Numerical verification of a conjecture of Harris and Venkatesh

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Abstract

In [Ven16], Venkatesh conjectures a relation between the action of derived Hecke operators and an action by motivic cohomology groups. In [HV18], Harris and Venkatesh reformulate this conjecture for the case of cohomology groups of coherent sheaves associated to modular forms of weight one. We refer to the latter as the HV conjecture. Recently, the work of Darmon, Harris, Rotger and Venkatesh in [DHRV] proves that this conjecture holds for dihedral weight one forms. The following article focuses therefore on the case of exotic forms, describes methods to compute explicitly all key ingredients appearing in the HV conjecture and provides further numerical evidence for it.

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1 Introduction

Let g be a cuspidal newform of level $\mathfrak{c}$, weight 1 and nebentype χ_g. Denote its q-expansion at infinity by $\sum_{n\geq 1} a_n(g)q^n$ and assume that g is normalized so that $a_1(g) = 1$. One can associate to such a modular form an isomorphism class of irreducible two-dimensional odd Artin representations $G_{\bar{\mathbb{Q}}/\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$ with Artin conductor $\mathfrak{c}$, see [DS74, Theorem 4.1] or [Ser77, Theorem 1 and 2].

Fix a representation ρ_g in this isomorphism class. Because ρ_g is continuous, its image is finite and hence contained in $\mathrm{GL}_2(E_g)$ for some number field E_g. Denote the ring of integers of the latter by $\mathcal{O}_g$. This also implies that $\ker(\rho_g) = G_{\bar{\mathbb{Q}}/\mathcal{L}}$ for some finite Galois extension $\mathcal{L}/\mathbb{Q}$. We say that $\mathcal{L}$ is *the field cut out by* ρ_g.

Consider the projective representation $\bar{\rho}_g$ arising from the composition of ρ_g with the natural projection $\mathrm{GL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$, cutting out some possibly smaller number field $L \subset \mathcal{L}$. The image of $\bar{\rho}_g$ is a finite subgroup of $\mathrm{PGL}_2(\mathbb{C})$, namely is cyclic, dihedral or isomorphic to A_4, S_4 or A_5. One says that g is *cyclic*, *dihedral*, *tetrahedral*, *octahedral* and *icosahedral* accordingly. One also says that g is *exotic* if it is a member of any of the latter three families. The HV conjecture is only concerned with cuspidal newforms, i.e. with dihedral and exotic newforms (see [Kan12] and [Ser77]), and this article focusses solely on exotic newforms.

Notation : Let p, $N \geq 5$ denote primes such that $N \nmid \mathfrak{c}$ and $p \mid N-1$. Let $\mathfrak{N}$ be a prime of $\mathcal{L}$ above N and denote the corresponding Frobenius element by $\sigma_{\mathfrak{N}} \in G_{\mathcal{L}/\mathbb{Q}}$, well-defined modulo the inertia subgroup at $\mathfrak{N}$. Since ρ_g has conductor $\mathfrak{c}$, the image of ρ_g at $\sigma_{\mathfrak{N}}$ is well-defined, even if N is possibly ramified in $\mathcal{L}$. As this is all that matters to verify the HV conjecture numerically, $\sigma_{\mathfrak{N}}$ is always implicitely treated in this article as a well-defined element of $G_{\mathcal{L}/\mathbb{Q}}$. Denote its image in $G_{L/\mathbb{Q}}$ by $\bar{\sigma}_{\mathfrak{N}}$.

Given a representation τ of a group G, by abusing notation, we denote the associated vector space by τ again. Moreover, given a group G, a 1-dimensional character ψ taking values in a ring $R \subset \mathbb{C}$ and an $R[G]$-module M, we let $M^{(\psi)}$ be the submodule of M on which G acts by multiplication by ψ.

1.1 The Stark Unit Group

Consider Ad $\rho_g := \mathrm{End}(\rho_g)$, the adjoint representation of ρ_g, an Artin representation which also factors through $G_{L/\mathbb{Q}}$. The representation ρ_g maps the group ring $\mathcal{O}_g[G_{\mathcal{L}/\mathbb{Q}}]$ onto a Galois-stable $\mathcal{O}_g$-lattice in Ad ρ_g, denoted by $\mathrm{Ad}^\circ\rho_g$. In the litterature, one sometime finds the symbols Ad ρ_g or $\mathrm{Ad}^\circ\rho_g$ for the three-dimensional subrepresentation of $\mathrm{End}(\rho_g)$ given by the trace-free endomorphisms. We will instead denote the latter by $\mathrm{Ad}^0\rho_g$ (see section 2.2). Hopefully, the similar symbols for these three objects is not confusing.

