

p -ADIC L -FUNCTIONS FOR P -ORDINARY HIDA FAMILIES ON UNITARY GROUPS

DAVID MARCIL

ABSTRACT. We construct a p -adic L -function for P -ordinary Hida families of cuspidal automorphic representations on a unitary group G . The main new idea of our work is to incorporate the theory of Schneider-Zink types for the Levi quotient of P , to allow for the possibility of higher ramification at primes dividing p , into the study of (p -adic) modular forms and automorphic representations on G . For instance, we describe the local structure of such a P -ordinary automorphic representation π at p using these types, allowing us to analyze the geometry of P -ordinary Hida families. Furthermore, these types play a crucial role in the construction of certain Siegel Eisenstein series designed to be compatible with such Hida families in two specific ways : Their Fourier coefficients can be p -adically interpolated into a p -adic Eisenstein measure on $d + 1$ variables and, via the doubling method of Garrett and Piatetski-Shapiro-Rallis, the corresponding zeta integrals yield special values of standard L -functions. Here, d is the rank of the Levi quotient of P . Lastly, the doubling method is reinterpreted algebraically as a pairing between modular forms on G , whose nebentype are types, and viewed as the evaluation of our p -adic L -function at classical points of a P -ordinary Hida family.

CONTENTS

Introduction	3
Part I. P-(anti-)ordinary theory on unitary groups.	8
1. Notation and conventions.	8
1.1. CM types and local places.	8
1.2. Local theory of types for smooth representations.	9
2. Modular forms on unitary groups with P -Iwahoric level at p .	11
2.1. Unitary PEL datum.	11
2.2. Structure of G over \mathbb{Z}_p .	14
2.3. Structure of G over \mathbb{C} .	17
2.4. Shimura varieties of P -Iwahoric level at p .	22
2.5. P -nebenotypus theory of modular forms.	25

Date: September 11, 2024.

2010 Mathematics Subject Classification. Primary: 11F33, 11F55, 11F67, 11F70, 11F85, 11S40.
 Secondary: 11F03, 11F30, 11G18, 11R23, 14G10 .

Key words and phrases. Types, P -ordinary representations, Hida families, p -adic L -functions.

2.6. Complex Uniformization.	31
2.7. Duality and integrality.	33
2.8. P -(anti-)ordinary modular forms	37
3. P -(anti-)ordinary (anti-)holomorphic automorphic representations.	39
3.1. (Anti-)holomorphic automorphic representations.	39
3.2. P -(anti-)ordinary automorphic representations.	42
3.3. P -(anti-)weight-level-type	44
4. Compatibility and comparison between PEL data.	45
4.1. Unitary groups for the doubling method.	45
4.2. Comparisons between G_1 and G_2 .	49
5. P -ordinary p -adic modular forms.	55
5.1. Igusa tower.	55
5.2. Scalar-valued p -adic modular forms with respect to P	56
5.3. p -adic modular forms valued in locally algebraic representations	60
5.4. Hecke operators on p -adic modular forms.	63
Part II. Families of P-(anti-)ordinary automorphic representations.	63
6. Structure at p of P -(anti-)ordinary automorphic representations.	63
6.1. P -ordinary theory on G_1 .	63
6.2. P -anti-ordinary theory on G_1 .	69
6.3. P -ordinary theory on G_2 .	73
6.4. P -anti-ordinary theory on G_2 .	75
7. Explicit choice of P -(anti-)ordinary vectors.	76
7.1. Local test vectors at places away from p and ∞ .	76
7.2. Local test vectors at p	77
7.3. Local test vectors at ∞	83
8. P -(anti-)ordinary Hida families.	85
8.1. Hecke algebras for modular forms with respect to P .	85
8.2. Lattices of holomorphic P -ordinary forms.	87
8.3. Lattices of anti-holomorphic P -anti-ordinary forms.	89
8.4. Big Hecke algebra and P -anti-ordinary Hida families.	91
Part III. P-ordinary family of Siegel Eisenstein series	97
9. Siegel Eisenstein series for the doubling method.	97
9.1. Siegel Eisenstein series.	97
9.2. Local Siegel-Weil section at p .	99
9.3. Local Siegel-Weil section at ∞ .	104
9.4. Local Siegel-Weil section away from p and ∞ .	108
9.5. Siegel Eisenstein series as C^∞ -modular forms.	110
10. Zeta integrals and the doubling method.	112
10.1. Local zeta integrals at p .	112
10.2. Local zeta integrals at ∞ .	121
10.3. Local zeta integrals away from p and ∞ .	121

10.4. Global formula.	122
11. P -ordinary Eisenstein measure.	122
11.1. Fourier coefficients of Eisenstein series.	123
11.2. Calculations of local Fourier coefficients	124
11.3. p -adic differential operators	130
Part IV. p-adic L-functions for P-ordinary families.	135
12. Pairing, periods and main result.	135
12.1. Eisenstein measures and p -adic L -functions	135
12.2. Normalized periods and congruence ideals.	138
12.3. Statement of the main theorem	139
References	140

INTRODUCTION

In [HLS06], Michael Harris, Jian-Shu Li and Christopher Skinner initiated a project to construct a p -adic L -function for ordinary Hida families on a unitary group of arbitrary signature. In [EHLS20], together with Ellen Eischen, they completed this project p -adic L -function for ordinary families on unitary groups. This required the development of several technical results on p -adic differential operators, accomplished in great part by Eischen in [Eis12], to obtain a more general Eisenstein measure [Eis15] than the one originally constructed in [HLS06]. Fundamental properties of their p -adic L -function for families are obtained by carefully computing local zeta integrals related to the doubling method [GPSR87] as well as local coefficients of Siegel Eisenstein series [Eis15]. The most technical calculations are for local factors at places above the fixed prime p . Moreover, a theorem of Hida in [Hid98] establishing the uniqueness (up to scalar) of ordinary vectors plays a crucial role in their analysis.

In this article, we generalize the many steps involved in their construction to construct a p -adic L -function for a P -ordinary Hida family on G . Here, P is a parabolic subgroup of a product of general linear groups related to G . When P corresponds to (a product of) upper triangular Borel subgroups, the notion of π being “ P -ordinary” coincides with the usual notion of being “ordinary” studied in [EHLS20].

One advantage of working with P -ordinary families is that they are substantially more general than ordinary families. In particular, every cuspidal automorphic representation lies in a P -ordinary family for some choice of P . However, the dimension of families is inverse proportional to the size of the chosen parabolic P . Namely, if d is the rank of the center of the Levi of P , then a P -ordinary family is d -dimensional.

In this paper, we therefore construct a $(d + 1)$ -variable p -adic L -function on a P -ordinary Hida family associated to a P -ordinary automorphic representation. Here the extra variable corresponds to the cyclotomic variable.

Note that our work actually considers (*anti-holomorphic*) P -*anti-ordinary* automorphic representations, however we discuss the technicalities of this notion in details later in this paper.

Structure of this paper. In Part I, we introduce the necessary setup to discuss the geometry, representation theory and the complex analysis related to (P -ordinary) automorphic representations.

The fundamental difference between working with a representation π that is P -ordinary (at p) instead of ordinary representation is the necessity to consider local finite-dimensional representations, i.e. *types*, instead of characters to study the structure at p of π . Therefore, in Section 1, we discuss some of the work of Bushnell-Kutzko [BK93, BK98, BK99] and Schneider-Zink [SZ99] to study smooth representations of local p -adic groups via types. we generalize this theorem of Hida to construct a canonical finite-dimensional subspace in the space of P -ordinary vectors for a P -ordinary representation π on a unitary group G .

In Section 2, we study the geometry of the Shimura varieties associated to general unitary groups and vector bundles associated to a weight and a type. The automorphic representations of interest in this paper have a non-trivial contribution in the cohomology groups of such bundles. In particular, we introduce Iwahori subgroups $I_{P,r}^0$ and pro- p -Iwahori subgroup $I_{P,r}$, that depend on P , and whose quotient is a product of general linear group (and not a torus). The center of this quotient plays an important role to parameterize P -ordinary Hida families. Furthermore, a type (or a P -*nebenotypus*) is a smooth representation of $I_{P,r}^0$ that factors through $I_{P,r}$.

In Section 3, we introduce the notion of a P -ordinary representation π , namely a representation that is ordinary with respect to some parabolic subgroup P of G . The main goal is to explain how, for our purpose, their theory is encapsulated by the information of a weight κ , a level K_r and a P -nebenotypus τ . The weight holds archimedean information on π_∞ , the level holds information about ramified places and p , and τ holds information about π_p . We refer to the datum (κ, K_r, τ) as the P -*weight-level-type* of π .

In Section 4, we briefly recall the functors necessary to compare automorphic representations between the various unitary groups involved in the doubling method. Most of this work is well-established in [Ehls20], however one needs to make minor modifications to consider the P -ordinary setting when comparing level subgroups on different unitary groups.

In Section 5, we introduce p -adic modular forms with respect to the choice of parabolic P . To the best of the author's knowledge, this material has yet to be discussed in the literature when working with unitary groups.

We introduce two variations of such p -adic forms: scalar-valued ones and vector-valued ones. The first case, i.e. the scalar-valued case, is relatively similar to the usual notion of p -adic modular forms, as discussed in [Hid04] and [Ehls20]. However, the global sections on the Igusa tower considered are not fixed by the unipotent radical subgroup of a Borel but rather by the unipotent radical of P .

This allows us to study the smooth action of the Levi of P on this space of p -adic L -functions and decompose it as a direct sum over types.

The second case, i.e. the vector-valued case, considers global sections over the Igusa tower of a non-trivial vector bundle. The vector bundles involved are closely related to the P -nebentypus introduced in the previous sections. After introducing the relevant notation, we then discuss some of the “standard” results of Hida theory, i.e. density, classicality and the vertical control theorem, in the P -ordinary setting. Note that some of these conjectures are later stated in Section 8.

As the goal of this paper is not to establish “ P -ordinary Hida theory”, we simply leave the necessary results as conjectures and the author plans to revisit each of these conjectures in a subsequent paper to complete the theory developed in this section.

In Part II, we dedicate Section 6 to study of the local representation theory at p of P -ordinary (and P -anti-ordinary) representations. The goal is to generalize the theorem of Hida mentioned above in the introduction to the P -ordinary setting and interpret it in various settings necessary for applications with the doubling method in later sections.

We obtain the uniqueness (up to scalar) of an embedding of certain “Schneider-Zink types” (discussed in Section 1) inside the space of P -ordinary vectors of a P -ordinary representations. In other words, although P -ordinary vectors are not unique (up to scalar) as in the ordinary case, we can construct a canonical subspace associated to a type whose dimension equals the dimension of this type. This represents the first main accomplishment of this paper.

In later section, it becomes clear that the construction of our p -adic L -function does not depend on the choice of a P -ordinary vector in this subspace associated to a type but only on the unique (up to scalar) embedding of that type.

Note that to obtain this result, we impose a certain hypothesis on the supercuspidal support of the P -ordinary representations involved. The author expects that this hypothesis is completely superficial and can be removed. We use it to simplify the analysis of the filtration obtained by the Bernstein-Zelevinsky geometric lemma of local representations involved. However, the results and the remainder of the paper are phrased with as little dependency as possible to this hypothesis. The author plans to revisit this issue in a later paper to explain how the results of Section 6 should still holds without this hypothesis.

We discuss in Sections 7-8 the relation between such subspaces of P -ordinary vectors and analogous subspaces of P -ordinary modular forms and how these subspaces vary nicely over a d -dimensional P -ordinary family of representations over a weight space associated to P . Here, d is the rank of the center of the Levi of P , as mentioned in the introduction above. This requires the study of certain Hecke algebras of infinite level at p associated to a weight and P -nebentypus. In particular, this analysis demonstrates that the P -nebentypus cuts out a branch of an infinite dimensional p -adic space containing such a P -ordinary family. The p -adic L -function constructed in this paper is a function on such a branch.

In Part III, we present all the necessary computations to construct our p -adic L -function using the doubling method of Garrett and Piatetski–Shapiro–Rallis. This first requires the construction in Section 9 of certain Siegel Eisenstein series depending on various inputs, most importantly on the P -weight-level-type of a P -ordinary representation as well as a Hecke character.

To accomplish this, we build the Eisenstein series from local Siegel–Weil sections, one for every place of \mathbb{Q} . In the literature, such local sections have been well studied, especially at archimedean places and at unramified places. However our construction of local sections at p in Section 9.2 is considerably more involved and represents the core of the computations that follow.

The main goal for this task consists of finding a local section (at p), given a fixed type τ , whose contribution to local zeta integrals yields the right Euler factors at p of L -functions, see Theorem 10.6, and whose contribution to the Fourier coefficients of the corresponding Eisenstein series fit in a p -adic measure, see Proposition 11.2. Our construction generalizes the already complicated machinery developed in [EHLS20, Section 4.3.1] (where τ is only allowed to be a character). However, the author hopes that the systematic use of types helps to simplify the exposition to some extent.

In Section 10, we then compute the local zeta integrals associated to the factorization of doubling method integral between the Siegel Eisenstein series constructed in the previous section and the test vectors constructed in Section 7.

This yields the second main accomplishment of this paper, briefly mentioned above, namely the calculations of local zeta integrals at p in the P -ordinary setting. Our approach owes a great deal to the precise details explained in [EHLS20, Section 4.3.6]. Nonetheless, our analysis requires to resolve many issues related to the dimension of the types. In particular, the “ordinary characters” involved in *loc.cit* are replaced by P -nebenotypes. The main novelty of our work is to use matrix coefficients of these P -nebenotypes to compute the necessary integrals explicitly and relate them to special values of standard L -functions.

Then, in Section 11, we p -adically interpolate the Siegel Eisenstein series previously constructed to obtain an *Eisenstein measure* which generalizes an analogous construction in [EFMV18]. The latter is also a generalization of analogous Eisenstein measure in various papers of Eischen and ultimately finds its roots in the seminal work of Katz [Kat78]. Our approach is to p -adically interpolate the Fourier coefficient of Eisenstein series. Inspired by the computations presented in [Eis15], the third main accomplishment of this paper is the explicit computation of local Fourier coefficients at p of our Siegel Eisenstein series, generalizing one of the main results in *loc.cit*. Once more, the use of matrix coefficients leads to simple formulae for local Fourier coefficients.

Lastly, in Part IV, we reinterpret the Eisenstein measure constructed in previous sections as an element \mathcal{L} of a certain Hecke algebra tensored with an Iwasawa algebra (related to the cyclotomic variable). This follows an approach, adapted to the P -ordinary setting, parallel to the one discussed in [EHLS20, Section 7.4]. We interpret

the results of previous sections algebraically to interpolate special values of standard L -functions as the evaluation of \mathcal{L} at classical points of a P -ordinary Hida family.

Main result. The main result of this paper is Theorem 12.6 which can be summarized as follows.

Theorem. For a general unitary group G associated to a Hermitian vector space over a CM field \mathcal{K} , fix a parabolic subgroup P of $G(\mathbb{Z}_p)$ as in Section 2.2.2.

Let π be an (anti-)holomorphic, P -(anti-)ordinary cuspidal automorphic form on a general unitary group $G(\mathbb{A})$. Let (κ, K_r, τ) be its P -(anti-)weight-level-type, where τ is a certain (fixed) Schneider-Zink type of π . Assume that the “standard conjectures of P -ordinary Hida theory” hold and that π satisfy various other standard hypotheses discussed in Sections 6 and 8.

Let $\mathbb{T} = \mathbb{T}_{\pi, [\kappa, \tau]}$ be the P -ordinary Hecke algebra associated to π as in Section 8.4.1, which only depends on κ and τ up to “ P -parallel shifts” as discussed in Sections 2.3.2 and 2.5.2 respectively.

Let Λ_{X_p} denote the \mathbb{Z}_p -Iwasawa algebra of the ray class group X_p of conductor p^∞ over \mathcal{K} .

Given test vectors $\varphi \in \widehat{I}_\pi$, $\varphi^b \in \widehat{I}_{\pi^b}$ as in Section 8.4.4, there exists a unique element

$$L(\text{Eis}^{[\kappa, \tau]}, P\text{-ord}; \varphi \otimes \varphi^b) \in \Lambda_{X_p, R} \widehat{\otimes} \mathbb{T}_\pi$$

satisfying the following property :

Let $\chi = \|\cdot\|_{\frac{n-k}{2}} \chi_u : X_p \rightarrow R^\times$ be the p -adic shift of a Hecke character as in Section 11.2.5. Let $\pi' \in \mathcal{S}(K^p, \pi)$ be a classical point of the P -ordinary Hida family \mathbb{T}_π as in Section 8.4.2. Let $\lambda_{\pi'}$ be the Hecke character of \mathbb{T} associated to π' as in Sections 8.2–8.3.

Then, $L(\text{Eis}^{[\kappa, \tau]}, P\text{-ord}; \varphi \otimes \varphi^b)$ is mapped under the character $\chi \otimes \lambda_{\pi'}$ to

$$\begin{aligned} & c(\pi', \chi) \Omega_{\pi', \chi}(\varphi, \varphi^b) L_p \left(\frac{k-n+1}{2}, P\text{-ord}, \pi', \chi_u \right) \\ & \times L_\infty \left(\frac{k-n+1}{2}; \chi_u, \kappa' \right) I_S \frac{L^S(\frac{k-n+1}{2}, \pi', \chi_u)}{P_{\pi', \chi}}, \end{aligned}$$

where $P_{\pi', \chi} = Q_{\pi', \chi}^{-1}$. Here, $c(\pi', \chi)$, $\Omega_{\pi', \chi}$ and $Q_{\pi', \chi}$ are algebraic numbers related to periods and congruence ideals associated to π discussed in Section 12.2. Furthermore, L_p , L_∞ , and L^S are various Euler factors associated to standard L -functions discussed in Section 10. An explicit formula for L_p , one of the main accomplishment of this paper, is given in Theorems 10.6–10.7. On the other hand, I_S is a constant volume factor fixed to “simplify” the theory at ramified places.

