G, there exists a $(d+1)$-variable p-adic L-function $\mathcal{L}_p^{P-\mathrm{ord}} \in \Lambda \widehat{\otimes} \mathbb{T}$ whose value at "critical" points $(|\cdot|^{-k/2}\chi_u, \pi)$ is

$$(\star) E_{p,P-\mathrm{ord}}\left(\frac{k+1}{2}; \pi, \chi_u\right) E_\infty\left(\frac{k+1}{2}; \pi, \chi_u\right) L^{S,p,\infty}\left(\frac{k+1}{2}; \pi, \chi_u\right) I_S$$

Here, $\frac{k+1}{2}$ is critical, S = set of "bad" primes, and $E_{p,P-\mathrm{ord}}(\cdots)$ & $E_\infty(\cdots)$ are modified Euler factors at p and ∞ respectively.

- Next : Explain where $E_{p,P-\mathrm{ord}}$ comes from + brief comments about the other factors.

Step 2) Comparison

- If $P = B$ is "minimal" $\Rightarrow$ recovers the result of Eischen, Harris, Li and Skinner (2020) for $(B-)$*ordinary* Hida families.
- In fact, my results are deeply related to/inspired by [EHLS]
- Difference : P-ordinary is far more general but $(d+1)$-variables $<$ $(n+1)$-variables
- If $G = GU(1)$ $\Rightarrow$ we recover Katz's p-adic L-function for CM fields (1978).
- Liu-Rosso (2020) have a similar result over symplectic groups
- Difference : They deal with *non-cuspidal families* but their notion of "P-ordinary" is weaker (they only allow $\dim \tau = 1$).

Step 3) Strategy

- $G = G_1 = GU(V) = GU(a, b)$ and $G_2 = GU(-V) = GU(b, a)$
- $G_4 = GU(V \oplus (-V)) = GU(n, n)$ and

$$G_3 = \{(g_1, g_2) \in G_1 \times G_2 : \nu_1(g_1) = \nu_2(g_2)\},$$

  where ν_i is similitude character.
- IDEA : G_1 and $G_2 \rightsquigarrow$ small Shimura varieties. $G_4 \rightsquigarrow$ big Shimura variety. $G_3 \rightsquigarrow$ middle man
- Everything that "happens" for π on $G_1 \rightsquigarrow$ also "happens" for $\pi^\vee$ for G_2. (Technically, need to work with a twist $\pi^\flat$ of $\pi^\vee$ but we ignore this here)
- Doubling method : By *carefully* choosing an Eisenstein series $\mathcal{E}_{\chi,\pi}$ on G_4, $\phi \in \pi$ (and $\phi^\vee \in \pi^\vee$), we have

$$Z := \int_{[H']} \mathcal{E}_{\chi,\pi}(g_1, g_2) \cdot \phi(g_1) \cdot \phi^\vee(g_2) \approx L(s + \frac{1}{2}; \pi, \chi)$$

Step 3) Strategy (cont.)

- Idea : go 1 prime at a time $\Leftrightarrow \pi = \pi_\infty \otimes \pi_p \otimes \pi^{p,\infty}$, $\chi = \chi_\infty \otimes \chi_p \otimes \chi^{p,\infty} \Leftrightarrow Z = Z_\infty \times Z_p \times Z^{p,\infty}$
- There are "standard" test vectors $\phi_\infty \in \pi_\infty$ and $\phi^{p,\infty} \in \pi^{p,\infty}$.
- NEW : $v \in \tau \rightsquigarrow \iota_\tau(v) = \phi_{p,v} \in \pi_p^{P-\mathrm{ord}} \rightsquigarrow \phi_{\pi,v} := \phi_\infty \otimes \phi_{p,v} \otimes \phi^{p,\infty}$ (small issue : depends on v. Fixed later.)
- Similarly, Eisenstein series is built 1 prime at a time.
- At ∞ : all choices mostly depends on κ and built with certain differential operators.
- Away from p, ∞ : standard unramified choices + pretty much ignore set S of bad primes
- NEW : construct the factor at p out of τ (key idea : build a Schwartz function on a "big" compact open subset of $G(\mathbb{Q}_p)$ out of matrix coefficients of τ)