Definition. The *Stark unit group associated to* ρ_g is the $\mathcal{O}_g$-module

$$\mathrm{U}_g^\circ := \left(\mathcal{O}_L^\times \otimes (\mathrm{Ad}^\circ\rho_g)^\vee\right)^{G_{L/\mathbb{Q}}} \cong \mathrm{Hom}_{\mathcal{O}_g[G_{L/\mathbb{Q}}]}(\mathrm{Ad}^\circ\rho_g, \mathcal{O}_L^\times \otimes \mathcal{O}_g),$$

where the superscript $\vee$ denotes $\mathcal{O}_g$-dual. This is an $\mathcal{O}_g$-module of rank 1, see [HV18, Lemma 2.7].

We recall here the "reduction map" $\mathrm{U}_g^\circ \to (\mathbb{Z}/N\mathbb{Z})^\times \otimes \mathcal{O}_g$, see [HV18, section 2.9] or the introduction of [DHRV] for more details. The evaluation at $\rho_g(\sigma_{\mathfrak{N}})$ gives a $\sigma_{\mathfrak{N}}$-equivariant map $(\mathrm{Ad}^\circ\rho_g)^\vee \to \mathcal{O}_g$. Together with reduction modulo $\mathfrak{N}_L := \mathfrak{N} \cap L$, this induces a map

$$\mathrm{red}_{\mathfrak{N}} : \mathrm{U}_g^\circ \xrightarrow{\langle -,\sigma_{\mathfrak{N}}\rangle} \left((\mathcal{O}_L/\mathfrak{N}_L)^\times \otimes \mathcal{O}_g\right)^{\sigma_{\mathfrak{N}}=1} = (\mathbb{Z}/N\mathbb{Z})^\times \otimes \mathcal{O}_g$$

Choosing another prime above N amounts to conjugating $\sigma_{\mathfrak{N}}$ by some member of $G_{\mathcal{L}/\mathbb{Q}}$. Using this and the definition of the Galois action on $(\mathrm{Ad}^\circ\rho_g)^\vee$, one readily sees that the construction above is independent of our choice of $\mathfrak{N} \mid N$, see [HV18, Section 2.9]. Hence from now on, we simply write σ_N, $\bar{\sigma}_N$ and red_N to emphasize this fact.

1.2 The Shimura Class and the HV Conjecture

Consider the module $M_2(N)$ of modular forms of weight 2 and level $\Gamma_0(N)$ with q-expansions having integer coefficients and $S_2(N)$, its submodule of cusp forms. Let $\mathbb{T}$ be the Hecke algebra generated from the Hecke operators T_l, for $l \neq N$ prime, and U_N acting on $M_2(N)$. Seen as a $\mathbb{T}$-module, the space $M_2(N)^*$ (where $*$ denotes $\mathbb{Z}$-dual) has a canonical Eisenstein element $\mathfrak{S}_0$ defined by

$$\langle \sum_{n\geq 0} a_n q^n, \mathfrak{S}_0 \rangle = a_0$$

Fix a discrete logarithm

$$\log : (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\sim} \mathbb{Z}/(N-1)\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$$

Definition. The *Shimura class* $\mathfrak{S}_1$ is the element of $M_2(N)^* \otimes (\mathbb{Z}/p\mathbb{Z})$, unique up to additive multiples of $\mathfrak{S}_0$, such that $U_N\mathfrak{S}_1 = \mathfrak{S}_1$ and for all primes $l \neq N$,

$$(T_l - (l+1))\mathfrak{S}_1 = (l-1)\log(l)\mathfrak{S}_0 \mod p\ .$$

Remark 1.1. The existence of such a class is established by Lecouturier in his work on *higher Eisenstein elements*, see [Lec18] or [DHRV]. As $\mathfrak{S}_0$ is identically zero on $S_2(N)$, $\mathfrak{S}_1$ is well-defined in $S_2(N)^* \otimes (\mathbb{Z}/p\mathbb{Z})$. Furthermore, choosing a different discrete logarithm, say picking $a \cdot \log$ instead of log for some $a \in (\mathbb{Z}/p\mathbb{Z})^\times$, changes $\mathfrak{S}_1$ to $a\mathfrak{S}_1$. This class plays a crucial role in the construction of derived Hecke operators by Venkatesh, see [Ven16], [HV18] or [DHRV].

Set $g' = \sum_{n\geq 1} \overline{a_n(g)} q^n$, where $\bar{\cdot}$ denotes complex conjugation, a newform of weight 1, level $\mathfrak{c}$ and nebentype χ_g^{-1}. Pairing $\mathfrak{S}_1$ with

$$G(z) := \mathrm{Tr}^{X_0(N\mathfrak{c})}_{X_0(N)}(g(z)g'(Nz)) \in S_2(N) \otimes \mathcal{O}_g$$

naturally yields an invariant in $(\mathbb{Z}/p\mathbb{Z}) \otimes \mathcal{O}_g$.