Additional comments. As mentioned above, the author plans to establish the necessary results of P -ordinary theory on unitary groups in a subsequent paper. Similar results have been obtained and used when working with symplectic groups, for instance see [Pil12] and [LR20]. However, in both cases, their work only considers

a version of P -ordinary representations where all types involved are 1-dimensional. More precisely, the pro- p -Iwahori subgroups considered are larger than the one involved in this paper. Nonetheless, the geometry of the Igusa tower (and the relevant vector bundles) is relatively unaffected by the dimension of the types involved. Therefore, the author plans to adapt the proofs of [Pil12] to unitary groups to prove the conjectures discussed in Section 5.

Acknowledgments. I thank my advisor Michael Harris who suggested that I look at the work of [EHLS20] and adapt it to the P -ordinary setting for my doctoral thesis. His countless insights and comments greatly helped me to obtain the results of this article. His encouragements and endless support helped me tremendously to complete this project. I also thank Ellen Eischen, Zheng Liu, Christopher Skinner and Eric Urban for many helpful conversations about their work and the various complexities related to Hida theory and p -adic L -functions.

Part I. P -(anti)-ordinary theory on unitary groups.

1. NOTATION AND CONVENTIONS.

Let $\overline{\mathbb{Q}} \subset \mathbb{C}$ be the algebraic closure of \mathbb{Q} in \mathbb{C} . For any number field $F \subset \overline{\mathbb{Q}}$, let Σ_F denote its set of complex embeddings $\text{Hom}(F, \mathbb{C}) = \text{Hom}(F, \overline{\mathbb{Q}})$.

Throughout this article, we fix a CM field $\mathcal{K} \subset \overline{\mathbb{Q}}$ with ring of integers $\mathcal{O} = \mathcal{O}_{\mathcal{K}}$. Let \mathcal{K}^+ be the maximal real subfield of \mathcal{K} and denote its ring of integers as $\mathcal{O}^+ = \mathcal{O}_{\mathcal{K}^+}$. Let $c \in \text{Gal}(\mathcal{K}/\mathcal{K}^+)$ denote complex conjugation, the unique nontrivial automorphism. Given a place w of \mathcal{K} , we usually denote $c(w)$ as \bar{w} .

Given a representation ρ of some group G , we always denote its contragredient representation by ρ^\vee . We write $\langle \cdot, \cdot \rangle_\rho$ for the tautological pairing between ρ and ρ^\vee .

We denote the kernel $\mathbb{Z} \cdot (2\pi i) \subset \mathbb{C}$ of the exponential map $\exp : \mathbb{C} \rightarrow \mathbb{C}^\times$ by $\mathbb{Z}(1)$. Given any commutative ring R , we set $R(1) := R \otimes \mathbb{Z}(1)$.

Given a map $R \rightarrow R'$ of commutative rings and an R -module M , we write $M_{R'}$ for the base change $M \otimes_R R'$ of M to R' .

Let M be an R -module endowed with an action of some group G . For any representation τ of G , we denote the τ -isotypic component of M by $M[\tau]$. We say that τ occurs in M if $M[\tau] \neq 0$.

1.1. CM types and local places. Fix an integer prime p that is unramified in \mathcal{K} . Throughout this paper, we assume the following :

HYPOTHESIS 1.1. Each place v^+ of \mathcal{K}^+ above p totally splits as $v^+ = w\bar{w}$ in \mathcal{K} , for some place w of \mathcal{K} .

Fix an algebraic closure $\overline{\mathbb{Q}}_p$ of \mathbb{Q}_p and an embedding $\text{incl}_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Define

$$\overline{\mathbb{Z}}_{(p)} = \{z \in \overline{\mathbb{Q}} : \nu_p(\text{incl}_p(z)) \geq 0\} ,$$

where ν_p is the canonical extension to $\overline{\mathbb{Q}}_p$ of the normalized p -adic valuation on \mathbb{Q}_p .

Let \mathbb{C}_p be the completion of $\overline{\mathbb{Q}}_p$. The map incl_p yields an isomorphism between its valuation ring $\mathcal{O}_{\mathbb{C}_p}$ and the completion of $\overline{\mathbb{Z}}_{(p)}$ which extends to an isomorphism $\iota : \mathbb{C} \xrightarrow{\sim} \mathbb{C}_p$. In particular, \mathbb{C} is viewed as an algebra over \mathbb{Z}_p (or even $\overline{\mathbb{Z}}_{(p)}$) via ι .

Fix an embedding $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ such that $\text{incl}_p = \iota \circ \iota_\infty$. Identify $\overline{\mathbb{Q}}$ with its images $\iota_\infty(\overline{\mathbb{Q}}) \subset \mathbb{C}$ and $\text{incl}_p(\overline{\mathbb{Q}}) \subset \mathbb{C}_p$.

Given $\sigma \in \Sigma_{\mathcal{K}}$, the embedding $\text{incl}_p \circ \sigma$ determines a prime ideal \mathfrak{p}_σ of $\Sigma_{\mathcal{K}}$. There may be several embeddings inducing the same prime ideal. Similarly, given a place w of \mathcal{K} , let \mathfrak{p}_w denote the corresponding prime ideal of \mathcal{O} .

Under Hypotesis 1.1, for each place v^+ of \mathcal{K}^+ above p , there are exactly two primes of \mathcal{O} above v^+ . Fix a set Σ_p containing exactly one of these prime ideals for each place $v^+ \mid p$. Moreover, let

$$(1) \quad \Sigma = \{ \sigma \in \Sigma_{\mathcal{K}} \mid \mathfrak{p}_\sigma \in \Sigma_p \},$$

a CM type of \mathcal{K} , see [Kat78, p.202].

1.2. Local theory of types for smooth representations. Let F be a non-archimedean local field. Denote its ring of integers by \mathcal{O}_F , and set $G = \text{GL}_n(F)$ and $\mathcal{G} = \text{GL}_n(\mathcal{O}_F)$.

1.2.1. Parabolic inductions. For any parabolic subgroup P of G , let P^u be its unipotent radical and $L = P/P^u$, its Levi factor. Let $\delta_P : P \rightarrow \mathbb{C}^\times$ denote its modulus character.

Recall that δ_P factors through L . Moreover, if P is the standard parabolic subgroup associated to the partition $n = n_1 + \dots + n_s$, one has

$$(2) \quad \delta_P(l) = \prod_{k=1, \dots, s} |\det(l_k)|^{-\sum_{i < k} n_i + \sum_{j > k} n_j}$$

for any $l = (l_1, \dots, l_s)$ in $L = \prod_{k=1}^s \text{GL}_{n_k}(F)$.

Given a smooth representation σ of L , we often consider σ as a representation of P without comments. Let $\text{Ind}_P^G \sigma$ denote the classical parabolic induction functor from P to G . Similarly, we let

$$\iota_P^G \sigma = \text{Ind}_P^G(\sigma \otimes \delta_P^{1/2})$$

denote the *normalized* parabolic induction functor.

In our work (especially Sections 6 and 7.2), we prefer to work with the normalized version but the main calculations of Section 10.1 can entirely be done with unnormalized parabolic induction as well.

1.2.2. Supercuspidal support. A theorem of Jacquet (see [Cas95, Theorem 5.1.2]) implies that given any irreducible representation π of G , one may find a parabolic subgroup P of G with Levi subgroup L and a supercuspidal representation σ of L such that $\pi \subset \iota_P^G \sigma$.

The pair (L, σ) is uniquely determined by π , up to G -conjugacy and one refers to this conjugacy class as the *supercuspidal support* of π .

Consider two pairs (L, σ) and (L', σ') consisting of a Levi subgroup of G and one of its supercuspidal representation. One says that they are G -inertially equivalent if there exists some $g \in G$ such that $L' = g^{-1}Lg$ and some unramified character ψ of L' such that ${}^g\sigma \cong \sigma' \otimes \psi$, where ${}^g\sigma(x) = \sigma(gxg^{-1})$. We write $[L, \sigma]_G$ for the G -inertial equivalence class of (L, σ) .

For such an equivalence class \mathfrak{s} , let $\text{Rep}^{\mathfrak{s}}(G)$ denote the full subcategory of $\text{Rep}(G)$ whose objects are the representations such that all their irreducible subquotients have inertial equivalence class \mathfrak{s} . The Bernstein-Zelevinsky geometric lemma, see [Ren10, subsection VI.5.1], implies that $\iota_P^G \sigma \in \text{Rep}^{\mathfrak{s}}(G)$, where $\mathfrak{s} = [L, \sigma]_G$.

Definition 1.2 ([BK98]). Let J be a compact open subgroup of G and τ be an irreducible representation of J . Let $\text{Rep}_{\tau}(G)$ denote the full subcategory of $\text{Rep}(G)$ whose objects are the representations generated over G by their τ -isotypic subspace. We say that (J, τ) is an \mathfrak{s} -type if $\text{Rep}_{\tau}(G) = \text{Rep}^{\mathfrak{s}}(G)$.

The work of Bushnell-Kutzko in [BK99] constructs a type for every supercuspidal support. In fact, [BK98] and [BK99] establish the core theory of using *types* to study the category of smooth complex representations of G . However, the fact that the compact group J acting on a given type need not be maximal is inconvenient for the calculus in Section 10.1. Therefore, we prefer to work with Schneider-Zink types, which are refinements of Bushnell-Kutzko types, see [SZ99].

1.2.3. *Schneider-Zink types.* Using the local Langlands correspondence, the types introduced by Schneider and Zink refines the ones of Bushnell-Kutzko by also studying the monodromy and the associated Weil-Deligne representations of a given smooth representation of G .

Although we do not need the full depth of this point of view for our purposes, we use [BC09, Theorem 6.5.3] which imply that for each admissible irreducible representation σ of G , there exists a smooth irreducible representation τ of \mathcal{G} such that τ has multiplicity one in $\sigma|_{\mathcal{G}}$. The other properties of τ provided by [BC09, Theorem 6.5.3] (see also [HLLM23, Theorem 2.5.4]) play no role in our work and we omit them.

Remark 1.3. We later use types (or more precisely, their inertial equivalence class) to construct “branches P -ordinary Hida families” associated to some particular automorphic representations, see Definition 8.20. As mentioned above, we strictly use their multiplicity one property. However, it could be interesting to see how the additional properties of these types can be used to study these Hida families.

Remark 1.4. These Schneider-Zink types are essentially constructed by studying irreducible components of $\text{Ind}_J^G \tau'$, where (J, τ') is some Bushnell-Kutzko type for the supercuspidal support of σ .

Note that the above does not mention anything about the uniqueness of such a representation τ of \mathcal{G} . Therefore, for later purposes, we fix a choice of such a

representation $\tau = \tau_\sigma$ for each σ and refer to it as our *fixed choice of Schneider-Zink type* for σ . We also say that τ is the (chosen) *SZ-type* of σ .

Remark 1.5. We choose them compatibly so that for given an unramified character ψ of G , the SZ-types of σ and $\sigma \otimes \psi$ satisfy $\tau_{\sigma \otimes \psi} = \tau_\sigma \otimes \psi$. We also choose them so that $\tau_{\sigma^\vee} = (\tau_\sigma)^\vee$.

2. MODULAR FORMS ON UNITARY GROUPS WITH P -IWAHORIC LEVEL AT p .

Let V be an n -dimensional \mathcal{K} -vector space, equipped with a non-degenerate Hermitian pairing $\langle \cdot, \cdot \rangle_V$ with respect to the quadratic imaginary extension $\mathcal{K}/\mathcal{K}^+$ fixed in the previous section.

2.1. Unitary PEL datum. Let $\delta \in \mathcal{O}$ be totally imaginary and prime to p . Define $\langle \cdot, \cdot \rangle = \text{tr}_{\mathcal{K}/\mathbb{Q}}(\delta \langle \cdot, \cdot \rangle_V)$. This choice of δ and our Hypothesis 1.1 ensure the existence of an \mathcal{O} -lattice $L \subset V$ such that the restriction of $\langle \cdot, \cdot \rangle$ to L is integral and yields a perfect pairing on $L \otimes \mathbb{Z}_p$.

For each $\sigma \in \Sigma_{\mathcal{K}}$, let V_σ denote $V \otimes_{\mathcal{K}, \sigma} \mathbb{C}$. Fix a \mathbb{C} -basis diagonalizing the pairing $\langle \cdot, \cdot \rangle$. We assume that the basis is chosen so that the corresponding diagonal matrix is $\text{diag}(1, \dots, 1, -1, \dots, -1)$ with a_σ entries equal to 1 and $b_\sigma = n - a_\sigma$ entries equal to -1 . Fixing such a basis, let $h_\sigma : \mathbb{C} \rightarrow \text{End}_{\mathbb{R}}(V_\sigma)$ be $h_\sigma = \text{diag}(z 1_{a_\sigma}, \bar{z} 1_{b_\sigma})$.

Let $h = \prod_{\sigma \in \Sigma} h_\sigma : \mathbb{C} \rightarrow \text{End}_{\mathcal{K}^+ \otimes \mathbb{R}}(V \otimes \mathbb{R})$, using the canonical identification

$$\prod_{\sigma \in \Sigma} \text{End}_{\mathbb{R}}(V_\sigma) = \text{End}_{\mathcal{K}^+ \otimes \mathbb{R}}(V \otimes \mathbb{R})$$

provided by our fixed choice of CM type Σ of \mathcal{K} . The *signature* of h is defined as the collection of pairs $\{(a_\sigma, b_\sigma)\}_{\sigma \in \Sigma_{\mathcal{K}}}$.

The signature of h is naturally related to the pure Hodge structure of weight -1 on $V_{\mathbb{C}} = L \otimes \mathbb{C}$ determined by h . Namely, we have $V_{\mathbb{C}} = V^{-1,0} \oplus V^{0,-1}$ where $h(z)$ acts as z on $V^{-1,0}$ and as \bar{z} on $V^{0,-1}$. By definition, the complex dimension of $V^{-1,0} \otimes_{\mathcal{O} \otimes \mathbb{C}, \sigma} \mathbb{C}$ is equal to a_σ if $\sigma \in \Sigma$ and b_σ if $\sigma \in \Sigma_{\mathcal{K}} \setminus \Sigma$.

Throughout this paper, we assume the following two hypothesis :

HYPOTHESIS 2.1 (Standard hypothesis). We assume that h is standard, as defined in [Ehls20, Section 2.3.2]. Namely, there is a \mathcal{K} -basis of V that simultaneously diagonalizes the matrix associated to $\langle \cdot, \cdot \rangle_V$ as well as the image of h_σ (with respect to the induced basis of $V \otimes_{\mathcal{K}, \sigma} \mathbb{C}$), for each $\sigma \in \Sigma$.

HYPOTHESIS 2.2 (Ordinary hypothesis). For all embeddings $\sigma, \sigma' \in \Sigma_{\mathcal{K}}$, if $\mathfrak{p}_\sigma = \mathfrak{p}_{\sigma'}$, then $a_\sigma = a_{\sigma'}$.

Using the second hypothesis, given a place w of \mathcal{K} above p , we can define $(a_w, b_w) := (a_\sigma, b_\sigma)$, where $\sigma \in \Sigma_{\mathcal{K}}$ is any embedding such that $\mathfrak{p}_\sigma = \mathfrak{p}_w$.

The tuple

$$\mathcal{P} = (\mathcal{K}, c, \mathcal{O}, L, 2\pi\sqrt{-1}\langle \cdot, \cdot \rangle, h)$$

is a PEL datum of unitary type, as defined in [Ehls20, Section 2.1-2.2]. One can associate a group scheme $G = G_{\mathcal{P}}$ over \mathbb{Z} to \mathcal{P} whose R -points are

$$(3) \quad G(R) = \{(g, \nu) \in \mathrm{GL}_{\mathcal{O} \otimes R}(L \otimes R) \times R^\times \mid \langle gx, gy \rangle = \nu \langle x, y \rangle, \forall x, y \in L \otimes R\},$$

for any commutative ring R . We define the signature of G as the signature of the underlying homomorphism h .

Note that G/\mathbb{Q} is a reductive group. Moreover, our assumptions on p imply that G/\mathbb{Z}_p is smooth and $G(\mathbb{Z}_p)$ is a hyperspecial maximal compact of $G(\mathbb{Q}_p)$.

Remark 2.3. Here and in what follows, we only introduce the relevant theory for a PEL datum \mathcal{P} as above associated to a single Hermitian vector space. In later sections, we also need to consider more general PEL data (and the associated objects) obtained from a pair of Hermitian vector spaces, see \mathcal{P}_3 in Section 4.1. The necessary modifications to construct the relevant objects for such PEL data are obvious, hence we do not address them explicitly to lighten our notation. See [Ehls20, Section 2] for precise details on the theory of unitary PEL data associated to any (finite) number of Hermitian vectors spaces over \mathcal{K} .

2.1.1. *Unitary moduli spaces.* Let $F = F_{\mathcal{P}}$ be the reflex field of the PEL datum \mathcal{P} introduced above, as defined in [Lan13, 1.2.5.4]. Let \mathcal{O}_F be its ring of integers and let $S_p = \mathcal{O}_F \otimes \mathbb{Z}_{(p)}$.

Remark 2.4. In later sections, we work with models of Shimura varieties that are integral away-from- p . The existence of such integral models over S_p is obtained by restricting our attention to level subgroups at p satisfying certain conditions, see Sections 2.4 and 2.5.

One may remove these conditions and consider more general level subgroups by working over F instead. For more details about the differences (and similarities) between working over S_p or F , see [Ehls20, Section 2]. We prefer (and need) to work with models over S_p as we only work with with level subgroups at p satisfying the conditions briefly mentioned above.