Step 3) Consequences of our choices

- At ∞, from work of Eischen-Liu (2024) :

$$Z_\infty = (\textit{algebraic term}) \cdot E_\infty\left(\frac{k+1}{2}; \pi_\infty, \chi_\infty\right)$$

  and E_∞ is a long product of Gamma factors.
- Away from p and ∞, from [Jacquet79], [GPSR87] and [Li92]

$$Z^{p,\infty} = L^{S,p,\infty}\left(\frac{k+1}{2}; \pi, \chi\right) I_S$$

  is "almost" a standard L-factor away from p and ∞
- At p, inspired by [EHLS], I obtain

$$Z_p = (*) \cdot \frac{\epsilon(\frac{k+1}{2}, \pi_a \otimes \chi_w) L(\frac{k+1}{2}, \pi_a^\vee \otimes \chi_w^{-1}) L(\frac{k+1}{2}, \pi_b \otimes \chi_{\overline{w}}^{-1})}{L(\frac{k+1}{2}, \pi_a \otimes \chi_w) \epsilon(\frac{k+1}{2}, \pi_b \otimes \chi_{\overline{w}}^{-1}) L(\frac{k+1}{2}, \pi_b^\vee \otimes \chi_{\overline{w}})}$$

  where $\chi_p = \chi_w \otimes \chi_{\overline{w}}$ and π_p is "built out" of $\pi_a \otimes \pi_b$.

STEP 3) P-ORDINARY EISENSTEIN MEASURE

- The Eisenstein series $\mathcal{E}_{\chi,\pi} = \mathcal{E}_{\chi,\kappa,\tau}$ is essentially a family of (p-adic) Eisenstein series $\mathcal{E}_{\bullet,\bullet,\bullet}$.
- This is a well-defined p-adic object, i.e. a p-adic measure.
- More precisely, by using p-adic differential operators and computing a Fourier coefficients, we can p-adically interpolate the whole family into a measure dEis s.t.

$$\int_{\Lambda \times Z(L)} (\chi, \kappa \cdot \tau)\mathrm{dEis} \approx \mathcal{E}_{\chi,\kappa,\tau}$$

- ADVANTAGE : dEis makes sense over all of $\Lambda \otimes \mathcal{W}(Z(L))$ but is uniquely determined by its evaluation at "classical" points $(\chi, \tau \cdot \kappa)$

STEP 4) DOUBLING METHOD VS. dEis

- dEis "evaluated" at "critical" (π, χ) = $\mathcal{E}_{\chi,\kappa,\tau}$.
- $\mathcal{E}_{\chi,\kappa,\tau}$ is a p-adic modular form on G_3... P-ordinary Hida theory + localization at $\mathfrak{m}_\pi \Rightarrow$ becomes classical $\in H^0(\mathrm{Sh}(G_3))$
- Our choice of $\phi_{\pi,v} = \phi_\infty \otimes \phi_v \otimes \phi^{p,\infty} \in \pi$ is equiv. to some

$$\Phi_{\pi,v} \in \widehat{S}_\kappa(K_r; R) = H^{top}(\mathrm{Sh}(G_1); \omega_\kappa)$$

- Better yet! $\phi_{\pi,\tau} := \phi_\infty \otimes \iota_\tau \otimes \phi^{p,\infty} \in \mathrm{Hom}_{L(\mathbb{Z}_p)}(\tau, \pi)$ is equiv. to some

$$\Phi_{\pi,\tau} \in \widehat{S}_\kappa(K_r, \tau; R) = H^{top}(\mathrm{Sh}(G_1); \omega_{\kappa,\tau})$$

- Serre duality pairing is equiv. to doubling method integral

$$H^0(\mathrm{Sh}(G_3)) \otimes (H^{top}(\mathrm{Sh}(G_1)) \otimes H^{top}(\mathrm{Sh}(G_2))) \to R$$

  and pairs Eisenstein series with $\Phi_\tau \otimes \Phi_\tau^\vee$ (advantage : doesn't need to choose $v \in \tau$).

STEP 4) CONCLUSION

- dEis + the knowledge of pairing it with $\Phi_\tau \otimes \Phi_\tau^\vee$ is equivalent to an element
$$\mathcal{L}_p^{P-\mathrm{ord}} \in \mathbb{T} \otimes \Lambda$$
- Roughly :
$$\int_{[H']} \mathcal{E}_{\bullet,\bullet,\bullet} \cdot \phi_\bullet \cdot \phi_\bullet^\vee \iff \mathcal{L}_p^{P-\mathrm{ord}} \in \mathbb{T} \otimes \Lambda$$
- LHS : plug in & RHS : map through $\lambda_\pi \otimes \chi : \mathbb{T} \otimes \Lambda \to \mathcal{O}_{\mathbb{C}_p}$
- Evaluation $\mathcal{L}_p^{P-\mathrm{ord}}$ at $(\chi', \pi') \rightsquigarrow (\chi', \kappa', \tau') = (\chi', \kappa + \rho, \tau \otimes \psi)$ is
$$(\textit{algebraic factor}) \cdot E_{p,P-\mathrm{ord}}\left(\frac{k+1}{2}; \pi', \chi'\right)$$
$$\times E_\infty\left(\frac{k+1}{2}; \pi', \chi'\right) L^{S,p,\infty}\left(\frac{k+1}{2}; \pi', \chi'\right) I_S$$

The End

Questions? Comments?