Conjecture (HV)**.** *There exists some $u_g \in \mathrm{U}_g^\circ$ such that given any primes p, $N \geq 5$ with $N \nmid \mathfrak{c}$ and $p \mid N-1$, then*

$$\langle G, \mathfrak{S}_1 \rangle = \log(\mathrm{red}_N(u_g))$$

where this equality holds in $(\mathbb{Z}/p\mathbb{Z}) \otimes \mathcal{O}_g$.

Remark 1.2. Note that both sides of this equation depend linearly on the choice of log and hence the statement is well-defined. Since U_g° has rank 1, it is equivalent to fix any $u_g \in \mathrm{U}_g^\circ$ and conjecture that both quantities are proportional modulo p by a factor that depends neither on p nor on N.

The following section describes in details how to compute the right hand side of this equation. For the left hand side, the definition of $\mathfrak{S}_1$ above includes an explicit method to construct it and computing the trace of $g(z)g'(Nz)$ from level $\mathfrak{c}N$ to level N does not require much work, see [HV18, section 8]. In section 3, we verify the HV conjecture for four newforms of weight 1.

Remark 1.3. The work of Darmon, Harris, Rotger and Venkatesh in [DHRV] shows that the HV conjecture is true when g is dihedral. Their proof fundamentally depends on the fact that dihedral Artin representations are induced from characters of quadratic number fields. Therefore, one does not expect to be able to generalize it for the cases of A_4, S_4 and A_5. The motivation behind this article is to provide numerical evidence for the exotic cases, even if theoretical evidence is lacking.

2 The HV Conjecture for Exotic Newforms

Let $g \in S_1(\mathfrak{c}, \chi_g)$ be an exotic newform, i.e. if $\rho_g : G_{\bar{\mathbb{Q}}/\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$ is its associated Artin representation and $\bar{\rho}_g : G_{\bar{\mathbb{Q}}/\mathbb{Q}} \to \mathrm{PGL}_2(\mathbb{C})$ is obtained by composing ρ_g with the natural surjection $\mathrm{GL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$, then the image of $\bar{\rho}_g$ is isomorphic to $\Gamma = A_4$, S_4 or A_5. This section describes how to test numerically the HV conjecture for such g while only requiring an explicit description of the field L cut out by $\bar{\rho}_g$ and the first few coefficients of the q-expansion of g. The precision to which the latter is known bounds the primes N that can be used to verify the HV conjecture.

2.1 Constructing the Associated Artin Representation

The construction of $\bar{\rho}_g$ below is based on the theory of finite subgroups in $\mathrm{PGL}_2(\mathbb{C})$, which dates back to Klein [Kle56]. The construction of ρ_g that follows is loosely based on techniques and ideas presented in [DS74] and part II of [Ser77].

Fix an isomorphism $\varphi : G_{L/\mathbb{Q}} \to \Gamma$. Firstly, we construct $\bar{\rho}_g : G_{L/\mathbb{Q}} \xrightarrow{\varphi} \Gamma \hookrightarrow \mathrm{PGL}_2(\mathbb{C})$. One may describe all possible embeddings $\Gamma \hookrightarrow \mathrm{PGL}_2(\mathbb{C})$ using platonic solids. Recall that A_4, S_4 and A_5 are the groups of rigid automorphisms of a tetrahedron, octahedron and icosahedron respectively. Writting down their action with respect to an orthonormal basis embeds Γ in $\mathrm{SO}_3(\mathbb{C})$. Finally, lifting this via the two fold cover $\mathrm{SU}_2(\mathbb{C}) \to \mathrm{SO}_3(\mathbb{C})$ and then projecting $\mathrm{SU}_2(\mathbb{C})$ in $\mathrm{PGL}_2(\mathbb{C})$ yields an embedding of Γ in $\mathrm{PGL}_2(\mathbb{C})$. Ultimately, this gives a map $G_{L/\mathbb{Q}} \xrightarrow{\varphi} \Gamma \hookrightarrow \mathrm{PGL}_2(\mathbb{C})$.

In $\mathrm{PGL}_2(\mathbb{C})$, there is a unique conjugacy class of subgroups isomorphic to A_4 and the same holds true for S_4. Therefore, the above constructs $\bar{\rho}_g$ if $\Gamma = A_4$ or S_4. On the other hand, there are exactly two conjugacy classes of subgroups isomorphic to A_5. If $\Gamma = A_5$, see Remark 2.1 below to distinguish which one corresponds to the image of $\bar{\rho}_g$.

The following table shows explicit projective representations $G_{L/\mathbb{Q}} \xrightarrow{\sim} \Gamma \hookrightarrow \mathrm{PGL}_2(\mathbb{C})$ obtained this way for all three possible cases. It serves as a reference to understand the numerical values in section 3. Only the images of generators of Γ are included. Set $\varphi_\pm = \frac{1 \pm \sqrt{5}}{2}$.