Let $K^p \subset G(\mathbb{A}_f^p)$ be any open compact subgroup and set $K = G(\mathbb{Z}_p)K^p$. Define the moduli problem $\mathrm{M}_K = \mathrm{M}_K(\mathcal{P})$ as the functor that assigns, to any locally noetherian S_p -scheme T , the set of equivalence classes of quadruples $\underline{A} = (A, \lambda, \iota, \alpha K^p)$, where

- (i) A is an abelian scheme over T ;
- (ii) $\lambda : A \rightarrow A^\vee$ is a prime-to- p polarization;
- (iii) $\iota : S_p \hookrightarrow \mathrm{End}_T A \otimes \mathbb{Z}_{(p)}$ such that $\iota(b)^\vee \circ \lambda = \lambda^\vee \circ \iota(\bar{b})$;
- (iv) αK^p is a K^p -level structure, in the sense of [Ehls20, Section 2.1]. Namely, α is a rule that assigns, to each connected component T° of T , an isomorphism

$$\alpha_t : L \otimes \mathbb{A}_f^p \xrightarrow{\sim} H^1(A_t, \mathbb{A}_f^p)$$

over $\mathcal{O}_K \otimes \mathbb{A}_f^p$ such that the K^p -orbit αK^p is $\pi_1(T, t)$ -stable (where t is an arbitrary geometric point of T°). Furthermore, it identifies the pairing $\langle \cdot, \cdot \rangle$ with a $\mathbb{A}_f^{p, \times}$ -multiple of the symplectic pairing on $H^1(A_t, \mathbb{A}_f^p)$ induced by the Weil pairing and the polarization λ ;

- (v) $\text{Lie}_T A$ satisfies the Kottwitz determinant condition defined by $(L \otimes R, \langle \cdot, \cdot \rangle, h)$, see [Lan13, Definition 1.3.4.1];

and two quadruples $(A, \lambda, \iota, \alpha)$ and $(A', \lambda', \iota', \alpha')$ are equivalent if there exists some prime-to- p isogeny $f : A \rightarrow A'$ such that

- (i) λ and $f^\vee \circ \lambda' \circ f$ are equal, up to multiplication by some positive element in $\mathbb{Z}_{(p)}^\times$;
- (ii) $\iota'(b) \circ f = f \circ \iota(b)$, for all $b \in \mathcal{O}_K$;
- (iii) $\alpha' K^p = f \circ \alpha K^p$.

When K^p is clear from context, we often denote the orbit αK^p simply by α . Furthermore, for a generalization (over F instead of S_p) of this moduli problem for all open compact subgroups $K \subset G(\mathbb{A}_f)$, see [EHLS20, Section 2.1].

In this article, we always assume that K is *neat*, in the sense of [Lan13, Definition 1.4.1.8.]. Then, [Lan13, Corollary 7.2.3.10] implies that there is a smooth, quasi-projective S_p -scheme that represents this moduli problem M_K . By abuse of notation, we denote this scheme by M_K again. If $K' = G(\mathbb{Z}^p)K'^p \subset K$, there is a natural homomorphism $M_{K'} \rightarrow M_K$ induced by the “forgetful map” $\alpha K'^p \mapsto \alpha K^p$. Similarly, given $g \in G(\mathbb{A}_f^p)$, there is a canonical map $[g] : M_{gKg^{-1}} \rightarrow M_K$ induced by the functor $(A, \lambda, \iota, \alpha) \mapsto (A, \lambda, \iota, \alpha g)$.

2.1.2. Toroidal compactifications. We now briefly recall the existence of toroidal compactifications of the moduli spaces above constructed in [Lan13]. These are associated to *smooth projective polyhedral cone decompositions*, a notion whose exact definition plays no role later in this article. Hence, we do not introduce this notion precisely.

The only properties relevant for this paper are that given such a polyhedral cone decomposition Ω , there exists a smooth toroidal compactification $M_{K, \Omega}^{\text{tor}}$ of M_K over S_p , and that there exists a partial ordering on the set of such Ω 's by *refinements*.

Given two polyhedral cone decompositions Ω and Ω' , if Ω' refines Ω , then there is a canonical proper surjective map $\pi_{\Omega', \Omega} : M_{K, \Omega'}^{\text{tor}} \rightarrow M_{K, \Omega}^{\text{tor}}$ which restricts to the identity on M_K . We denote the tower $\{M_{K, \Omega}^{\text{tor}}\}_\Omega$ by M_K^{tor} .

Remark 2.5. We often refer to the tower as if it were a single scheme and do not emphasize the specific compatible choices of Ω in some constructions. This is essentially justified by the K ocher’s principle in many cases, see Remark 2.17. See [EHLS20, Section 2.4] for more details.

Furthermore, if $K' \subset K$, the map $M_K \rightarrow M_{K'}$ extends canonically to maps $M_{K, \Omega}^{\text{tor}} \rightarrow M_{K', \Omega}^{\text{tor}}$, for each Ω , and hence to a map $M_K^{\text{tor}} \rightarrow M_{K'}^{\text{tor}}$. Similarly, the maps

$[g] : M_{gKg^{-1}} \rightarrow M_K$ also extend canonically to maps $[g] : M_{gKg^{-1}}^{\text{tor}} \rightarrow M_K^{\text{tor}}$, for all $g \in G(\mathbb{A}_f^p)$. Hence, $G(\mathbb{A}_f^p)$ acts on the tower (of towers) $\{M_{G(\mathbb{Z}_p)K^p}^{\text{tor}}\}_{K^p \subset G(\mathbb{A}_f^p)}$.

2.2. Structure of G over \mathbb{Z}_p .

2.2.1. *Comparison to general linear groups.* For each prime $w \mid p$ of \mathcal{K} , denote the localization of \mathcal{K} at w by \mathcal{K}_w and its ring of integers by \mathcal{O}_w .

The factorization $\mathcal{O} \otimes \mathbb{Z}_p = \prod_{w \mid p} \mathcal{O}_w$, over primes $w \mid p$, yields a decomposition $L \otimes \mathbb{Z}_p = \prod_{w \mid p} L_w$. Using Hypothesis 1.1, we fix identifications $\mathcal{K}_w = \mathcal{K}_{\bar{w}}$ and $\mathcal{O}_w = \mathcal{O}_{\bar{w}}$. We consider both L_w and $L_{\bar{w}}$ as \mathcal{O}_w -lattices.

The above factorization of $L \otimes \mathbb{Z}_p$ corresponds to

$$(4) \quad \text{GL}_{\mathcal{O} \otimes \mathbb{Z}_p}(L \otimes \mathbb{Z}_p) \xrightarrow{\sim} \prod_{w \mid p} \text{GL}_{\mathcal{O}_w}(L_w), \quad g \mapsto (g_w)_{w \mid p},$$

a canonical \mathbb{Z}_p -isomorphism. From the above, one obtains the identification

$$(5) \quad G_{/\mathbb{Z}_p} \xrightarrow{\sim} \mathbb{G}_m \times \prod_{w \in \Sigma_p} \text{GL}_{\mathcal{O}_w}(L_w), \quad (g, \nu) \mapsto (\nu, (g_w)_{w \in \Sigma_p}).$$

Furthermore, our assumption above on the pairing $\langle \cdot, \cdot \rangle$ implies that for each $w \mid p$, there is an \mathcal{O}_w -decomposition of $L_w = L_w^+ \oplus L_w^-$ such that

- (i) $\text{rank}_{\mathcal{O}_w} L_w^+ = a_w$ and $\text{rank}_{\mathcal{O}_w} L_w^- = b_w$;
- (ii) Upon restricting $\langle \cdot, \cdot \rangle$ to $L_w \times L_{\bar{w}}$, the annihilator of L_w^\pm is $L_{\bar{w}}^\pm$. Hence, one has a perfect pairing $L_w^+ \oplus L_{\bar{w}}^- \rightarrow \mathbb{Z}_p(1)$, again denoted $\langle \cdot, \cdot \rangle$.

Fix dual \mathcal{O}_w -bases (with respect to the perfect pairing above) for L_w^+ and $L_{\bar{w}}^-$. They yield isomorphisms

$$(6) \quad \text{GL}_{a_w}(\mathcal{O}_w) \xrightarrow{\sim} \text{GL}_{\mathcal{O}_w}(L_w^+) \xrightarrow{\text{dual}} \text{GL}_{\mathcal{O}_w}(L_{\bar{w}}^-) \xrightarrow{\sim} \text{GL}_{b_{\bar{w}}}(\mathcal{O}_w)$$

such that the composition is the adjoint map $A \mapsto A^* = {}^t \bar{A}$ on $\text{GL}_{a_w}(\mathcal{O}_w) = \text{GL}_{b_{\bar{w}}}(\mathcal{O}_w)$. Furthermore, this induces an identification $\text{GL}_{\mathcal{O}_w}(L_w) = \text{GL}_n(\mathcal{O}_w)$ such that the obvious map

$$(7) \quad \text{GL}_{\mathcal{O}_w}(L_w^+) \times \text{GL}_{\mathcal{O}_w}(L_w^-) \hookrightarrow \text{GL}_{\mathcal{O}_w}(L_w)$$

is simply the diagonal embedding of block matrices.

Let $L^\pm = \prod_{w \mid p} L_w^\pm$ and let $H := \text{GL}_{\mathcal{O} \otimes \mathbb{Z}_p}(L^+)$. The identification (6) above induces a canonical isomorphism

$$(8) \quad H \cong \prod_{w \mid p} \text{GL}_{a_w}(\mathcal{O}_w) = \prod_{w \in \Sigma_p} \text{GL}_{a_w}(\mathcal{O}_w) \times \text{GL}_{b_w}(\mathcal{O}_w)$$

Remark 2.6. Here, we view H as an algebraic group over $\mathcal{O} \otimes \mathbb{Z}_p$. Namely, for any algebra S over $\mathcal{O} \otimes \mathbb{Z}_p$, we have $H(S) = \text{GL}_S(L^+ \otimes_{\mathcal{O} \otimes \mathbb{Z}_p} S)$. This technically leads to the confusion in notation since $H(\mathcal{O} \otimes \mathbb{Z}_p)$ is equal to the set $\text{GL}_{\mathcal{O} \otimes \mathbb{Z}_p}(L^+)$

(also denoted H above). However, we keep this convention of denoting an algebraic group by its set of points over its base ring, ignoring this minor abuse in notation.

For instance, the algebraic group denoted $\mathrm{GL}_{a_w}(\mathcal{O}_w)$ above technically stands for $\mathrm{GL}(a_w)_{/\mathcal{O}_w}$. We use such a convention in many instance in what follows without comments. The only exception is for \mathbb{G}_m which we refrain from denoting $\mathrm{GL}_1(\mathcal{O}_w)$ or \mathcal{O}_w^\times .

2.2.2. *Parabolic subgroups of G over \mathbb{Z}_p .* For $w \mid p$, let

$$(9) \quad \mathbf{d}_w = (n_{w,1}, \dots, n_{w,t_w})$$

be a partition of $a_w = b_w$. Let $P_{\mathbf{d}_w} \subset \mathrm{GL}_{a_w}(\mathcal{O}_w)$ denote the standard parabolic subgroup corresponding to \mathbf{d}_w . Define $P_H \subset H$ as the \mathbb{Z}_p -parabolic that corresponds to the products of all the $P_{\mathbf{d}_w}$ via the isomorphism (8). We denote the unipotent radical of P_H by P_H^u and its maximal subtorus by T_H .

We identify the elements of the Levi factor $L_H = P_H/P_H^u$ of P_H with collections of block-diagonal matrices, with respect to the partitions \mathbf{d}_w , via (8). In other words, we embed $L_{\mathbf{d}_w} := \mathrm{GL}_{n_{w,1}}(\mathcal{O}_w) \times \dots \times \mathrm{GL}_{n_{w,t}}(\mathcal{O}_w)$ in $\mathrm{GL}_{a_w}(\mathcal{O}_w)$ diagonally and identify L_H with $\prod_{w \mid p} L_{\mathbf{d}_w}$.

Define $\det_{\mathbf{d}_w} : L_{\mathbf{d}_w} \rightarrow (\mathbb{G}_m)^{t_w}$ as the homomorphism taking determinant of each GL -block of $L_{\mathbf{d}_w}$ individually (in the obvious order). Let $SL_{\mathbf{d}_w} \subset L_{\mathbf{d}_w}$ denote the kernel of $\det_{\mathbf{d}_w}$ and identify $SL_H = \prod_{w \mid p} SL_{\mathbf{d}_w}$ as a subgroup of H via (8). We define SP_H as the product $SL_H \cdot P_H^u$ in P_H and identify P_H/SP_H with $\prod_{w \mid p} (\mathbb{G}_m)^{t_w}$.

Note that the center Z_{L_H} of L_H is also canonically isomorphic to $\prod_{w \mid p} (\mathbb{G}_m)^{t_w}$. The identity map between these two copies of $\prod_{w \mid p} (\mathbb{G}_m)^{t_w}$ yields an identification that sends an element $g = (g_w)_{w \mid p} \in P_H/SP_H$ such that $\det_{\mathbf{d}_w}(g_w) = (g_{w,1}, \dots, g_{w,t_w})$ with

$$(\mathrm{diag}(g_{w,1}, \dots, g_{w,1}; g_{w,2}, \dots, g_{w,2}; \dots; g_{w,t_w}, \dots, g_{w,t_w}))_{w \mid p} \in Z_{L_H},$$

where the entry $g_{w,i}$ appears $n_{w,i}$ -times.

Remark 2.7. We use this identification later to view a character χ of Z_{L_H} as a character of P_H that factors through $\prod_{w \mid p} \det_{\mathbf{d}_w}$.

We can write such a character χ as a product $\prod_{w \mid p} \chi_w$ via the canonical identification $Z_{L_H} = \prod_{w \mid p} (\mathbb{G}_m)^{t_w}$. Then, the corresponding character χ' of $P_H/SP_H = \prod_{w \mid p} (\mathbb{G}_m)^{t_w}$ is

$$\chi' = \prod_{w \mid p} \chi_w \circ \det_{\mathbf{d}_w}.$$

In particular, the reader should keep in mind that the restriction of χ' to Z_{L_H} is *not* χ . Nonetheless, by abuse of notation, we often denote χ' as χ again. We remind the reader of this convention when necessary to avoid confusion.

Let $P^+ \subset G/\mathbb{Z}_p$ be the parabolic subgroup that stabilizes L^+ and such that

$$(10) \quad P^+ \rightarrow \mathbb{G}_m \times P_H \subset \mathbb{G}_m \times H$$

is surjective, where the map to the first factor is the similitude character ν and the map to the second factor is projection to H .

For $w \in \Sigma_p$, let P_w be the parabolic subgroup of $\mathrm{GL}_{\mathcal{O}_w}(L_w)$ given by

$$(11) \quad P_w = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \mathrm{GL}_n(\mathcal{O}_w) \mid A \in P_{\mathbf{d}_w}, D \in P_{\mathbf{d}_w}^{\mathrm{op}} \right\},$$

via the isomorphisms (6) and (7).

We identify $P = \prod_{w \in \Sigma_p} P_w$ as a subgroup of G/\mathbb{Z}_p via (5). Our choices of bases above imply that under the isomorphisms (5) and (6), P^+ corresponds to

$$(12) \quad P^+ \xrightarrow{\sim} \mathbb{G}_m \times P.$$

This induces an isomorphism $L_H \cong L_P := P/P^u$, where P^u is the unipotent radical of P . We again identify L_P as the subgroup of P consisting of collections of block-diagonal matrices (the sizes of the blocks are determined by the partitions \mathbf{d}_w).

Let $SL_P \subset L_P$ be the subgroup corresponding to SL_H via this isomorphism $L_H \cong L_P$ and let $SP = SL_P \cdot P$. Proceeding as above, we obtain a natural identification between the center Z_{L_P} of L_P and the quotient P/SP .

Remark 2.8. The trivial partition of a_w is $(1, \dots, 1)$ (of length $t_w = a_w$). If the partitions fixed above are all trivial, we write B_w , B and B^+ instead of P_w , P and P^+ . In this case, $L_B = B/B^u$ is equal to Z_{L_B} and identified with the maximal torus subgroup of $\prod_{w \in \Sigma_p} \mathrm{GL}_n(\mathcal{O}_w)$

Definition 2.9. We define the P -Iwahori subgroup of G of level $r \geq 0$ as

$$I_r^0 = I_{P,r}^0 := \{g \in G(\mathbb{Z}_p) \mid g \bmod p^r \in P^+(\mathbb{Z}_p/p^r\mathbb{Z}_p)\}$$

and the pro- p P -Iwahori subgroup $I_r = I_{P,r}$ of G of level r as

$$I_r = I_{P,r} := \{g \in G(\mathbb{Z}_p) \mid g \bmod p^r \in (\mathbb{Z}_p/p^r\mathbb{Z}_p)^\times \times P^u(\mathbb{Z}_p/p^r\mathbb{Z}_p)\}.$$

Remark 2.10. We refrain from referring to I_r^0 as a *parahoric* subgroup of G . This terminology is usually reserved for stabilizers of points in Bruhat-Tits building. We make no attempt here to introduce our construction from the point of view of these combinatorial and geometric structures.

The inclusion of $L_P(\mathbb{Z}_p)$ in I_r^0 yields a canonical isomorphism

$$(13) \quad L_P(\mathbb{Z}_p/p^r\mathbb{Z}_p) \xrightarrow{\sim} I_r^0/I_r.$$

For each $w \in \Sigma_p$, one similarly defines I_w^0 and $I_{w,r}$ by replacing P^+ by P_w and working in $\mathrm{GL}_n(\mathcal{O}_w)$ instead of $G(\mathbb{Z}_p)$. Let

$$(14) \quad I_r^{\mathrm{GL}} = \prod_{w \in \Sigma_p} I_{w,r} \quad \text{and} \quad I_r^{0,\mathrm{GL}} = \prod_{w \in \Sigma_p} I_{w,r}^0,$$

so that I_r and I_r^0 correspond to $\mathbb{Z}_p^\times \times I_{P,r}^{\mathrm{GL}}$ and $\mathbb{Z}_p^\times \times I_{P,r}^{0,\mathrm{GL}}$ respectively, via the isomorphisms (5) and (6).