Generators of $\Gamma = A_4$	$(1\ 2\ 3)$	$(1\ 2)(3\ 4)$
Image in $\mathrm{PGL}_2(\mathbb{C})$	$\frac{-1}{2}\begin{pmatrix} -1 & \sqrt{-3} \\ \sqrt{-3} & -1 \end{pmatrix}$	$\frac{1}{3}\begin{pmatrix} 0 & \sqrt{6}+\sqrt{-3} \\ -\sqrt{6}+\sqrt{-3} & 0 \end{pmatrix}$
Generators of $\Gamma = S_4$	$(1\ 2)$	$(1\ 2\ 3\ 4)$
Image in $\mathrm{PGL}_2(\mathbb{C})$	$\frac{\sqrt{-2}}{2}\begin{pmatrix} 0 & 1+i \\ -1+i & 0 \end{pmatrix}$	$\frac{\sqrt{-2}}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$
Generators of $\Gamma = A_5$	$(1\ 2\ 3)$	$(1\ 2\ 3\ 4\ 5)$
Image in $\mathrm{PGL}_2(\mathbb{C})$	$\frac{1}{2}\begin{pmatrix} 1-i & -1+i \\ 1+i & 1+i \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} \varphi_\pm - 1 & \varphi_\pm - i \\ -\varphi_\pm - i & \varphi_\pm - 1 \end{pmatrix}$

Table 1: Embeddings $\Gamma \hookrightarrow \mathrm{PGL}_2(\mathbb{C})$ used in section 3.

Secondly, we compute $\rho_g(\sigma_N)$ for primes $N \nmid \mathfrak{c}$. Let $s \in \mathrm{GL}_2(\mathbb{C})$ be one of the two matrices satisfying the following conditions :

(a) The image of s in $\mathrm{PGL}_2(\mathbb{C})$ equals $\bar{\rho}_g(\bar{\sigma}_N)$.

(b) $\det(s) = 1$.

It follows from (a) that $\rho_g(\sigma_N)$ differs from s by some $\lambda \in \mathbb{C}^\times$, say $\rho_g(\sigma_N) = \lambda s$. The correspondence between g and ρ_g ensures that

$$a_N(g) = \mathrm{Tr}(\rho_g(\sigma_N)) = \lambda \cdot \mathrm{Tr}(s) \quad \text{and} \quad \chi_g(N) = \det(\rho_g(\sigma_N)) = \lambda^2 ,$$

see [DS74, Theorem 4.1].

Remark 2.1. In particular, if g is icosahedral, the above implies that $|a_N(g)| = |\varphi_\pm - 1|$ for any prime $N \nmid \mathfrak{c}$ such that $\bar{\sigma}_N \in G_{L/\mathbb{Q}}$ has order 5, where $|\cdot|$ denotes absolute value. Therefore, the coefficients of the q-expansion of g identifies which conjugacy class in $\mathrm{PGL}_2(\mathbb{C})$ of subgroups isomorphic to A_5 corresponds to the image of $\bar{\rho}_g$.

If $a_N(g) \neq 0$, this uniquely determines λ and $\rho_g(\sigma_N)$. On the other hand, if $a_N(g) = 0$, this identifies $\rho_g(\sigma_N)$ up to a sign.

Remark 2.2. For primes $N \nmid \mathfrak{c}$ such that $a_N(g) = 0$, one can therefore only compute the constant of proportion between $\langle G, \mathfrak{S}_1 \rangle$ and $\log(\mathrm{red}_N(u_g))$ modulo its sign, see Remark 1.2. Thus, testing the HV conjecture amounts to verifying that one of the two signs leads to a constant that agrees with all the other primes $N' \nmid \mathfrak{c}$ where the value of $\rho_g(\sigma_{N'})$ is not ambiguous.

By repeating this procedure for sufficiently many primes $N \nmid \mathfrak{c}$, one constructs ρ_g and E_g. The resulting representation only depends on the choice of embedding $\Gamma \hookrightarrow \mathrm{PGL}_2(\mathbb{C})$.

2.2 Constructing Stark Units Associated to a Weight 1 Newform

In this section, we construct a non-trivial element of U_g°, modulo torsion coming from roots of unity in L, that can be used to verify the HV conjecture. First, consider the E_g-vector space

$$\mathrm{U}_g := (\mathcal{O}_L^\times \otimes \mathrm{Hom}_{E_g}(\mathrm{Ad}\ \rho_g, E_g))^{G_{L/\mathbb{Q}}} = \mathrm{Hom}_{E_g[G_{L/\mathbb{Q}}]}(\mathrm{Ad}\ \rho_g, \mathcal{O}_L^\times \otimes E_g)$$

Observe that Ad ρ_g is the direct sum between the trivial irreducible representation obtained from scalar endomorphisms and a 3-dimensional subrepresentation $\mathrm{Ad}^0\rho_g$ formed by the trace-free endomorphisms. Since any element of U_g vanishes on the trivial subrepresentation, there is a canonical isomorphism

$$\mathrm{U}_g \cong \mathrm{Hom}_{E_g[G_{L/\mathbb{Q}}]}(\mathrm{Ad}^0\rho_g, \mathcal{O}_L^\times \otimes E_g)$$

Assume there exists some subfield $M \subset L$ and a 1-dimensional character ψ of $G_{L/M}$ such that $\mathrm{Ad}^0\rho_g$ is isomorphic to a subrepresentation of $\mathrm{Ind}_M^{\mathbb{Q}}\psi$ over $\mathbb{C}$. By enlarging E_g if necessary, one may assume that ψ and this isomorphism are both defined over E_g. Using Frobenius reciprocity, one therefore obtains

$$(\mathcal{O}_L^\times \otimes E_g)^{(\psi)} \cong \mathrm{Hom}_{E_g[G_{L/M}]}(\psi, \mathcal{O}_L^\times \otimes E_g) \to \mathrm{Hom}_{E_g[G_{L/\mathbb{Q}}]}(\mathrm{Ad}^0\rho_g, \mathcal{O}_L^\times \otimes E_g) \cong \mathrm{U}_g\ .$$

Let r be the order of the torsion subgroup of $\mathcal{O}_L^\times$ and identify $\mathcal{O}_L^\times \otimes \mathcal{O}_g \otimes \mathbb{Z}[\frac{1}{r}]$ with its natural image in $\mathcal{O}_L^\times \otimes E_g$. Moreover, given $n \in \mathbb{Z}$, let $[n] : \mathrm{U}_g \to \mathrm{U}_g$ be the map obtained functorially from the *multiplication-by-n* homomorphism on $\mathcal{O}_L^\times \otimes E_g$. By taking n sufficiently large, one may assume that any element of $(\mathcal{O}_L^\times \otimes \mathcal{O}_g \otimes \mathbb{Z}[\frac{1}{r}])^{(\psi)} \cong (\mathcal{O}_L^\times \otimes \mathcal{O}_g)^{(\psi)} \otimes \mathbb{Z}[\frac{1}{r}]$ is sent by the following composition

$$(\mathcal{O}_L^\times \otimes \mathcal{O}_g)^{(\psi)} \otimes \mathbb{Z}[1/r] \subset (\mathcal{O}_L^\times \otimes E_g)^{(\psi)} \to \mathrm{U}_g \xrightarrow{[n]_*} \mathrm{U}_g \xrightarrow{\text{restriction}} \mathrm{Hom}_{\mathcal{O}_g[G_{L/\mathbb{Q}}]}(\mathrm{Ad}^\circ\rho_g, \mathcal{O}_L^\times \otimes E_g)$$

to some element mapping $\mathrm{Ad}^\circ\rho_g$ inside $\mathcal{O}_L^\times \otimes \mathcal{O}_g \otimes \mathbb{Z}[\frac{1}{r}]$. Hence, the above yields a map

$$\alpha_g : (\mathcal{O}_L^\times \otimes \mathcal{O}_g)^{(\psi)} \otimes \mathbb{Z}[1/r] \to \mathrm{U}_g^\circ \otimes \mathbb{Z}[1/r]\ .$$

Then, α_g can be used to construct non-trivial elements of $\mathrm{U}_g^\circ \otimes \mathbb{Z}[\frac{1}{r}]$. The reduction map red_N can naturally be extended to a homomorphism $\mathrm{red}_N : \mathrm{U}_g^\circ \otimes \mathbb{Z}[\frac{1}{r}] \to \mathbb{Z}/p\mathbb{Z} \otimes \mathcal{O}_g \otimes \mathbb{Z}[\frac{1}{r}]$ without affecting the validity of the HV conjecture. The proportionality predicted in Remark 1.2 is also independent of the choice of $n \in \mathbb{Z}$.

To apply this approach to the three possible types of exotic newforms, we first introduce some notation. By enlarging E_g if necessary, we consider all of the following representations as defined over E_g.

For a symmetric group S_n, let sgn_{S_n} denote its *sign* character and $\rho^{st}_{S_n}$, its *standard representation* obtained by letting S_n permute the coordinates of an n-dimensional E_g-vector space. Let $\rho^{st,0}_{S_n}$ be the unique irreducible $(n-1)$-dimensional subrepresentation of the latter and denote its restriction to an A_n-representation by $\rho^{st,0}_{A_n}$. Finally, let $\rho^{\pm}_{ico}$ denote the two irreducible 3-dimensional representations of A_5. We assume that E_g contains both roots of x^2-5 when working with $\rho^{\pm}_{ico}$.