In subsequent sections, we study certain P -ordinary Hecke operators at p associated to the parabolic subgroups introduced above. Therefore, for later purposes, let us define the following matrices :

Given $w \in \Sigma_p$ and $1 \leq j \leq n$, let $t_{w,j} \in \mathrm{GL}_n(\mathcal{O}_w)$ denote the diagonal matrix

$$(15) \quad t_{w,j} = \begin{cases} \mathrm{diag}(p1_j, 1_{n-j}), & \text{if } j \leq a_w \\ \mathrm{diag}(p1_{a_w}, 1_{n-j}, p1_{j-a_w}), & \text{if } j > a_w \end{cases}$$

It corresponds to an element of $G(\mathbb{Q}_p)$ under (5) and (7), which we denote $t_{w,j}^+$ (namely, all its other components are equal to 1). We set $t_{w,j}^- = (t_{w,j}^+)^{-1}$.

Furthermore, let $r_w = t_w + t_w^-$ and consider

$$\tilde{\mathbf{d}}_w = \left(\tilde{\mathbf{d}}_{w,1}, \dots, \tilde{\mathbf{d}}_{w,t_w}; \tilde{\mathbf{d}}_{w,t_w+1}, \dots, \tilde{\mathbf{d}}_{w,r_w} \right) := (n_{w,1}, \dots, n_{w,t_w}; n_{\bar{w},t_w}, \dots, n_{\bar{w},1}),$$

a partition of $n = a_w + b_w$. For $j = 1, \dots, r_w$, let $D_w(j)$ be the partial sum $\sum_{i=1}^j \tilde{\mathbf{d}}_{w,i}$. We define

$$(16) \quad t_{P,p}^\pm = \prod_{w \in \Sigma_p} \prod_{j=1}^{r_w} t_{w,D_w(j)}^\pm$$

By construction, $t_{P,p}^\pm$ lies in the center $Z_{LP}(\mathbb{Q}_p)$ of $LP(\mathbb{Q}_p)$.

Remark 2.11. The reader should not confuse t_w and $t_{w,i}$ (or $t_{w,D_w(j)}$). The former is only ever used to denote an integer while the latter denotes an $n \times n$ -matrix over \mathcal{O}_w .

2.3. Structure of G over \mathbb{C} . Consider the pure Hodge decomposition $V_{\mathbb{C}} = L \otimes \mathbb{C} = V^{-1,0} \oplus V^{0,-1}$ of weight -1 , as in Section 2.1, for the \mathcal{O} -lattice L associated to \mathcal{P} . By definition of the reflex field of \mathcal{P} , the graded piece $W = V/V^{0,-1}$ of the corresponding Hodge filtration is defined over F .

Fix an S_p -submodule Λ_0 of W that is stable under the \mathcal{O} -action and such that $\Lambda_0 \otimes_{S_p} \mathbb{C} = W$. The module $\Lambda_0^\vee = \mathrm{Hom}_{\mathbb{Z}_{(p)}}(\Lambda_0, \mathbb{Z}_{(p)}(1))$ has a natural $\mathcal{O} \otimes S_p$ -action via

$$(b \otimes s)f(x) = f(\bar{b}sx),$$

for all $b \in \mathcal{O}$ and $s \in S_p$.

Define $\Lambda = \Lambda_0 \oplus \Lambda_0^\vee$ and

$$\begin{aligned} \langle \cdot, \cdot \rangle_{can} : \Lambda \times \Lambda &\rightarrow \mathbb{Z}_{(p)}(1) \\ \langle (f_1, x_1), (f_2, x_2) \rangle_{can} &= f_2(x_1) - f_1(x_2) \end{aligned}$$

so that both Λ_0 and Λ_0^\vee are isotropic submodules of Λ . One has $\langle bx, y \rangle_{can} = \langle x, \bar{b}y \rangle_{can}$, for $b \in \mathcal{O}$.

The pair $(\Lambda, \langle \cdot, \cdot \rangle_{can})$ induces an S_p -group scheme G_0 whose R -points are given by

$$G_0(R) = \{ (g, \nu) \in \mathrm{GL}_R(\Lambda \otimes_{S_\square} R) \times R^\times \mid \langle gx, gy \rangle_{can} = \nu \langle x, y \rangle_{can}, x, y \in \Lambda \otimes R \} ,$$

for any S_p -algebra R . Let $P_0 \subset G_0$ denote the parabolic subgroup that stabilizes Λ_0 .

One readily checks that there is an isomorphism $V \cong \Lambda \otimes_{S_p} \mathbb{C}$ of \mathbb{C} -vector spaces that identifies $V^{-1,0}$ (resp. $V^{0,-1}$) with $\Lambda_0 \otimes_{S_p} \mathbb{C}$ (resp. $\Lambda_0^\vee \otimes_{S_p} \mathbb{C}$) and the pairing $\langle \cdot, \cdot \rangle$ with $\langle \cdot, \cdot \rangle_{can}$. In other words, it yields an identification between G/\mathbb{C} and G_0/\mathbb{C} . Clearly, it identifies $P_0(\mathbb{C})$ with $P_h(\mathbb{C})$, where P_h is the stabilizer of the Hodge filtration on $L \otimes \mathbb{R}$.

Remark 2.12. The advantage to introduce Λ_0 is that it is well-defined over S_p , as opposed to $V^{-1,0}$. This is necessary to later view classical algebraic weights p -adically, see Section 2.3.3.

Let $H_0 \subset G_0$ be the stabilizer of the polarization $\Lambda = \Lambda_0 \oplus \Lambda_0^\vee$. The natural projection

$$(17) \quad H_0 \rightarrow \mathbb{G}_m \times \mathrm{GL}_{S_p}(\Lambda_0^\vee)$$

is an isomorphism, and the isomorphism between G/\mathbb{C} and G_0/\mathbb{C} above identifies $H_0(\mathbb{C})$ with $C(\mathbb{C})$, where C/\mathbb{R} is the centralizer of h under the conjugation action of G/\mathbb{R} . We recall the classification of the algebraic representations of H_0 in the next section to later describe cohomological weights of automorphic representations.

2.3.1. Algebraic weights. Let \mathcal{K}' be the Galois closure of \mathcal{K} and $\mathfrak{p}' \subset \mathcal{O}_{\mathcal{K}'}$ be the prime above p determined by incl_p . From [Lan13, Corollary 1.2.5.6], \mathcal{K}' contains F . Therefore, we can view $S_0 := \mathcal{O}_{\mathcal{K}',(\mathfrak{p}')}$ as an algebra over $S_p = \mathcal{O}_{F,(p)}$.

By definition of \mathcal{K}' , we have $\mathcal{O} \otimes S_0 = \prod_{\sigma \in \Sigma_{\mathcal{K}}} S_{0,\sigma}$. This naturally induces decompositions $\Lambda_0 \otimes S_0 = \prod_{\sigma \in \Sigma_{\mathcal{K}}} \Lambda_{0,\sigma}$ and $\Lambda_0^\vee \otimes S_0 = \prod_{\sigma \in \Sigma_{\mathcal{K}}} \Lambda_{0,\sigma}^\vee$. Moreover, the identification (17) yields an isomorphism

$$(18) \quad H_0/S_0 \xrightarrow{\sim} \mathbb{G}_m \times \prod_{\sigma \in \Sigma_{\mathcal{K}}} \mathrm{GL}_{\mathcal{O} \otimes_{\mathcal{O},\sigma} S_0}(\Lambda_{0,\sigma}^\vee).$$

Since S_0 is a PID, one readily sees that $\Lambda_{0,\sigma}$ (resp. $\Lambda_{0,\sigma}^\vee$) is a free S_0 -module of rank a_σ (resp. b_σ). Furthermore, for each $\sigma \in \Sigma_{\mathcal{K}}$, the pairing $\langle \cdot, \cdot \rangle_{can}$ identifies

$\Lambda_{0,\sigma c}^\vee$ with $\mathrm{Hom}_{\mathbb{Z}(p)}(\Lambda_{0,\sigma}, \mathbb{Z}(p)(1))$. Fix dual bases for $\Lambda_{0,\sigma}$ and $\Lambda_{0,\sigma c}^\vee$, so that (18) induces an identification

$$(19) \quad H_{0/S_0} \xrightarrow{\sim} \mathbb{G}_m \times \prod_{\sigma \in \Sigma_{\mathcal{K}}} \mathrm{GL}_{b_\sigma}(S_0).$$

Let $B_{H_0} \subset H_{0/S_0}$ be the Borel subgroup that corresponds to the product of the lower-triangular Borel subgroups via the isomorphism (19). Let $T_{H_0} \subset B_{H_0}$ denote its maximal subtorus and let $B_{H_0}^u$ denote its unipotent radical subgroup.

Given an S_0 -algebra R , a character κ of T_{H_0} over R is identified via the isomorphism (19) with a tuple

$$\kappa = (\kappa_0, (\kappa_\sigma)_{\sigma \in \Sigma_{\mathcal{K}}}),$$

where $\kappa_0 \in \mathbb{Z}$ and $\kappa_\sigma = (\kappa_{\sigma,j}) \in \mathbb{Z}^{b_\sigma}$. Namely, for

$$t = (t_0, (\mathrm{diag}(t_{\sigma,i,1}, \dots, t_{\sigma,i,b_{\sigma,i}}))_{\sigma \in \Sigma_{\mathcal{K}}}) \in T_{H_0},$$

one has

$$(20) \quad \kappa(t) = t_0^{\kappa_0} \prod_{\sigma \in \Sigma_{\mathcal{K}}} \prod_{j=1}^{b_\sigma} t_{\sigma,j}^{\kappa_{\sigma,j}}.$$

We refer to κ as a *weight*. We say that κ is *dominant* if $\kappa_{\sigma,j-1} \geq \kappa_{\sigma,j}$ for all $\sigma \in \Sigma_{\mathcal{K}}$, $1 < j \leq b_\sigma$, or equivalently if it is dominant with respect to the opposite Borel $B_{H_0}^{\mathrm{op}}$ (of upper-triangular matrices).

We say that κ is *regular* if $\kappa_{\sigma,j-1} > \kappa_{\sigma,j}$ for all $\sigma \in \Sigma_{\mathcal{K}}$, $1 < j \leq b_\sigma$. Furthermore, we say that κ is *very regular* if κ is regular and $\kappa_{\sigma,b_\sigma} \gg 0$ for each σ .

Remark 2.13. Note that we do not include an explicit lower bound in the definition of *very regular* weights above. This is because we only use this notion in conjectures, see Conjecture 5.5. The author plans to study this notion in more details in the future.

Given a dominant character κ of T_{H_0} over an S_0 -algebra R , extend it trivially to B_{H_0} . Define

$$W_\kappa = W_\kappa(R) = \mathrm{Ind}_{B_{H_0}}^{H_0} \kappa = \{\phi : H_{0/R} \rightarrow \mathbb{G}_a \mid \phi(bh) = \kappa(b)\phi(h), \forall b \in B_{H_0}\}.$$

with its natural structure as a left H_0 -module via multiplication on the right.

As explained in [Jan03, Part II. Chapter 2] and [Hid04, Section 8.1.2], if R is flat over S_0 , this is an R -model for the highest weight representation of H_0 with respect to $(T_{H_0}, B_{H_0}^{\mathrm{op}})$ of weight κ .

2.3.2. P -parallel weights. Given $\sigma \in \Sigma_{\mathcal{K}}$, let w be the place of \mathcal{K} above p such that $\mathfrak{p}_\sigma = \mathfrak{p}_w$. In this section, we write \mathbf{d}_σ for the partition $\mathbf{d}_w = (n_{w,1}, \dots, n_{w,t_w})$ of $a_\sigma = a_w$ introduced in Section 2.2.2, t_σ for t_w and $n_{\sigma,j}$ for $n_{w,j}$.

We denote the standard lower-triangular parabolic subgroup of $\mathrm{GL}_{b_\sigma}(S_0)$ corresponding to $\mathbf{d}_{\sigma c}$ by $P_{0,\mathbf{d}_{\sigma c}}$. Define $P_{H_0} \subset H_0$ as the S_0 -parabolic subgroup corresponding to the product $\prod_{\sigma \in \Sigma_{\mathcal{K}}} P_{0,\mathbf{d}_{\sigma c}}$ via (19). We denote its unipotent radical by $P_{H_0}^u$ and its Levi factor by L_{H_0} .

We identify L_{H_0} with

$$(21) \quad \mathbb{G}_m \times \prod_{\sigma \in \Sigma_{\mathcal{K}}} \prod_{i=1}^{t_\sigma} \mathrm{GL}_{S_0}(n_{w,i})$$

via (18) and the obvious block-diagonal embeddings (for each $\sigma \in \Sigma_{\mathcal{K}}$). Let $SL_{H_0} \subset L_{H_0}$ be the kernel the *block-by-block* determinant map (analogous to the definition of $SL_H \subset L_H$ in Section 2.2.2).

We say that a weight $\kappa = (\kappa_0, (\kappa_\sigma)_{\sigma \in \Sigma_{\mathcal{K}}})$ of T_{H_0} is *P-parallel* if κ extends to a character of L_{H_0} that factors through L_{H_0}/SL_{H_0} . Using the conventions set in Remark 2.8, we see that every weight is *B-parallel*.

For $k = 1, \dots, t_{\sigma c}$, let $N_{\sigma,k}$ denote the partial sum $\sum_{j=1}^k n_{\sigma c,j}$ and define $N_{\sigma,0} = 0$. By identifying each κ_σ with a tuple in \mathbb{Z}^{b_σ} as above, κ is *P-parallel* if and only if

$$(22) \quad \kappa_{\sigma,1+N_{\sigma,k}} = \kappa_{\sigma,2+N_{\sigma,k}} = \dots = \kappa_{\sigma,N_{\sigma,k+1}},$$

for all $\sigma \in \Sigma_{\mathcal{K}}$ and $0 \leq k < t_{\sigma c}$.

The tuple $\kappa_{Z_0} = (\kappa_0, (\kappa_{N_{\sigma,1}}, \kappa_{N_{\sigma,2}}, \dots, \kappa_{N_{\sigma,t_{\sigma c}}})_{\sigma \in \Sigma_{\mathcal{K}}})$ naturally corresponds to a character of the center Z_0 of L_{H_0} . However, note that κ_{Z_0} is not the restriction of κ from T_{H_0} to Z_0 (see Remark 2.7).

For later purposes, let $B(L_{H_0})$ denote the S_0 -group given by the intersection of $B_{H_0} \cap L_{H_0}$. Equivalently, $B(L_{H_0})$ is the Borel of L_{H_0} corresponding to the product (over $\sigma \in \Sigma_{\mathcal{K}}$, $1 \leq i \leq t_\sigma$) of standard lower-triangular Borel subgroups via (21).

Let ρ_κ denote the L_{H_0} -representation $\mathrm{Ind}_{B(L_{H_0})}^{L_{H_0}} \kappa$ and write V_κ for the associated algebraic vector space. In particular, we have $W_\kappa = \mathrm{Ind}_{L_{H_0}}^{H_0} \rho_\kappa$.

Now, let β be some *P-parallel* weight of T_{H_0} and denote its extension to a character of L_{H_0} by β again. Note that $\rho_{\kappa+\beta}$ is canonically isomorphic to $\rho_\kappa \otimes \beta$.

Therefore, we view V_κ as the vector space associated to the representation $\rho_{\kappa'}$ for every algebraic weight κ' in the “*P-parallel* lattice”

$$(23) \quad [\kappa] := \{\kappa + \theta \mid \theta \text{ is } P\text{-parallel}\}$$

of algebraic weights containing κ . We sometimes write V_κ as $V_{[\kappa]}$ to emphasize this fact.

2.3.3. *p*-adic weights. Let \mathcal{O}' be the ring of integers of the smallest field $\mathcal{L}' \subset \overline{\mathbb{Q}}_p$ containing the image of all embeddings $\mathcal{K} \hookrightarrow \overline{\mathbb{Q}}_p$. In particular, \mathcal{L}' contains $\mathrm{incl}_p(\mathcal{K}')$, hence incl_p identifies \mathcal{O}' as an S_0 -algebra (and as an S_p -algebra).

Consider the factorization $\mathcal{O}_{(p)} = \prod_{w|p} \mathcal{O}_w$. Then, we have

$$\mathcal{O}_{(p)} \otimes \mathcal{O}' = \prod_{w|p} \mathcal{O}_w \otimes \mathcal{O}' \xrightarrow{\sim} \prod_{w|p} \prod_{\substack{\sigma \in \Sigma_{\mathcal{K}} \\ \mathfrak{p}_\sigma = \mathfrak{p}_w}} \mathcal{O}' = \prod_{\sigma \in \Sigma_{\mathcal{K}}} \mathcal{O}' ,$$

by definition of \mathcal{O}' . This identification, together with the choice of basis for L^+ in Section 2.2.1, yields a decomposition

$$L^+ \otimes \mathcal{O}' = \prod_{w|p} L_w \otimes \mathcal{O}' = \prod_{\sigma \in \Sigma_{\mathcal{K}}} (\mathcal{O}')^{a_w} .$$

Similarly, the choice of basis for Λ_0 in Section 2.3.1 induces

$$\Lambda_0 \otimes_{S_p} \mathcal{O}' = \prod_{\sigma \in \Sigma_{\mathcal{K}}} \Lambda_{0,\sigma} \otimes_{S_0} \mathcal{O}' = \prod_{\sigma \in \Sigma_{\mathcal{K}}} (\mathcal{O}')^{a_\sigma} .$$

From the above, we obtain an identification $L^+ \otimes \mathcal{O}' = \Lambda_0 \otimes_{S_p} \mathcal{O}'$ over $\mathcal{O} \otimes \mathcal{O}' = \mathcal{O}_{(p)} \otimes \mathcal{O}'$. Therefore, using the duality between Λ_0 and Λ_0^\vee , we have an isomorphism $H_{0/\mathcal{O}'} \xrightarrow{\sim} \mathbb{G}_m \times H_{/\mathcal{O}'}$ given by

$$(24) \quad (\nu, (g_\sigma)_{\sigma \in \Sigma_{\mathcal{K}}}) \mapsto \left(\nu, \left(\prod_{\substack{\sigma \in \Sigma_{\mathcal{K}} \\ \mathfrak{p}_\sigma = \mathfrak{p}_w}} \nu \cdot {}^t g_{\sigma c}^{-1} \right)_{w|p} \right)$$

using the isomorphisms (6), (8), (18) and (19).

In particular, it induces a natural inclusion $L_H(\mathbb{Z}_p) \hookrightarrow L_{H_0}(\mathcal{O}')$ and allows us to view V_κ as a representation of $L_H(\mathbb{Z}_p)$. We write ρ_{κ_p} instead of ρ_κ when referring to V_κ as an $L_H(\mathbb{Z}_p)$ -module. For instance, given $l \in L_H(\mathbb{Z}_p)$, we write

$$(25) \quad \rho_\kappa({}^t l^{-1}) = \rho_{\kappa_p}(l) ,$$

where we abuse notation to denote the element of $L_{H_0}(\mathcal{O}')$ corresponding to l under the isomorphism (24) by ${}^t l^{-1}$.

Similarly, the identification (24) induces an embedding $T_H(\mathbb{Z}_p) \hookrightarrow T_{H_0}(\mathcal{O}')$. Given $t = (\text{diag}(t_{w,1}, \dots, t_{w,a_w})_{w|p}) \in T_H(\mathbb{Z}_p)$, its image in $T_{H_0}(\mathcal{O}')$ is naturally identified with $x = (1, t^{-1})$ and we have

$$(26) \quad \kappa(x) = \kappa_p(t) ,$$

where

$$\kappa_p(t) = \prod_{w|p} \prod_{\substack{\sigma \in \Sigma_{\mathcal{K}} \\ \mathfrak{p}_\sigma = \mathfrak{p}_w}} \prod_{j=1}^{a_\sigma} \sigma(t_{w,j})^{\kappa_{\sigma c,j}} .$$

We sometimes write

$$(27) \quad \kappa_p = (\kappa_{\sigma c})_{\sigma \in \Sigma_{\mathcal{K}}} \in \prod_{\sigma \in \Sigma_{\mathcal{K}}} \mathbb{Z}^{a_\sigma} ,$$

for convenience and refer to κ_p as a p -adic weight.

We say that κ_p is a P -parallel if κ is P -parallel. Clearly, P -parallel p -adic weights extend to characters of $L_H(\mathbb{Z}_p)$ (that factor through $L_H(\mathbb{Z}_p)/SL_H(\mathbb{Z}_p)$).

If β is an algebraic P -parallel weight and κ is any algebraic weight, then $\rho_{\kappa_p+\beta_p}$ is canonically isomorphic to $\rho_{\kappa_p} \otimes \beta_p$. Thus, we again view V_κ as the space on which $\rho_{\kappa_p+\beta_p}$ acts for all P -parallel p -adic weights β_p .

Remark 2.14. The representation ρ_{κ_p} and the character κ_p coincide with one another when $P = B$ as in Remark 2.8. This is what occurs in [EHLS20, Section 2.9.4].

2.4. Shimura varieties of P -Iwahoric level at p . We first recall the familiar theory of integral away-from- p models of Shimura variety for the unitary group G . We use them to define holomorphic and anti-holomorphic automorphic representations of G .

Remark 2.15. In Section 2.5, we introduce more general level structures at p related to the P -Iwahori subgroups constructed in Section 2.2.2 and define the notion of P -nebenspecies for both modular forms and automorphic representations.

Let $X = X_{\mathcal{P}}$ denote the conjugacy class of h via the natural action of $G(\mathbb{R})$ on $\text{End}_{\mathcal{K}^+ \otimes \mathbb{R}}(V \otimes \mathbb{R})$. It is well-known that the pair (G, X) defines a *Shimura datum* in the usual sense whose reflex field is again F .

Let $K = G(\mathbb{Z}_p)K^p$ as in the beginning of Section 2.1.1. Let $Sh_K(G, X)$ be the canonical model of the Shimura variety of level K over F associated to (G, X) . Then, the moduli space $M_{K/F}$ is the union of finitely many copies of $Sh_K(G, X)$, see [Kot92, Section 8] for details.

More precisely, let $V^{(1)}, \dots, V^{(k)}$ be representatives for the isomorphism classes of all hermitian vector spaces that are locally isomorphic to V at every place of \mathbb{Q} . As explained in [CEF⁺16, Section 2.3.2], there are finitely many such classes, in fact $k = |\ker^1(\mathbb{Q}, G)|$, where

$$\ker^1(\mathbb{Q}, G) = \ker \left(H^1(\mathbb{Q}, G) \rightarrow \prod_v H^1(\mathbb{Q}_v, G) \right).$$

The base change of M_K over F is the disjoint union of F -schemes $M_{K, V^{(j)}}$, naturally indexed by the $V^{(j)}$ and all isomorphic to $Sh_K(G, X)$.

Assume that $V^{(1)} = V$. To work integrally (away-from- p), denote the scheme-theoretic closure of $M_{K, V}$ in M_K by ${}_K\text{Sh}(V)$. When V is clear from context, we simply write ${}_K\text{Sh}$.

It is well-known that ${}_K\text{Sh}$ is a smooth, quasi-projective S_p -scheme. We refer to ${}_K\text{Sh}$ as a *Shimura variety* of level K (associated to \mathcal{P}) and M_K as a *moduli space*. Denote the natural inclusion ${}_K\text{Sh} \hookrightarrow M_K$ over S_p by s_K .

Furthermore, to work with compactified Shimura varieties, let Ω be a polyhedral cone decomposition, as in Section 2.1.2, and denote the scheme-theoretic closure of ${}_K\text{Sh}$ in $M_{K, \Omega}^{\text{tor}}$ by ${}_K\text{Sh}_{\Omega}^{\text{tor}}$. This is the natural smooth toroidal compactification of

$_K\text{Sh}$ discussed in [Lan12, Sections 3-4] and, over F , it recovers the usual toroidal compactification of $Sh_K(G, X)$.

We often treat the tower $_K\text{Sh}^{\text{tor}} := \{ {}_K\text{Sh}_{\Omega}^{\text{tor}} \}_{\Omega}$ as a single scheme. We denote the natural inclusions ${}_K\text{Sh}_{\Omega}^{\text{tor}} \hookrightarrow M_{K,\Omega}^{\text{tor}}$ and ${}_K\text{Sh}^{\text{tor}} \hookrightarrow M_K^{\text{tor}}$ by $s_{K,\Omega}$ and s_K respectively.

Given a neat compact open subgroup $K'^p \subset K^p$, let $K' = G(\mathbb{Z}_p)K'^p$. The map $M_{K'} \rightarrow M_K$ is compatible with the inclusions s_K and $s_{K'}$, hence induces an analogous homomorphism ${}_{K'}\text{Sh} \rightarrow {}_K\text{Sh}$. The latter extends canonically to a map (of towers) ${}_{K'}\text{Sh}^{\text{tor}} \rightarrow {}_K\text{Sh}^{\text{tor}}$ on toroidal compactifications.

A similar statement holds true for $[g] : {}_gKg^{-1}\text{Sh} \rightarrow {}_K\text{Sh}$, given any $g \in G(\mathbb{A}_f^p)$. This induces a natural action of $G(\mathbb{A}_f^p)$ on the towers $\{ {}_K\text{Sh} \}_{K^p}$ and $\{ {}_K\text{Sh}^{\text{tor}} \}_{K^p}$.

Lastly, we set $\text{Sh}(V) := \varprojlim_K {}_K\text{Sh}(V)$ and $\text{Sh}(V)^{\text{tor}} := \varprojlim_K {}_K\text{Sh}(V)^{\text{tor}}$ when working with the profinite Shimura variety of infinite level and its compactification.

2.4.1. The canonical bundle. The following section recalls some of the material of [EHLS20, Sections 2.6 and 6.1].

Let ω be the $\mathcal{O}_{M_K^{\text{tor}}}$ -dual of $\text{Lie}_{M_K^{\text{tor}}} \mathcal{A}^{\vee}$ over S_p . The Kottwitz determinant condition mentioned in the definition of the moduli problem $M_K(\mathcal{P})$ implies that ω is locally isomorphic to $\Lambda_0^{\vee} \otimes_{S_p} \mathcal{O}_{M_K^{\text{tor}}}$ over $\mathcal{O} \otimes \mathcal{O}_{M_K^{\text{tor}}}$. Define the canonical bundle \mathcal{E} as the scheme

$$\text{Isom}_{\mathcal{O}_{\mathcal{K}} \otimes \mathcal{O}_{M_K^{\text{tor}}}}(\mathcal{O}_{M_K^{\text{tor}}}(1), \mathcal{O}_{M_K^{\text{tor}}}(1)) \times \text{Isom}_{\mathcal{O}_{\mathcal{K}} \otimes \mathcal{O}_{M_K^{\text{tor}}}}(\omega, \Lambda_0^{\vee} \otimes_{S_p} \mathcal{O}_{M_K^{\text{tor}}}),$$

over M_K^{tor} .

The natural structure map $\pi : \mathcal{E} \rightarrow M_K^{\text{tor}}$ is an H_0 -torsor and is defined over S_p when K is a neat open compact subgroup of $G(\mathbb{A}_f)$ of the form $G(\mathbb{Z}_p)K^p$. Note that the first factor in the definition of \mathcal{E} is included to keep track of the action of the (similitude) \mathbb{G}_m -factor of H_0 , however it does not play a significant role in the rest of the paper.

2.4.2. Modular forms of weight κ . Let R be an algebra over $S_0 = \mathcal{O}_{\mathcal{K}',(\mathfrak{p})}$, and let κ be a dominant character of T_{H_0} over R as in Section 2.3.1. Consider the vector bundle

$$\omega_{\kappa} = \omega_{\kappa,\Omega} = s_{K,\Omega}^* \pi_*(\mathcal{O}_{\mathcal{E}}[\kappa]),$$

above ${}_K\text{Sh}_{\Omega}$ defined over S_0 . Here, we extend κ to an algebraic character of B_{H_0} trivially and $\mathcal{O}_{\mathcal{E}}[\kappa]$ denotes the κ -isotypic part of $\mathcal{O}_{\mathcal{E}}$. By taking limits over K and Ω , we often view ω_{κ} over $\text{Sh}(V)^{\text{tor}}$ without comment.

Remark 2.16. Recall that given an irreducible representation of P_0 over \mathbb{C} that factors through H_0 , one can view it as a G -equivariant vector bundle on the compact dual \widehat{X} of X and thus define an automorphic vector bundle ω_W on $\text{Sh}(V)_{/\mathbb{C}}$ using the usual \otimes -functor

$$G\text{-Bun}(\widehat{X}) \rightarrow \text{Bun}(\text{Sh}(V)),$$

see [EHLS20, Section 6.1.1] for further details.

It is well-known that each such ω_W has a canonical model over a number field $F(W)/F$ such that $F(W) \subset \mathcal{K}'$. For instance, the base change of ω_{W_κ} from $F(W_\kappa)$ to \mathcal{K}' is actually canonically isomorphic to the restriction from $\mathrm{Sh}(V)^{\mathrm{tor}}$ to $\mathrm{Sh}(V)$ of ω_κ .

For each polyhedral cone decomposition Ω , let D_Ω be the Cartier divisor ${}_K\mathrm{Sh}_\Omega^{\mathrm{tor}} - {}_K\mathrm{Sh}$ equipped with its structure of a reduced closed subscheme. Implicitly, we restrict our attention to choices of Ω for which this complement D_Ω is a divisor with normal crossing. Let $\omega_\kappa(-D_\Omega)$ be the twist of ω_κ by the ideal sheaf of the boundaries corresponding to D_Ω .

Then, cuspidal cohomology (of degree i) of level K with respect to Ω is defined as

$$H_!^i({}_K\mathrm{Sh}(V)_\Omega^{\mathrm{tor}}, \omega_\kappa) := \mathrm{Im} \left(H^i({}_K\mathrm{Sh}(V)_\Omega^{\mathrm{tor}}, \omega_\kappa(-D_\Omega)) \rightarrow H^i({}_K\mathrm{Sh}(V)_\Omega^{\mathrm{tor}}, \omega_\kappa) \right),$$

and we mainly work with

$$H_!^i(\mathrm{Sh}(V), \omega_\kappa) = H_!^i(\mathrm{Sh}(V)^{\mathrm{tor}}, \omega_\kappa) := \varinjlim_{K, \Omega} H_!^i({}_K\mathrm{Sh}(V)_\Omega^{\mathrm{tor}}, \omega_\kappa),$$

where the limit is restricted to subgroups K of the form $G(\mathbb{Z}_p)K^p$, so that the above is defined over S_0 . We first review the theory of degree $i = 0$ in what follows and discuss the middle degree cohomology in Section 2.7.

Remark 2.17. If the reflex field F is different from \mathbb{Q} or the derived group G^{der} of G (over \mathbb{Q}) has no irreducible factor isomorphic to $\mathrm{SU}(1, 1)$, then we can invoke the K ocher principle, namely

$$H^0({}_K\mathrm{Sh}(V)_\Omega^{\mathrm{tor}}, \omega_\kappa) = H^0({}_K\mathrm{Sh}(V), \omega_\kappa),$$

see [Lan16]. Therefore, in that case, we can ignore the toroidal compactification and omit the limit over Ω in the definitions above.

Otherwise, the toroidal compactifications are canonical; they are simply the minimal compactification. We ignore the details needed to treat this case and implicitly view the tower ${}_K\mathrm{Sh}(V)^{\mathrm{tor}}$ as a single scheme, see the remarks at the end of Section 2.1.2 (or [EHLS20, Section 2.6.5] for more details and a similar treatment).

The action of $G(\mathbb{A}_f^p)$ on

$$H^0(\mathrm{Sh}(V), \omega_\kappa) := \varinjlim_{K, \Omega} H^0({}_K\mathrm{Sh}(V)_\Omega^{\mathrm{tor}}, \omega_\kappa)$$

induced by its action on the tower $\{{}_K\mathrm{Sh}^{\mathrm{tor}}\}_{K^p}$ stabilizes $H_!^0(\mathrm{Sh}(V), \omega_\kappa)$.

The R -modules of $M_\kappa(K; R)$ and $S_\kappa(K; R)$ of modular forms and cusp forms of weight κ and level $K = G(\mathbb{Z}_p)K^p$ are defined by taking K^p -fixed points of this action, namely

$$M_\kappa(K; R) := H^0(\mathrm{Sh}(V)/_R, \omega_\kappa)^{K^p} = H^0({}_K\mathrm{Sh}(V), \omega_\kappa)$$

and

$$S_\kappa(K; R) := H_!^0(\mathrm{Sh}(V)/_R, \omega_\kappa)^{K^p} = H_!^0({}_K\mathrm{Sh}(V), \omega_\kappa),$$

respectively.

Via the moduli interpretation of M_K , we view a modular form $f \in M_\kappa(K; R)$ as a rule on the set of pairs $(\underline{A}, \varepsilon) \in \mathcal{E}(S)$, for any R -algebra S , such that $f(\underline{A}, \varepsilon) \in S$, the rule is functorial in S , and

$$f(\underline{A}, b\varepsilon) = \kappa(b)f(\underline{A}, \varepsilon),$$

for all $b \in B_{H_0}(S)$.

2.4.3. Hecke operators away from p . Given $K = G(\mathbb{Z}_p)K^p$ as above, $g \in G(\mathbb{A}_f^p)$ and an S_0 -algebra R , the double coset KgK naturally defines an operator

$$[KgK] : M_\kappa(K; R) \rightarrow M_\kappa(K; R)$$

induced by viewing KgK as a correspondence on M_K . More precisely, given $f \in M_\kappa(K; R)$ and writing $K^p g K^p$ as finite disjoint union $\bigsqcup_i g_i K^p$ of right cosets, we have

$$(28) \quad ([KgK]f)(A, \lambda, \iota, \alpha, \varepsilon) = \sum_i f(A, \lambda, \iota, \alpha \circ g_i, \varepsilon),$$

which is obviously independent of the choice of representatives g_i . When the level K is clear from context, we simply write $T(g)$ instead of $[KgK]$. One readily checks that $T(g)$ stabilizes $S_\kappa(K; R)$.

2.5. P -nebentypus theory of modular forms. We now introduce a more general level structure at p via covers of M_K and M_K^{tor} .

2.5.1. Level subgroup $K_{P,r}$. Let $\underline{\mathcal{A}} = (\mathcal{A}, \lambda, \iota, \alpha)$ be the universal abelian scheme over M_K . Using [Lan13, Theorem 6.4.1.1], \mathcal{A} can be extended to a semiabelian scheme over M_K^{tor} that is part of a degenerating family and which we still denote \mathcal{A} .

By [Lan13, Theorem 3.4.3.2], there exists a dual semiabelian scheme \mathcal{A}^\vee together with homomorphisms $\mathcal{A} \rightarrow \mathcal{A}^\vee$, $S_p \rightarrow \text{End}_{M_K^{\text{tor}}} \mathcal{A}$ and a $K^{(p)}$ -level structure on \mathcal{A} that extend λ , ι and α respectively.

Define an S_p -scheme \overline{M}_{K_r} over M_K^{tor} whose S -points is the set of $P_H^u(\mathbb{Z}_p)$ -orbits of injections $\phi : L^+ \otimes \mu_{p^r} \hookrightarrow \mathcal{A}^\vee[p^r]_S$ of group schemes over $\mathcal{O} \otimes \mathbb{Z}_p$ such that the image of ϕ is an isotropic subgroup scheme. The natural action of $\mathcal{L}_r = L_H(\mathbb{Z}_p/p^r \mathbb{Z}_p)$ on $L^+ \otimes \mu_{p^r}$ induces a structure of \mathcal{L}_r -torsor on $\overline{M}_{K_r} \rightarrow M_K^{\text{tor}}$.

Let M_{K_r} denote the pullback of \overline{M}_{K_r} over M_K , i.e. we have the Cartesian commutative diagram

$$(29) \quad \begin{array}{ccc} M_{K_r} & \hookrightarrow & \overline{M}_{K_r} \\ \downarrow & & \downarrow \\ M_K & \hookrightarrow & M_K^{\text{tor}} \end{array}$$

and the vertical arrows are \mathcal{L}_r -torsors. We set $K_{P,r} := I_{P,r}K^p \subset G(\mathbb{A}_f)$

Remark 2.18. Recall that one can define an F -rational moduli problem generalizing the one in Section 2.1.1 for each neat open compact subgroup $K \subset G(\mathbb{A}_f)$ (by essentially dropping all “prime-to- p ” conditions). We again denote the corresponding moduli space by M_K . The $G(\mathbb{A}_f^p)$ -action on $\{M_K\}_{K^p}$ extends to an action of $G(\mathbb{A}_f)$ on $\{M_K\}_K$. We do not include the exact details needed to modify the prior theory to $M_{K/F}$ and instead refer the reader to [EHLS20, Section 2.1] or [Lan13, Corollary 7.2.3.10].