One easily verifies that

$$\mathrm{Ad}^0\rho_g \cong \begin{cases} \rho^{st,0}_{A_4}\ , & \text{if } g \text{ is tetrahedral} \\ \rho^{st,0}_{S_4}\otimes \mathrm{sgn}_{S_4}\ , & \text{if } g \text{ is octahedral} \\ \rho^{\pm}_{ico}\ , & \text{if } g \text{ is icosahedral} \end{cases}$$

Assume that ρ^{+}_{ico} (resp. ρ^{-}_{ico}) is the representation occuring in the decomposition of $\mathrm{Ad}\,\rho_g$ when $\bar\rho_g$ corresponds to the embedding $A_5 \to \mathrm{PGL}_2(\mathbb{C})$ involving φ_+ (resp. φ_-).

Tetrahedral newforms : If g is of type A_4, one has $\mathrm{Ad}\,\rho_g \cong \mathrm{Ind}^{\mathbb{Q}}_M \mathbf{1}$, where M is any quartic subfield of L and $\mathbf{1}$ is the trivial character of $G_{L/M}$. Here, Frobenius reciprocity immediately gives an isomorphism $\mathcal{O}^{\times}_M \otimes E_g \xrightarrow{\sim} \mathrm{U}_g$ and hence, one obtains

$$\alpha_g : \mathcal{O}^{\times}_M \otimes \mathcal{O}_g \otimes \mathbb{Z}[1/r] \to \mathrm{U}^{\circ}_g \otimes \mathbb{Z}[1/r]$$

Otherwise, letting M' be any cubic subextension of L, K' be any quadratic extension of M' contained in L and ψ be the unique non-trivial character of $G_{K'/M'}$, one finds $\mathrm{Ad}^0\rho_g \cong \mathrm{Ind}^{\mathbb{Q}}_{M'}\psi$ and

$$\alpha_g : \mathcal{O}^{(1)}_{K'/M'} \otimes \mathcal{O}_g \otimes \mathbb{Z}[1/r] \to U^{\circ}_g \otimes \mathbb{Z}[1/r]$$

where $\mathcal{O}^{(1)}_{K'/M'} = (\mathcal{O}^{\times}_{K'})^{(\psi)}$ denotes the groups of units of $\mathcal{O}_{K'}$ with relative norm to M' equal to 1.

Octahedral newforms: If g is of type S_4, let K be a subfield of L of degree 8, $M \subset K$ a subfield of degree 4 and H the unique quadratic subfield of L. We have $\mathrm{Ind}^{\mathbb{Q}}_M\psi = \mathrm{sgn}_{S_4} \oplus \mathrm{Ad}^0\rho_g$, where ψ is the unique non-trivial character of $G_{L/M}$. The above gives us

$$\alpha_g : \mathcal{O}^{(1)}_{K/M} \otimes \mathcal{O}_g \otimes \mathbb{Z}[1/r] \to \mathrm{U}^{\circ}_g \otimes \mathbb{Z}[1/r]\ ,$$

where $\mathcal{O}^{(1)}_{K/M} = (\mathcal{O}^{\times}_K)^{(\psi)}$ again denotes the units in $\mathcal{O}^{\times}_K$ with relative norm to M equal to 1. The kernel of α_g is $\mathcal{O}^{(1)}_H \otimes \mathcal{O}_g \otimes \mathbb{Z}[1/r]$, where $\mathcal{O}^{(1)}_H = (\mathcal{O}^{\times}_L)^{(\mathrm{sgn}_{S_4})}$ is the group of units of norm 1 in $\mathcal{O}_H$. Hence, the kernel is trivial if H is imaginary and has rank 1 otherwise.

Similarly, one may consider any cubic subextension M' of L and K', the unique quadratic extension of M' contained in L such that $G_{L/K'}$ is cyclic. Then, $\mathrm{Ad}^{\circ}\rho_g \cong \mathrm{Ind}^{\mathbb{Q}}_{M'}\psi$, where ψ is the unique non-trivial character of $G_{K'/M'}$. Once more, this yields an homomorphism

$$\alpha_g : \mathcal{O}^{(1)}_{K'/M'} \otimes \mathcal{O}_g \otimes \mathbb{Z}[1/r] \to \mathrm{U}^{\circ}_g \otimes \mathbb{Z}[1/r]$$

where $\mathcal{O}^{(1)}_{K'/M'} = (\mathcal{O}^{\times}_{K'})^{(\psi)}$ is the group of units of $\mathcal{O}_{K'}$ with relative norm to M' equal to 1.

Icosahedral newforms : If g is of type A_5, let M be any subfield of L of degree 12. Here, we enlarge E_g if necessary to ensure it contains 5-th roots of unity (and hence also contains $\pm\sqrt{5}$). For exactly two of the four non-trivial characters ψ of $G_{L/M} \cong \mathbb{Z}/5\mathbb{Z}$, the induced representation $\mathrm{Ind}_M^{\mathbb{Q}}\psi$ contains ρ^+_{ico} as a subrepresentation, and it contains ρ^-_{ico} for the other two.