A choice of basis of $\mathbb{Z}_p(1)$ induces a natural isomorphism between the scheme $M_{K_r/F}$, defined as a pullback in (29), and the moduli space $M_{I_r K^p/F}$ representing the F -rational moduli problem mentioned in the previous paragraph. Furthermore, this same choice identifies $\overline{M}_{K_r/F}$ with the normalization of $M_{K/F}^{\text{tor}}$ in $M_{K_r/F}$. Therefore, we can write $K_{P,r} = K_r$ without risk of confusion when P is clear from context.

Over S_p , define ${}_{K_r}\text{Sh}$ (resp. ${}_{K_r}\overline{\text{Sh}}$) as the pullback of M_{K_r} (resp. \overline{M}_{K_r}) via s_K . Hence, we have the commutative diagrams

$$\begin{array}{ccc} {}_{K_r}\text{Sh} & \hookrightarrow & M_{K_r} \\ \downarrow & & \downarrow \\ {}_K\text{Sh} & \hookrightarrow & M_K \end{array} \quad , \quad \begin{array}{ccc} {}_{K_r}\overline{\text{Sh}} & \hookrightarrow & \overline{M}_{K_r} \\ \downarrow & & \downarrow \\ {}_K\text{Sh}^{\text{tor}} & \hookrightarrow & M_K^{\text{tor}} \end{array}$$

and by abusing notation, we denote all four horizontal inclusions by s_K . All four vertical arrows are again \mathcal{L}_r -torsors.

Remark 2.19. As in Remark 2.18, a choice of basis of $\mathbb{Z}_p(1)$ identifies $\text{Sh}_{K_r/F}$ with $Sh_{I_r K^p}(G, X)_{/F}$ (the analogue of $Sh_K(G, X)$ introduced in Section 2.4 for $K = I_r K^p$), and identifies ${}_{K_r}\overline{\text{Sh}}_{/F}$ with the normalization of ${}_K\text{Sh}_{/F}^{\text{tor}}$ in ${}_{K_r}\text{Sh}_{/F}$.

The action of $G(\mathbb{A}_f^p)$ on the tower $\{{}_K\text{Sh}\}_{K^p}$ naturally induces an action on $\{{}_{K_r}\text{Sh}\}_{K^p}$. Analogous statements hold true for $\{M_{K_r}\}_{K^p}$, $\{{}_{K_r}\overline{\text{Sh}}\}_{K^p}$, and $\{\overline{M}_{K_r}\}_{K^p}$.

Furthermore, let $\mathcal{E}_r = \mathcal{E} \times_{M_K^{\text{tor}}} \overline{M}_{K_r}$, so

$$\begin{array}{ccc} \mathcal{E}_r & \xrightarrow{H_0} & \overline{M}_{K_r} \\ \downarrow \mathcal{L}_r & & \downarrow \mathcal{L}_r \\ \mathcal{E} & \xrightarrow{H_0} & M_K^{\text{tor}} \end{array}$$

and denote the structure map $\mathcal{E}_r \rightarrow \overline{M}_{K_r}$ by π_r .

Given a dominant weight κ of T_{H_0} over some S_0 -algebra R , we define

$$\omega_{\kappa,r} := s_{K_r}^*(\pi_r)_*(\mathcal{O}_{\mathcal{E}_r}[\kappa])$$

as a sheaf over ${}_{K_r}\overline{\text{Sh}}_{/R}$.

We define the space of modular forms on G over R of level K_r and weight κ as

$$(30) \quad M_\kappa(K_r; R) := H^0({}_{K_r}\overline{\text{Sh}}, \omega_{\kappa,r})$$

and its subspace of cusp forms as

$$(31) \quad S_\kappa(K_r; R) := H_1^0(K_r, \overline{S\mathfrak{h}}, \omega_{\kappa, r}),$$

where H_1^0 denotes cuspidal cohomology as in Section 2.4.1.

It follows from Remarks 2.18 and 2.19 that $M_\kappa(K_r; S_p)$ (resp. $S_\kappa(K_r; S_p)$) is an S_p -integral structure of the usual space of modular (resp. cusp) forms over F on G of level $I_r K^p$ and weight κ .

We view a modular form $f \in M_\kappa(K_r; R)$ as a rule on the set of pairs $(\underline{A}, \phi, \varepsilon) \in \mathcal{E}_r(S)$, for any R -algebra S , such that $f(\underline{A}, \phi, \varepsilon) \in S$, the rule is functorial in S , and

$$f(\underline{A}, \phi, b\varepsilon) = \kappa(b)f(\underline{A}, \phi, \varepsilon),$$

for all $b \in B_{H_0}(S)$.

Given a $\overline{\mathbb{Q}}_p$ -valued multiplicative character ψ_B of the maximal torus $T_H(\mathbb{Z}_p)$ of $H(\mathbb{Z}_p)$ that factors through $T_H(\mathbb{Z}_p/p^r\mathbb{Z}_p)$, let $S_p[\psi_B]$ denote the smallest ring extension of S_p containing the values of ψ_B . Given an $S_p[\psi_B]$ -algebra R , we define the R -module of modular forms over R , weight κ , level K_r , and (classical) *nebentypus* ψ_B as

$$M_\kappa(K_r, \psi_B; R) := \{f \in H^0(K_r, \overline{S\mathfrak{h}}, \omega_{\kappa, r}) : t \cdot f = \psi_B(t)f, \forall t \in T_H(\mathbb{Z}_p)\},$$

and we define the analogous R -module of cusp forms $S_\kappa(K_r, \psi_B; R)$ similarly.

Given $g \in G(\mathbb{A}_f)$, the formula (28) can similarly be adapted to define an operator $M_\kappa(K_r; R)$ via

$$(32) \quad ([K_r g K_r]f)(A, \lambda, \iota, \alpha, \phi, \varepsilon) = \sum_i f(A, \lambda, \iota, \alpha \circ g_i, \phi, \varepsilon),$$

using the same notation as in Section 2.4.3. By abuse of notation, we again denote this operator by $T(g)$ when K_r is clear from context.

Furthermore, if R contains the reflex field F , then $M_\kappa(K_r; R)$ is also obtained as the K_r -fixed points of the R -module

$$\varinjlim_{K \subset G(\mathbb{A}_f)} H^0(K, \overline{S\mathfrak{h}}, \omega_\kappa),$$

and the same holds true for $S_\kappa(K_r; R)$ upon replacing $H^0(-)$ by $H_1^0(-)$.

2.5.2. P -nebentypus of modular forms. Let τ be a smooth irreducible representation of $L_H(\mathbb{Z}_p)$ acting on a module \mathcal{M}_τ over some S_p -algebra $S_p[\tau] \subset \mathbb{C}$.

Definition 2.20. We say that τ is a P -nebentypus of level r if it factors through $\mathcal{L}_r = L_H(\mathbb{Z}_p/p^r\mathbb{Z}_p)$. In this case, we can always assume $S_p[\tau]$ is finite over S_p , hence contained in $\overline{\mathbb{Q}}$.

We do not assume that r is minimal for this property. In fact, if τ is of level r , it is obviously of level r' for every $r' \geq r$, and therefore we sometimes write that τ is a “ P -nebentypus of level $r \gg 0$ ”.

Define $\mathcal{E}_{r,\tau}$ as the $S_p[\tau]$ -scheme over \mathcal{E}_r whose R -points are given by

$$\mathcal{E}_{r,\tau}(R) = \mathcal{E}_r(R) \times^\tau \mathcal{M}_{\tau,R} := (\mathcal{E}_r(R) \times \mathcal{M}_{\tau,R}) / \sim^\tau$$

for any $S_p[\tau]$ -algebra R , where the equivalence relation \sim^τ is

$$((\varepsilon, \phi), m) \sim^\tau ((\varepsilon, \phi \circ l), \tau(l)m),$$

for all $(\varepsilon, \phi) \in \mathcal{E}_r$, $m \in \mathcal{M}_{\tau,R}$ and $l \in L_H(\mathbb{Z}_p)$. We denote the structure map $\mathcal{E}_{r,\tau} \rightarrow \overline{\mathcal{M}}_{K_r}$ by $\pi_{r,\tau}$.

Let $S_0[\tau] \subset \overline{\mathbb{Q}}$ be the compositum of $S_p[\tau]$ and $S_0 = \mathcal{O}_{\mathcal{K}',(\mathfrak{p})}$. Given a dominant weight κ of T_{H_0} over an $S_0[\tau]$ -algebra R , we define

$$\omega_{\kappa,r,\tau} = s_K^*(\pi_{r,\tau})_*(\mathcal{O}_{\mathcal{E}_{r,\tau}}[\kappa])$$

as a sheaf on ${}_{K_r}\overline{\text{Sh}}$ over R . We denote its restriction to ${}_{K_r}\text{Sh}$ by $\omega_{\kappa,r,\tau}$ as well.

Definition 2.21. We define the space of modular forms on G over R , level K_r , weight κ and P -nebenotypus τ as

$$M_\kappa(K_r, \tau; R) := H^0({}_{K_r}\overline{\text{Sh}}, \omega_{\kappa,r,\tau})$$

and its subspace of cusp forms as

$$S_\kappa(K_r, \tau; R) := H^0_!({}_{K_r}\overline{\text{Sh}}, \omega_{\kappa,r,\tau}),$$

where $H^0_!$ again denotes cuspidal cohomology as in Section 2.4.1.

Remark 2.22. Classically, the nebenotypus of a modular form is a finite-order character of the maximal torus $T_H(\mathbb{Z}_p)$ of H . In our terminology, see Remark 2.8, this is equivalent to a B -nebenotypus.

Remark 2.23. The reader should note that in this notation τ is always a P -nebenotypus, i.e. a smooth finite-dimensional representation of the Levi subgroup of $P_H(\mathbb{Z}_p)$. On the other hand, when writing $M_\kappa(K_r, \psi_B; R)$, we always use the symbol ψ_B for a character of the maximal torus $T_H(\mathbb{Z}_p)$ of the Borel subgroup $B_H(\mathbb{Z}_p)$. The subscript B is to remind the reader of the relation between ψ_B and B_H and help distinguish between the *similar yet different* spaces $M_\kappa(K_r, \psi_B; R)$ and $M_\kappa(K_r, \tau; R)$. The two notions overlap exactly when $P = B$, as in Remark 2.8, in which case the notation is not ambiguous.

A modular form $f \in M_\kappa(K_r, \tau; R)$ can be interpreted as a functorial rule that assigns to a tuple $(\underline{A}, \phi, \varepsilon) \in \mathcal{E}_r(S)$, over an R -algebra S , an element

$$f(\underline{A}, \phi, \varepsilon) \in \text{Hom}_S(\mathcal{M}_{\tau,S}, S) = \mathcal{M}_{\tau,S}^\vee$$

such that

$$f(\underline{A}, \phi \circ l^{-1}, b\varepsilon) = \kappa(b)\tau^\vee(l)f(\underline{A}, \phi, \varepsilon)$$

for all $b \in B_{H_0}(S)$ and $l \in L_H(\mathbb{Z}_p)$.

Equivalently, using Frobenius reciprocity, f can be interpreted as a functorial rule such that

$$f(\underline{A}, \phi, \varepsilon) \in V_{\kappa,S} \otimes \mathcal{M}_{\tau,S}^\vee$$

and

$$f(\underline{A}, \phi \circ l^{-1}, l_0 \epsilon) = f(\underline{A}, \phi \circ l^{-1}, l_0 \epsilon)(v) = (\rho_\kappa(l_0) \otimes \tau^\vee(l))f(\underline{A}, \phi, \epsilon)$$

for all $l_0 \in L_{H_0}(S)$ and $l \in L_H(\mathbb{Z}_p)$.

Given $g \in G(\mathbb{A}_f^p)$, one can again define a Hecke operator $T(g)$ on $M_\kappa(K_r, \tau; R)$ which stabilizes the subspace of cusp forms via (32).

More generally, view $\mathcal{M} = \mathcal{M}_\tau$ simply as a module over $S_p[\mathcal{M}] = S_p[\tau]$, forgetting the representation τ momentarily.

We define $\mathcal{E}_{r, \mathcal{M}}$ as the $S_p[\mathcal{M}]$ -scheme over \mathcal{E}_r whose R -points are given by

$$\mathcal{E}_{r, \mathcal{M}}(R) = \mathcal{E}_r(R) \times \mathcal{M}_R$$

for any $S_p[\mathcal{M}]$ -algebra R , without any equivalence relation. We denote the structure map $\mathcal{E}_{r, \mathcal{M}} \rightarrow \overline{\mathcal{M}}_{K_r}$ by $\pi_{r, \mathcal{M}}$.

Let $S_0[\mathcal{M}] \subset \overline{\mathbb{Q}}$ be the compositum of $S_p[\tau]$ and $S_0 = \mathcal{O}_{\mathcal{K}', (\mathfrak{p}')}$. Given a dominant weight κ of T_{H_0} over an $S_0[\mathcal{M}]$ -algebra R , we define

$$\omega_{\kappa, r, \mathcal{M}} = s_K^*(\pi_{r, \mathcal{M}})_*(\mathcal{O}_{\mathcal{E}_{r, \mathcal{M}}}[\kappa])$$

as a sheaf on $_{K_r}\overline{\text{Sh}}$ over R . We denote its restriction to $_{K_r}\text{Sh}$ by $\omega_{\kappa, r, \mathcal{M}}$ as well.

Definition 2.24. For any $S_0[\mathcal{M}]$ -algebra R , we define the space of modular forms over R on G of weight κ , level K_r and P -type \mathcal{M} as

$$M_\kappa(K_r, \mathcal{M}; R) := H^0({}_{K_r}\overline{\text{Sh}}, \omega_{\kappa, r, \mathcal{M}})$$

and its subspace of cusp forms as

$$S_\kappa(K_r, \mathcal{M}; R) := H_!^0({}_{K_r}\overline{\text{Sh}}, \omega_{\kappa, r, \mathcal{M}}).$$

In particular, $f \in M_\kappa(K_r, \mathcal{M}; R)$ can be viewed as a functorial rule on the set of tuples $(\underline{A}, \phi, \epsilon) \in \mathcal{E}_r(S)$, for any R -algebra S , such that

$$f(\underline{A}, \phi, \epsilon) \in V_{\kappa, S} \otimes \mathcal{M}_S^\vee$$

and

$$f(\underline{A}, \phi, l_0 \epsilon) = \rho_\kappa(l_0)f(\underline{A}, \phi, \epsilon).$$

Remark 2.25. When working with $P = B$, as in Remark 2.8, then $M_\kappa(K_r, [\tau]; R) = M_\kappa(K_r; R)$ and $S_\kappa(K_r, [\tau]; R) = S_\kappa(K_r; R)$.

Going back to the representation τ on $\mathcal{M} = \mathcal{M}_\tau$, consider an algebra R over $S_0[\tau] := S_0[\mathcal{M}_\tau]$. Naturally, $M_\kappa(K_r, \mathcal{M}; R)$ contains $M_\kappa(K_r, \tau; R)$ but it also contains $M_\kappa(K_r, \tau'; R)$ for any representation τ' on $\mathcal{M}_{\tau, R}$.

In this work, we are mostly concern with twists of τ by finite-order characters of \mathcal{L}_r , all viewed as acting on the same module \mathcal{M} (over a sufficiently large ring). This leads to the following definition.

Definition 2.26. We say that two P -nebentype τ and τ' of level r are *equivalent*, and write $\tau \sim_r \tau'$, if $\tau = \tau' \otimes \psi$ for some finite-order character ψ of \mathcal{L}_r . We let $[\tau]_r$

denote the (finite) equivalence class of τ as a P -nebenotypus of level r . This notion obviously depends on r but we sometimes write $[\tau]$ when r is clear from the context.

For each $r \gg 0$, fix a ring $S_r[\tau]$ large enough to contain $S_0[\tau']$ for all $\tau' \sim_r \tau$. After base change, if necessary, we view \mathcal{M}_τ as the $S_r[\tau]$ -module on which τ' acts, for all $\tau' \sim_r \tau$. To emphasize this convention, we now refer to \mathcal{M}_τ as $\mathcal{M}_{[\tau]}$. Similarly, given any $S_r[\tau]$ -algebra R , we set $\mathcal{M}_{[\tau],R} = \mathcal{M}_{\tau,R} := \mathcal{M}_\tau \otimes_{S_r[\tau]} R$. Note that the contragredient module $\mathcal{M}_{[\tau]}^\vee = \mathcal{M}_\tau^\vee$ and the tautological pairing $(\cdot, \cdot)_\tau = (\cdot, \cdot)_{[\tau]}$ on $\mathcal{M}_\tau \otimes \mathcal{M}_\tau^\vee$ are both well-defined up to equivalence of P -nebenotype.

Therefore, one readily sees that

$$(33) \quad M_\kappa(K_r, [\tau]; R) := \bigoplus_{\tau' \in [\tau]_r} M_\kappa(K_r, \tau'; R).$$

is a subspace of $M_\kappa(K_r, \mathcal{M}; R)$.

Remark 2.27. One similarly defines $S_\kappa(K_r, [\tau]; R)$ and $\omega_{\kappa,r,[\tau]}$. We refer to $f \in M_\kappa(K_r, [\tau]; R)$ (resp. $S_\kappa(K_r, [\tau]; R)$) as a modular (resp. cusp) form over R on G of weight κ , level K_r and P -type class $[\tau]$.