3 Numerical Evidence

3.1 Example with a Tetrahedral Newform

Example 1 : Let $\mathfrak{c} = 133 = 7 \cdot 19$ and χ be the Dirichlet character of conductor 133 with order 2 at 7 and order 3 at 19. According to Darmon, Lauder and Rotger in [DLR15],

$$g = q + (\nu^2 - 1)q^2 - \nu q^3 + \nu q^5 - \nu(\nu^2-1)q^6 + \nu^3 q^7 - q^8 + \nu(\nu^2-1)q^{10} + O(q^{13})$$

is an A_4-exotic newform of level 133 and nebentype $\chi_g = \chi$, where ν is a primitive 12-th root of unity. Its associated Artin representation ρ_g can be realized over the 24-th cyclotomic field (see section 2.1). Moreover, $\bar{\rho}_g$ cuts out the splitting field L of $x^4 + 3x^2 - 7x + 4$. Using section 2.2, one may construct a non-trivial Stark unit by selecting any root of $x^4 - 4x^3 + 9x^2 - 3x + 1$ in L.

In the appendix, Table 1 verifies the HV conjecture with this newform for primes p, $N \leq 130$. In this case, one obtains

$$\log(\mathrm{red}_N(u_g)) = -\sqrt{3} \cdot \langle G, \mathfrak{S}_1 \rangle \mod p$$

3.2 Examples with Octahedral Newforms

Example 2 : According to Darmon, Lauder and Rotger in [DLR15], the space $S_1(283, \chi_{283})$, where χ_{283} is the unique quadratic character of conductor 283, is spanned by the following three Hecke newforms

$$f_0 = q + q^4 - q^7 + q^9 - q^{11} - q^{13} + q^{16} - q^{23} + \dots$$
$$f_\pm = q \pm \sqrt{-2}q^2 \mp \sqrt{-2}q^3 - q^4 \mp \sqrt{-2}q^5 + 2q^6 + \dots$$

and the latter two are S_4-exotic newforms. Consider $g = f_+$. Its associated Artin representation ρ_g can be realised over the 8th cyclotomic field and $\bar{\rho}_g$ cuts out the splitting field L of $x^4 - x - 1$. Finally, selecting any root of $x^8 - 18x^7 + 81x^6 + 18x^5 + 119x^4 + 18x^3 + 81x^2 - 18x + 1$ in L allows one to construct a non-trivial Stark unit, see section 2.2.

In the Appendix, Table 2 verifies the HV conjecture with this newform for primes p, $N \leq 61$. In this case, one obtains

$$\log(\mathrm{red}_N(u_g)) = \frac{\zeta_8^3}{2} \cdot \langle G, \mathfrak{S}_1 \rangle \mod p$$

where ζ_8 is a primitive 8-th root of unity.

Examples 3 and 4 : According to Serre [Ser77], the space $S_1(229, \chi_{229})$, where χ_{229} is the unique quartic character of conductor 229 such that $\chi_{229}(2) = i$, is spanned by the following two Hecke newforms

$$g_1 = q + q^3 - iq^4 + iq^5 + (i-1)q^7 - iq^{11} - iq^{12} + \dots$$
$$g_2 = q + (1+i)q^2 - q^3 + iq^4 + iq^5 - (1+i)q^6 + \dots$$

and are both S_4-exotic newforms.

Their associated Artin representations ρ_{g_1} and ρ_{g_2} can both be realized over the 8-th cyclotomic field. The fields cut out by $\bar{\rho}_{g_1}$ and $\bar{\rho}_{g_2}$ are

$$L_1 := \mathbb{Q}(\sqrt{8x_i - 3} : i = 1,2,3) \quad \text{and} \quad L_2 := \mathbb{Q}(\sqrt{4 - 3x_i^2} : i = 1,2,3)$$

respectively, where x_1, x_2 and x_3 are the roots of $x^3 - 4x + 1$. Finally, one can find some non-trivial Stark unit by selecting any root of $f_i(x)$ in L_i, where

$$f_1(x) = x^8 + 4x^7 + 10x^6 + 1848x^5 + 214363x^4 + 1848x^3 + 10x^2 + 4x + 1 \quad \text{and}$$
$$f_2(x) = x^8 - 232x^7 + 22249x^6 - 1046784x^5 + 20376493x^4 - 1046784x^3 + 22249x^2 - 232x + 1$$

In the Appendix, Table 3 and 4 verify the HV conjecture on the newforms g_1 and g_2 respectively, for primes p, $N \leq 71$. One finds

$$\log(\mathrm{red}_N(u_{g_1})) = 2(1-i) \cdot \langle G, \mathfrak{S}_1 \rangle \mod p$$

and

$$\log(\mathrm{red}_N(u_{g_2})) = 2(i-1) \cdot \langle G, \mathfrak{S}_1 \rangle \mod p$$