Remark 2.28. In general, $M_\kappa(K_r, \mathcal{M}; R)$ is strictly larger than $M_\kappa(K_r, [\tau]; R)$. Indeed, if ψ and ψ' are two characters of \mathcal{L}_r that are congruent modulo p , and $f \in M_\kappa(K_r, \tau; R)$, then

$$(34) \quad \frac{1}{p}(f \otimes \psi - f \otimes \psi')$$

lies in $M_\kappa(K_r, [\tau]; R)$ but not in the direct sum of (33).

Remark 2.29. In all that follows, we almost exclusively work with $M_\kappa(K_r, [\tau]; R)$. Effectively, in Section 8, this leads us to consider P -ordinary Hida families, viewed as closed subschemes of the spectrum of certain P -ordinary Hecke algebras, containing a dense set of classical points. This set of classical points corresponds to P -ordinary automorphic representations whose P -nebenotypus at p are members $\tau' \in [\tau]$ that all congruent modulo p . Although there are additional details omitted in this comment, the notions above are all defined properly later in the text. See (125) for a concrete description of this set of classical points.

It is certainly interesting to work with $M_\kappa(K_r, \mathcal{M}; R)$ instead. In this case, one obtains larger Hida family whose dense set of classical points corresponds to all P -ordinary automorphic representations whose P -nebenotypus at p are all representations τ' on \mathcal{M} that are congruent modulo p . These families are sensitive to the existence of congruences as in (34).

However, our computations in this paper are only worked out when the types are all in the same P -class, i.e. twists of each other by finite-order characters. The author hopes to generalize the necessary computation in later work to consider these larger families.

2.6. Complex Uniformization. The coherent cohomology group defining the various spaces of algebraic modular forms introduced in the previous sections can be computed with Lie algebra cohomology groups, at least over \mathbb{C} .

2.6.1. *Complex structure.* Recall that X denotes the $G(\mathbb{R})$ -conjugacy class of h . Let $C \subset G/\mathbb{R}$ denote the centralizer of h , so that there is a natural identification $G(\mathbb{R})/C(\mathbb{R}) \xrightarrow{\sim} X$. In particular, this induces a structure of a real manifold on X . In what follows, we set $U_\infty := C(\mathbb{R})$.

Furthermore, recall that under the identification of $G_{\mathbb{C}}$ with G_0/\mathbb{C} from Section 2.3, $P_h(\mathbb{C}) \subset G(\mathbb{C})$ corresponds to $P_0(\mathbb{C})$, and $C(\mathbb{C})$ corresponds to $H_0(\mathbb{C})$. It is well-known that X then corresponds to an open subspace of $G_0(\mathbb{C})/P_0(\mathbb{C})$ and hence also admits the structure of a complex manifold.

Let $r \geq 0$ and $K^p \subset G(\mathbb{A}_f^p)$ be a neat open compact subgroup. Let $\text{Sh} = \text{Sh}(V)$ be the pro-finite tower of Shimura varieties associated to G .

Given $(h', g) \in X \times G(\mathbb{A}_f)$, with $g_p \in G(\mathbb{Z}_p)$, one can naturally define a tuple

$$X_{h',g} = (A_{h'}, \lambda_{h'}, \iota_{h'}, \alpha_g, \phi_g) \in_{K_r} \text{Sh}(\mathbb{C})$$

as well as an $\mathcal{O} \otimes \mathbb{C}$ -isomorphism $\varepsilon_{h'} : \omega_{A_{h'}^\vee} \xrightarrow{\sim} \Lambda_0 \otimes_{S_0} \mathbb{C}$. The precise descriptions of $X_{h',g}$ and $\varepsilon_{h'}$ plays no role in what follows, see [EHLS20, Sections 2.7.1-2.7.2] for details.

In fact, the map $(h', g) \rightarrow X_{h',g}$ provides a bijection

$$(35) \quad G(\mathbb{Q}) \backslash G(\mathbb{R}) \times G(\mathbb{A}_f) / U_\infty K_r = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K_r \xrightarrow{\sim}_{K_r} \text{Sh}(\mathbb{C})$$

which identifies the complex analytic structures on both side. In particular, the dimension d of $\text{Sh}(V)$ is just the \mathbb{C} -dimension of X , i.e.

$$d = \sum_{\sigma \in \Sigma_\kappa} a_\sigma b_\sigma.$$

2.6.2. *Complex modular forms.* Similarly, there is an identification

$$(36) \quad G(\mathbb{Q}) \backslash G(\mathbb{R}) \times H_0(\mathbb{C}) \times G(\mathbb{A}_f) / U_\infty K_r \xrightarrow{\sim} \mathcal{E}_r(\mathbb{C})$$

given by sending $(h', h_0, g) \in X \times H_0 \times G(\mathbb{A}_f)$ to $(X_{h',g}, (h_0 \cdot \varepsilon_{h'}, \nu(h_0)))$.

Hence, according to (30), given an dominant character κ of $T_{H_0}(\mathbb{C})$, a modular form $\varphi \in M_\kappa(K_r; \mathbb{C})$ is a smooth holomorphic \mathbb{C} -valued function on $G(\mathbb{R}) \times H_0(\mathbb{C}) \times G(\mathbb{A}_f) = G(\mathbb{A}) \times H_0(\mathbb{C})$ such that

$$\varphi(\gamma g u k, b h_0 u) = \kappa(b) \varphi(g, h_0),$$

for all $\gamma \in G(\mathbb{Q})$, $g \in G(\mathbb{A})$, $u \in U_\infty$, $k \in K_r$, $b \in B_{H_0}(\mathbb{C})$ and $h_0 \in H_0(\mathbb{C})$.

Similarly, let (τ, \mathcal{M}_τ) be a P -nebensystem of level r over \mathbb{C} and view it as a representation of K_r^0 that factors through K_r . A modular form $\varphi \in M_\kappa(K_r, \tau; \mathbb{C})$ can be viewed as a smooth holomorphic function $\varphi : G(\mathbb{A}) \times H_0(\mathbb{C}) \rightarrow \mathcal{M}_\tau^\vee$ such that

$$\varphi(\gamma g u k, b h_0 u) = \kappa(b) \tau^\vee(k) \varphi(g, h_0),$$

for all $k \in K_r^0$, $\gamma \in G(\mathbb{Q})$, $g \in G(\mathbb{A})$, $u \in U_\infty$, $k \in K_r$, $b \in B_{H_0}(\mathbb{C})$ and $h_0 \in H_0(\mathbb{C})$.

2.6.3. *Lie algebra cohomology.* We now reinterpret the above using the algebraic representations W_κ of H_0 associated to κ as in Section 2.3.1.

Let $\mathfrak{g} = \text{Lie}(G(\mathbb{R}))_{\mathbb{C}} = \text{Lie}(G(\mathbb{C}))$ and write

$$\mathfrak{g} = \mathfrak{p}_h^- \oplus \mathfrak{k}_h \oplus \mathfrak{p}_h^+$$

for the Harish-Chandra decomposition corresponding to the $-1, 0$ and 1 eigenspaces of the involution $\text{ad } h(\sqrt{-1})$ respectively.

Then, $\mathfrak{k}_h = \text{Lie}(U_\infty)$ and $\mathfrak{P}_h = \mathfrak{p}_h^- \oplus \mathfrak{k}_h = \text{Lie}(P_h(\mathbb{R}))_{\mathbb{C}} = \text{Lie}(P_h(\mathbb{C}))$. Note that a function φ as above is holomorphic (with respect to the complex structure on $G(\mathbb{R})/U_\infty$) if and only if it vanishes under the action of \mathfrak{p}_h^- .

In what follows, when considering Lie algebra cohomology, we write K_h for $U_\infty = C(\mathbb{R})$ so that $\mathfrak{k}_h = \text{Lie}(K_h)$.

The Borel-Weil theorem states that the set of \mathbb{C} -points of W_κ is

$W_\kappa(\mathbb{C}) = \{\phi : H_0(\mathbb{C}) \rightarrow \mathbb{C} \mid \phi \text{ is holomorphic and } \phi(bx) = \kappa(b)\phi(x), \forall x \in B_{H_0}(\mathbb{C})\}$, which we view as a (\mathfrak{P}_h, K_h) -module, under the identification of $P_0(\mathbb{C})$ with $P_h(\mathbb{C})$ and $H_0(\mathbb{C})$ with $C(\mathbb{C})$.

It is well-known that over \mathbb{C} , one has a natural $G(\mathbb{A}_f)$ -equivariant isomorphism

$$(37) \quad H^i(\mathfrak{P}_h, K_h; \mathcal{A}_0(G) \otimes W_\kappa) = H^i_!(\text{Sh}(V), \omega_\kappa), \text{ for } i = 0 \text{ or } d,$$

where $\mathcal{A}_0(G)$ denotes the space of complex-valued cusp forms on $G(\mathbb{A})$.

Taking $i = 0$ and K_r -equivariance on both sides of (37), one obtains

$$(38) \quad H^0(\mathfrak{P}_h, K_h; \mathcal{A}_0(G) \otimes W_\kappa)^{K_r} = S_\kappa(K_r; \mathbb{C}),$$

which identifies $\varphi \in S_\kappa(K_r; \mathbb{C})$ as above with a function $f : G(\mathbb{A}) \rightarrow W_\kappa(\mathbb{C})$ such that $f(\gamma g u k) = u^{-1} f(g)$, for all $\gamma \in G(\mathbb{Q})$, $g \in G(\mathbb{A})$, $u \in K_h$ and $k \in K_r$. The correspondence is given by $f(g)(x) = \varphi(g, x)$, for all $(g, x) \in G(\mathbb{A}) \times H_0(\mathbb{C})$.

Similarly, taking tensor with \mathcal{M}_τ^\vee over \mathbb{C} (momentarily forgetting the action of \mathcal{L}_r), we obtain an isomorphism

$$(39) \quad \text{Hom}_{\mathbb{C}}(\mathcal{M}_{[\tau]}, H^i(\mathfrak{P}_h, K_h; \mathcal{A}_0(G) \otimes W_\kappa)) = H^i_!(\text{Sh}(V)_{/\mathbb{C}}, \omega_{\kappa, [\tau]}),$$

and taking tensor over $(\mathcal{L}_r, \tau^\vee)$ instead, we obtain

$$(40) \quad \text{Hom}_{\mathcal{L}_r}(\mathcal{M}_\tau, H^i(\mathfrak{P}_h, K_h; \mathcal{A}_0(G) \otimes W_\kappa)) = H^i_!(\text{Sh}(V)_{/\mathbb{C}}, \omega_{\kappa, \tau}),$$

Since $L_H(\mathbb{Z}_p)$ normalizes K_r , taking $i = 0$ as well as K_r -equivariance, we obtain

$$(41) \quad \text{Hom}_{\mathbb{C}}(\mathcal{M}_{[\tau]}, H^0(\mathfrak{P}_h, K_h; \mathcal{A}_0(G) \otimes W_\kappa)) = S_\kappa(K_r, [\tau]; \mathbb{C}),$$

and

$$(42) \quad \text{Hom}_{K_r^0}(\mathcal{M}_\tau, H^0(\mathfrak{P}_h, K_h; \mathcal{A}_0(G) \otimes W_\kappa)) = S_\kappa(K_r, \tau; \mathbb{C}).$$

Then as above, $\varphi_{[\tau]} \in S_\kappa(K_r, [\tau]; \mathbb{C})$ and $\varphi_\tau \in S_\kappa(K_r, \tau; \mathbb{C})$ corresponds via (42) to functions $f_{[\tau]}, f_\tau : G(\mathbb{A}) \rightarrow W_\kappa(\mathbb{C}) \otimes_{\mathbb{C}} \mathcal{M}_\tau^\vee$, respectively, such that

$$f_{[\tau]}(\gamma g u k) = u^{-1} f(g) \quad \text{and} \quad f_\tau(\gamma g u k_0) = \tau^\vee(k)(u^{-1} f(g)),$$

for all $\gamma \in G(\mathbb{Q})$, $g \in G(\mathbb{A})$, $u \in K_h$, $k \in K_r$, and $k_0 \in K_r^0$.

2.7. Duality and integrality. In the previous sections, we mostly dealt with holomorphic modular forms, i.e. degree 0 cohomology. For our purposes however, following the approach of [Ehls20, Sections 6-7], it is necessary to deal with anti-holomorphic modular forms as well, i.e. degree d cohomology where $d = \sum_{\sigma} a_{\sigma} b_{\sigma}$.

2.7.1. Convention on measures. In the following sections, we introduce pairings between modular forms via integration over $G(\mathbb{A})$. We first need to set various conventions.

We fix a Haar measure $dg = \prod_{l \leq \infty} dg_l$ on $G(\mathbb{A})$, where the product runs over all places of \mathbb{Q} , such that the following properties hold :

- (i) Given a finite prime l such that G is unramified at l , dg_l is the normalized Haar measure on $G(\mathbb{Q}_l)$ assigning volume 1 to any hyperspecial maximal compact subgroup.
- (ii) Given a finite prime l such that G splits over l (e.g. when $l = p$), i.e. $G(\mathbb{Q}_l) = \prod_{i=1}^k \mathrm{GL}_{n_i}(F_{v_i})$ where F_{v_i} is a finite extension of \mathbb{Q}_l with ring of integer \mathcal{O}_{v_i} , then dg_l is normalized so that $\prod_{i=1}^k \mathrm{GL}_{n_i}(\mathcal{O}_{v_i})$ has volume 1. In this case, we further write $dg_l = \prod_{i=1}^k dg_{v_i}$, where dg_{v_i} is the standard Haar measure on $\mathrm{GL}_{n_i}(F_{v_i})$ with the obvious normalization.
- (iii) At all finite primes l , the volume of any compact open subgroup of $G(\mathbb{Q}_l)$ with respect to dg_l is rational.
- (iv) For $l = \infty$, dg_{∞} is Tamagawa measure on $G(\mathbb{R})$. We write

$$dg_{\infty} = dk_h \times dx \times dt/t,$$

where dk_h is the unique measure on K_h with total mass equal to 1, dx is a differential form on \mathfrak{p}_h and dt/t is the Lebesgue measure on the center $Z_G(\mathbb{R}) \simeq \mathbb{R}^{\times}$ of $G(\mathbb{R})$.

2.7.2. Unnormalized and normalized Serre duality. We first work with $R = \mathbb{C}$ and introduce an integral version of Serre duality afterward.

Let $\kappa = (\kappa_0, (\kappa_{\sigma})_{\sigma})$ be a dominant weight of T_{H_0} , as in Section 2.3.1. Define

$$a(\kappa) := 2\kappa_0 + \sum_{\sigma} \sum_{j=1}^{b_{\sigma}} \kappa_{\sigma,j} \quad ; \quad \kappa_0^* := -\kappa_0 + a(\kappa) \quad ; \quad \kappa_{\sigma}^* := (-\kappa_{\sigma,b_{\sigma}}, \dots, -\kappa_{\sigma,1}).$$

The highest weight representation W_{κ^*} corresponding to the dominant weight $\kappa^* = (\kappa_0^*, (\kappa_{\sigma}^*)_{\sigma})$ is a representation of $H_0(\mathbb{C})$ such that

$$(43) \quad W_{\kappa^*} \cong W_{\kappa}^{\vee} \otimes \nu^{a(\kappa)},$$

where we recall that ν denotes the similitude character of G .

As briefly mentioned above, we later work with (Lie algebra) cohomology in degree d , hence we are most interested in the $H_0(\mathbb{C})$ -representation

$$\mathrm{Hom}_{\mathbb{C}}(\wedge^d \mathfrak{p}_h^+, W_{\kappa^*}),$$

where the notation is as in Section 2.6.

Since $\mathrm{Hom}_{\mathbb{C}}(\wedge^d \mathfrak{p}_h^+, \mathbb{C})$ is the highest weight representation associated to the dominant weight

$$\kappa_h^+ := (-d, (\kappa_{h,\sigma}^+)_{\sigma}),$$

where $\kappa_{h,\sigma}^+ = (2a_{\sigma}, \dots, 2a_{\sigma})$, it follows that

$$\mathrm{Hom}_{\mathbb{C}}(\wedge^d \mathfrak{p}_h^+, W_{\kappa^*}) = \mathrm{Hom}_{\mathbb{C}}(\wedge^d \mathfrak{p}_h^+, \mathbb{C}) \otimes W_{\kappa^*}$$

is canonically isomorphic to W_{κ^D} , where

$$\kappa^D = \kappa^* + \kappa_h^+.$$

The natural contraction $W_{\kappa} \otimes W_{\kappa}^{\vee} \rightarrow \mathbb{C}$ induces a pairing

$$(44) \quad W_{\kappa} \otimes W_{\kappa^D} \rightarrow \mathrm{Hom}_{\mathbb{C}}(\wedge^d \mathfrak{p}_h^+, \mathbb{C}) \otimes \nu^{a(\kappa)},$$

by definition of W_{κ^D} .