Appendix

p	N	$\log(\mathrm{red}_N(u_g))$	$\langle\ G, \mathfrak{S}_1\ \rangle$	Ratio
5	11	$3\nu(\zeta_3+3)$	$4i$	$-\sqrt{3}$
	31	$4\nu(\zeta_3-1)$	4	$-\sqrt{3}$
	41	$-\nu(\zeta_3+3)$	$3\zeta_3$	$-\sqrt{3}$
	61	$\nu(\zeta_3+2)$	$4(\zeta_3+1)$	$-\sqrt{3}$
	101	$4\nu(\zeta_3+2)$	ζ_3+1	$-\sqrt{3}$
7	29	$-(\zeta_3-1)$	$\nu\zeta_3$	$-\sqrt{3}$
	43	ζ_3+2	6ν	$-\sqrt{3}$
	71	$5(\zeta_3+2)$	2ν	$-\sqrt{3}$
	113	ζ_3+3	$3i$	$-\sqrt{3}$
	127	$3(\zeta_3-1)$	$4\nu\zeta_3$	$-\sqrt{3}$
11	23	$2(\zeta_3-1)$	$9\nu\zeta_3$	$-\sqrt{3}$
	67	$4(\zeta_3-1)$	$7\nu\zeta_3$	$-\sqrt{3}$
	89	$5\nu(\zeta_3+2)$	$6(\zeta_3+1)$	$-\sqrt{3}$
13	53	$2\zeta_3$	$11\nu\zeta_3$	$-\sqrt{3}$
	79	$4(\zeta_3+2)$	9ν	$-\sqrt{3}$
17	103	$-5\sqrt{3}$	12	$-\sqrt{3}$
23	47	$10\nu(\zeta_3+2)$	$13(\zeta_3+1)$	$-\sqrt{3}$
29	59	$2\nu\sqrt{-3}$	$27\zeta_3$	$-\sqrt{3}$
41	83	$-7\sqrt{3}$	7	$-\sqrt{3}$
53	107	$-3\sqrt{-3}$	$3i$	$-\sqrt{3}$

Table 1 : Numerical Evidence for Example 1.

Conventions :

$i=\nu^3$, $\zeta_3=\nu^2-1$, $\sqrt{3}=-i+2\nu$ and $\sqrt{-3}=\sqrt{3}i$.

p	N	$\log(\mathrm{red}_N(u_g))$	$\langle\ G, \mathfrak{S}_1\ \rangle$	Ratio
5	11	$3(1+i)$	$4\sqrt{-2}$	$3\zeta_8^3$
	31	$2\zeta_8^3$	4	$3\zeta_8^3$
	41	0	0	–
7	29	$2(1+i)$	$3\sqrt{-2}$	$4\zeta_8^3$
	43	ζ_8^3	2	$4\zeta_8^3$
11	23	$3(1+i)$	$5\sqrt{-2}$	$6\zeta_8^3$
13	53	$2\zeta_8^3$	9	$7\zeta_8^3$
23	47	$17\zeta_8^3$	11	$12\zeta_8^3$
29	59	$2(1+i)$	$18\sqrt{-2}$	$15\zeta_8^3$

Table 2 : Numerical Evidence for Example 2.

Conventions : $\zeta_8 = \sqrt{2}(1+i)/2$ and $\sqrt{-2} = \sqrt{2}i$.

p	N	$\log(\mathrm{red}_N(u_g))$	$\langle\ G, \mathfrak{S}_1\ \rangle$	Ratio
5	11	$2(1-i)$	1	$2(1-i)$
	31	0	0	–
	41	4	$1+i$	$2(1-i)$
	61	$1+i$	$3i$	$2(1-i)$
	71	$1+i$	$3i$	$2(1-i)$
7	29	i	$5(1-i)$	$2(1-i)$
	43	$3(1+i)$	$5i$	$2(1-i)$
	71	0	0	–
11	23	$7i$	$1-i$	$2(1-i)$
	67	9	$4(1+i)$	$2(1-i)$
23	47	4	$1+i$	$2(1-i)$
29	59	$17i$	$3(1-i)$	$2(1-i)$

Table 3 : Numerical Evidence for Example 3.

p	N	$\log(\mathrm{red}_N(u_g))$	$\langle\ G, \mathfrak{S}_1\ \rangle$	Ratio
5	11	$1-i$	2	$2(i-1)$
	31	$3i$	$2(1-i)$	$2(i-1)$
	41	3	$3(1+i)$	$2(i-1)$
7	29	$5i$	$3(1-i)$	$2(i-1)$
	43	$6(1+i)$	$4i$	$2(i-1)$
11	23	$7i$	$i-1$	$2(i-1)$
23	47	13	$14(1+i)$	$2(i-1)$
29	59	$9i$	$5(i-1)$	$2(i-1)$

Table 4 : Numerical Evidence for Example 4.

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