Let $L(\kappa)$ denote the automorphic line bundle over $\mathrm{Sh}(V)$ associated to the character $\nu^{a(\kappa)}$, as in Remark 2.16. Namely, $L(\kappa)$ is topologically isomorphic to $\mathcal{O}_{\mathrm{Sh}(V)}$ but the action of $G(\mathbb{A}_f)$ on $L(\kappa)$ is given by multiplication via $\nu^{a(\kappa)}$. Then, (44) induces a map

$$\omega_{\kappa} \otimes \omega_{\kappa^D} \rightarrow \Omega_{\mathrm{Sh}(V)}^d \otimes L(\kappa).$$

Naturally, one can descend this pairing to ${}_K\mathrm{Sh}$. Furthermore, this pairing extends over any toroidal compactification ${}_K\mathrm{Sh}_{\Omega}$ of ${}_K\mathrm{Sh}$, provided either automorphic vector bundle is replaced by its subcanonical vector bundle. Namely, we have

$$\omega_{\kappa}^{\mathrm{sub}} \otimes \omega_{\kappa^D} \rightarrow \Omega_{{}_K\mathrm{Sh}_{\Omega}}^d \otimes L(\kappa) \quad \text{and} \quad \omega_{\kappa} \otimes \omega_{\kappa^D}^{\mathrm{sub}} \rightarrow \Omega_{{}_K\mathrm{Sh}_{\Omega}}^d \otimes L(\kappa).$$

Then, the *unnormalized* Serre duality pairing is the composition of

$$H_{\dagger}^0(\mathrm{Sh}(V), \omega_{\kappa}) \otimes H_{\dagger}^d(\mathrm{Sh}(V), \omega_{\kappa^D}) \rightarrow \varinjlim_{K, \Omega} H^d({}_K\mathrm{Sh}_{\Omega}, \Omega_{{}_K\mathrm{Sh}_{\Omega}}^d \otimes L(\kappa))$$

with the isomorphism

$$\varinjlim_{K, \Omega} H^d({}_K\mathrm{Sh}_{\Omega}, \Omega_{{}_K\mathrm{Sh}_{\Omega}}^d \otimes L(\kappa)) \xrightarrow{\sim} \varinjlim_{K, \Omega} H^d({}_K\mathrm{Sh}_{\Omega}, \Omega_{{}_K\mathrm{Sh}_{\Omega}}^d)$$

given by multiplication by the global section $g \mapsto \|\nu(g)\|^{a(\kappa)}$ of $L(\kappa)^{\vee}$, and the maps

$$\varinjlim_{K, \Omega} H^d({}_K\mathrm{Sh}_{\Omega}, \Omega_{{}_K\mathrm{Sh}_{\Omega}}^d) \xrightarrow{\sim} C(\pi_0(V)) \rightarrow \mathbb{C},$$

where $C(\pi_0(V))$ is the space of functions on the compact space $\pi_0(V)$ of similitude components of $\mathrm{Sh}(V)$, the isomorphism is the trace map, and the last map is integration over $\pi_0(V)$ with respect to an invariant measure of rational total mass.

The resulting map

$$(45) \quad \langle \cdot, \cdot \rangle_{\kappa}^{\mathrm{Ser}} : H_{\dagger}^0(\mathrm{Sh}(V), \omega_{\kappa}) \otimes H_{\dagger}^d(\mathrm{Sh}(V), \omega_{\kappa^D}) \rightarrow \mathbb{C}$$

is a canonical perfect pairing by [Har90, Corollary 2.3]. As explained above, we can replace either H_{\dagger}^0 or H_{\dagger}^d by H^0 and H^d respectively, but not both at once.

From Definition 2.24 and our discussion in Section 2.6, we see that given a P -nebenotypus τ , the tautological pairing $\langle \cdot, \cdot \rangle_{[\tau]} : \mathcal{M}_{[\tau]} \otimes \mathcal{M}_{[\tau]}^\vee \rightarrow \mathbb{C}$ yields a map

$$\begin{aligned} & H_{\dagger}^0(\mathrm{Sh}(V), \omega_{\kappa, [\tau]}) \otimes_{\mathbb{C}} H_{\dagger}^d(\mathrm{Sh}(V), \omega_{\kappa^D, [\tau^\vee]}) \\ & \xrightarrow{\langle \cdot, \cdot \rangle_{[\tau]}} H_{\dagger}^0(\mathrm{Sh}(V), \omega_{\kappa}) \otimes_{\mathbb{C}} H_{\dagger}^d(\mathrm{Sh}(V), \omega_{\kappa^D}), \end{aligned}$$

Hence, composition of the above with $\langle \cdot, \cdot \rangle_{\kappa}^{\mathrm{Ser}}$ induces a duality

$$(46) \quad \langle \cdot, \cdot \rangle_{\kappa, [\tau]}^{\mathrm{Ser}} : H_{\dagger}^0(\mathrm{Sh}(V), \omega_{\kappa, [\tau]}) \otimes_{\mathbb{C}} H_{\dagger}^d(\mathrm{Sh}(V), \omega_{\kappa^D, [\tau^\vee]}) \rightarrow \mathbb{C}.$$

Upon restriction to holomorphic and anti-holomorphic modular forms of P -nebenotypus τ and τ^\vee respectively, we similarly obtain a perfect pairing

$$(47) \quad \langle \cdot, \cdot \rangle_{\kappa, \tau}^{\mathrm{Ser}} : H_{\dagger}^0(\mathrm{Sh}(V), \omega_{\kappa, \tau}) \otimes_{\mathbb{C}} H_{\dagger}^d(\mathrm{Sh}(V), \omega_{\kappa^D, \tau^\vee}) \rightarrow \mathbb{C}.$$

For our purposes, it is important to compute $\langle \cdot, \cdot \rangle_{\kappa}^{\mathrm{Ser}}$ above in terms of automorphic forms using the isomorphism (37). Let $\varphi \in H^0(\mathfrak{P}_h, K_h; \mathcal{A}_0(G) \otimes W_{\kappa})$ and $\varphi' \in H^d(\mathfrak{P}_h, K_h; \mathcal{A}_0(G) \otimes W_{\kappa^D})$. For every $g \in G(\mathbb{Q})Z_G(\mathbb{R}) \backslash G(\mathbb{A})$, we have $\varphi(g) \in W_{\kappa}$ and

$$\varphi'(g) \in \mathrm{Hom}_{\mathbb{C}}(\wedge^d \mathfrak{p}_h^-, W_{\kappa^D}) = \mathrm{Hom}_{\mathbb{C}}(\wedge^{2d} \mathfrak{p}_h, W_{\kappa}^\vee \otimes \nu^{a(\kappa)}),$$

where $\mathfrak{p}_h = \mathfrak{p}_h^- \oplus \mathfrak{p}_h^+$.

Fix a differential form dx on \mathfrak{p}_h as in Section 2.7.1, i.e. a basis of $(\wedge^{2d} \mathfrak{p}_h)^\vee$. Using dx , we identify this space with \mathbb{C} and obtain a natural map

$$[\cdot, \cdot]_{dx} : W_{\kappa} \otimes \mathrm{Hom}_{\mathbb{C}}(\wedge^d \mathfrak{p}_h^-, W_{\kappa^D}) \rightarrow \mathbb{C}(\nu^{a(\kappa)})$$

Then, the Serre pairing $\langle \cdot, \cdot \rangle_{\kappa}^{\mathrm{Ser}}$ can be normalized (i.e. the invariant measure on $\pi_0(V)$ can be normalized) so that

$$(48) \quad \langle \varphi, \varphi' \rangle_{\kappa}^{\mathrm{Ser}} = \int_{G(\mathbb{Q})Z_G(\mathbb{R}) \backslash G(\mathbb{A})} [\varphi(g), \varphi'(g)]_{dx} \|\nu(g)\|^{-a(\kappa)} dg,$$

where dg is as in Section 2.7.1.

Remark 2.30. We later use this formula when φ is essentially a value of the Eisenstein measure from Proposition 11.8, as a modular form on G_3 , and φ' is the tensor product of two P -anti-ordinary cusp forms, one on G_1 and the other on G_2 . The groups G_1 , G_2 and G_3 are defined in Section 4.

To define a normalized version of (45), fix a compact open subgroup $K_r = I_r K^p \subset G(\mathbb{A}_f)$. Denote the volume of $K_r^0 = I_r^0 K_p$ with respect to the Tamagawa measure dg above by $\mathrm{Vol}(I_r^0)$.

Then, the *normalized* Serre pairing is the perfect pairing

$$(49) \quad \langle \cdot, \cdot \rangle_{\kappa, K_r} : H_{\dagger}^0(K_r, \mathrm{Sh}(V), \omega_{\kappa}) \otimes H_{\dagger}^d(K_r, \mathrm{Sh}(V), \omega_{\kappa^D}) \rightarrow \mathbb{C}$$

defined via $\langle \cdot, \cdot \rangle_{\kappa, K_r} = \mathrm{Vol}(I_r)^{-1} \langle \cdot, \cdot \rangle_{\kappa}^{\mathrm{Ser}}$. Similarly, we set

$$\langle \cdot, \cdot \rangle_{\kappa, K_r, \tau} = \mathrm{Vol}(I_r)^{-1} \langle \cdot, \cdot \rangle_{\kappa, \tau}^{\mathrm{Ser}} \quad \text{and} \quad \langle \cdot, \cdot \rangle_{\kappa, K_r, [\tau]} = \mathrm{Vol}(I_r)^{-1} \langle \cdot, \cdot \rangle_{\kappa, [\tau]}^{\mathrm{Ser}}.$$

The advantage of this normalized pairing is that it commutes with change of level maps. Namely, fix $r' \geq r$, $\varphi \in H_!^0(K_r, \text{Sh}(V), \omega_\kappa)$ and $\varphi' \in H_!^d(K_{r'}, \text{Sh}(V), \omega_{\kappa^D})$. The trace map from level $K_{r'}$ to level K_r maps φ' to

$$(50) \quad \text{tr}_{K_r/K_{r'}}(\varphi') := \frac{\#(I_r^0/I_r)}{\#(I_{r'}^0/I_{r'})} \sum_{\gamma \in K_r/K_{r'}} \gamma \cdot \varphi' \in H_!^d(K_r, \text{Sh}(V), \omega_{\kappa^D})$$

By definition, we have $I_r^0/I_r \simeq K_r^0/K_r$ and

$$\text{Vol}(I_r^0) = \text{Vol}(I_{r'}^0) \cdot \#(I_r^0/I_{r'}^0) = \text{Vol}(I_{r'}^0) \cdot \frac{\#(I_r^0/I_r)}{\#(I_{r'}^0/I_{r'})} \cdot \#(K_r/K_{r'}),$$

therefore one readily obtains

$$(51) \quad \langle \varphi, \varphi' \rangle_{\kappa, K_{r'}} = \langle \varphi, \text{tr}_{K_r/K_{r'}}(\varphi') \rangle_{\kappa, K_r},$$

as well as an analogous formula when φ has level $K_{r'}$ and φ' has level K_r .

From (40) and the fact that $L_{H(\mathbb{Z}_p)}$ normalizes K_r , one readily sees that the formula (50) is also well-defined on $H_!^d(K_{r'}, \text{Sh}(V), \omega_{\kappa^D, r, \tau^\vee})$ and $H_!^d(K_r, \text{Sh}(V), \omega_{\kappa^D, r, [\tau^\vee]})$, for $r' > r$, and yields a trace maps

$$(52) \quad \text{tr}_{K_r/K_{r'}} : H_!^d(K_{r'}, \text{Sh}(V), \omega_{\kappa^D, r', \tau^\vee}) \rightarrow H_!^d(K_r, \text{Sh}(V), \omega_{\kappa^D, r, \tau^\vee}).$$

and

$$(53) \quad \text{tr}_{K_r/K_{r'}} : H_!^d(K_{r'}, \text{Sh}(V), \omega_{\kappa^D, r', [\tau^\vee]}) \rightarrow H_!^d(K_r, \text{Sh}(V), \omega_{\kappa^D, r, [\tau^\vee]}).$$

It follows from (51) that the pairings $\langle \cdot, \cdot \rangle_{\kappa, K_r, \tau}$ and $\langle \cdot, \cdot \rangle_{\kappa, K_r, [\tau]}$ are again stable under trace maps.

2.7.3. Integral structures on (anti-)holomorphic modular forms. Recall that $S_0 = \mathcal{O}_{\mathcal{K}'/(p')}$ as in Section 2.3.1. Naturally, we define S_0 -integral structure of the \mathbb{C} -vector space $H_!^0(K_r, \text{Sh}(V)/\mathbb{C}, \omega_\kappa)$ as $S_\kappa(K_r, S_0)$. More generally, for any S_0 -algebra R , we set its R -structure to be $S_\kappa(K_r, R)$. This is obviously the structure induced by the R -structure of $H_!^0(K_r, \text{Sh}(V))$.

On the other hand, we do not define the R -integral structure for the space of *anti-holomorphic* forms as the one induced by the R -structure of the underlying schemes. This is to avoid the singularities of the special fibers of $H_!^d(K_r, \text{Sh}(V))_{S_0}$ as r grows. We instead use duality with respect to $\langle \cdot, \cdot \rangle_{\kappa, K_r}$, following the approach of [Ehls20, Section 6.4.2].

Firstly, motivated by the identification (37), we refer to

$$\widehat{S}_\kappa(K_r; \mathbb{C}) := H_!^d(K_r, \text{Sh}(V)/\mathbb{C}, \omega_{\kappa^D})$$

as the space of *anti-holomorphic* cusp forms on G of weight κ and level K_r over \mathbb{C} (note the twist by κ^D). Similarly, given a P -nebenotypus τ of level r , we set

$$\widehat{S}_\kappa(K_r, \tau; \mathbb{C}) := H_!^d(K_r, \text{Sh}(V)/\mathbb{C}, \omega_{\kappa^D, r, \tau^\vee}).$$

Then, by definition of $\langle \cdot, \cdot \rangle_{\kappa, K_r}$ and $\langle \cdot, \cdot \rangle_{\kappa, K_r, \tau}$, we have perfect pairings

$$S_\kappa(K_r; \mathbb{C}) \otimes \widehat{S}_\kappa(K_r; \mathbb{C}) \rightarrow \mathbb{C} \quad \text{and} \quad S_\kappa(K_r, \tau; \mathbb{C}) \otimes \widehat{S}_\kappa(K_r, \tau; \mathbb{C}) \rightarrow \mathbb{C},$$

and

We define the S_0 -integral structure $\widehat{S}_\kappa(K_r; S_0)$ of $H_1^d(K_r \text{Sh}/\mathbb{C}, \omega_{\kappa^D})$ as the S_0 -dual of $S_\kappa(K_r; S_0)$ via the pairing (49). Similarly, we define the $S_0[\tau]$ -integral structure $\widehat{S}_\kappa(K_r, \tau; S_0[\tau])$ of $H_1^d(K_r \text{Sh}/\mathbb{C}, \omega_{\kappa^D, r, \tau^\vee})$ as the $S_0[\tau]$ -dual of $S_\kappa(K_r, \tau; S_0[\tau])$ via the pairing $\langle \cdot, \cdot \rangle_{\kappa, K_r, \tau}$.

Given any S_0 -algebra or $S_0[\tau]$ -algebra R , let $\widehat{S}_\kappa(K_r; R) := \widehat{S}_\kappa(K_r; S_0) \otimes_{S_0} R$ and $\widehat{S}_\kappa(K_r, \tau; R) := \widehat{S}_\kappa(K_r, \tau; S_0[\tau]) \otimes_{S_0[\tau]} R$. This yields identifications

$$\widehat{S}_\kappa(K_r; R) = \text{Hom}_{S_0}(S_\kappa(K_r; S_0), R)$$

and

$$\widehat{S}_\kappa(K_r, \tau; R) = \text{Hom}_{S_0[\tau]}(S_\kappa(K_r, \tau; S_0[\tau]), R),$$

via $\langle \cdot, \cdot \rangle_{\kappa, K_r}$ and $\langle \cdot, \cdot \rangle_{\kappa, K_r, \tau}$, respectively. The R -integral structure $\widehat{S}_\kappa(K_r, [\tau]; R)$ of $\widehat{S}_\kappa(K_r, [\tau]; \mathbb{C})$ is defined similarly using the pairing $\langle \cdot, \cdot \rangle_{\kappa, K_r, [\tau]}$.

2.8. P -(anti)-ordinary modular forms. We return to the notation of Section 2.2.2. For instance, let $t_{w, D_w(j)} \in \text{GL}_n(\mathcal{O}_w)$, for $w \in \Sigma_p$ and $j = 1, \dots, r_w$, be the matrix defined in (15), let $t_{w, D_w(j)}^+$ be the corresponding element of $G(\mathbb{Q}_p)$, and let $t_{w, D_w(j)}^- = (t_{w, D_w(j)}^+)^{-1}$.

Given any $r \geq 1$, we consider the double coset operators

$$U_{w, D_w(j)} = [K_r t_{w, D_w(j)}^+ K_r] \quad \text{and} \quad U_{w, D_w(j)}^- = [K_r t_{w, D_w(j)}^- K_r].$$

One can easily write down a set of right coset representatives for $K_r t_{w, D_w(j)}^+ K_r$ (resp. $K_r t_{w, D_w(j)}^- K_r$) that does not depend on r , (see Section 6.1.1 for instance). This partly motivates why we omit r from the notation $U_{w, D_w(j)}$ (resp. $U_{w, D_w(j)}^-$).

If R is an S_0 -algebra in which p is invertible and κ be a dominant character of T_{H_0} over R , then both $U_{w, D_w(j)}$ and $U_{w, D_w(j)}^-$ define Hecke operators on $M_\kappa(K_r; R)$ and $S_\kappa(K_r; R)$ via (32) by proceeding as in Section 2.4.3. Naturally, they also both define Hecke operators on $M_\kappa(K_r, \tau; R)$ and $S_\kappa(K_r, \tau; R)$ (if R is an $S_0[\tau]$ -algebra).

One usually normalize these operators as follows : First, define

$$(54) \quad \kappa_{\text{norm}, \sigma} = (\kappa_{\sigma, 1} - b_\sigma, \dots, \kappa_{\sigma, b_\sigma} - b_\sigma) \in \mathbb{Z}^{b_\sigma},$$

and consider the character $\kappa_{\text{norm}} = (\kappa_0, (\kappa_{\text{norm}, \sigma})_{\sigma \in \Sigma_\kappa})$ of T_{H_0} . Let κ' denote the p -adic weight associated to κ_{norm} as in (27), i.e. $\kappa' = (\kappa_{\text{norm}})_p$.

We define the j -th normalized Hecke operators at p of weight κ as

$$(55) \quad u_{w, j}^\pm = u_{w, j, \kappa}^\pm := \kappa'(t_{w, j}^\pm) U_{w, j}^\pm,$$

and the Hecke operators at p of weight κ with respect to P as

$$(56) \quad u_{P,p}^\pm = u_{P,p,\kappa}^\pm := \prod_{w \in \Sigma_p} \prod_{j=1}^{r_w} u_{w,D_w(j),\kappa}^\pm.$$

Remark 2.31. When working with $t_{w,j}^+$, we often omit the $+$ superscript in the notation above and simply write $u_{w,j}$ and $u_{P,p}$.

Remark 2.32. The operators $u_{w,D_w(j),\kappa}$ can be interpreted as correspondences on the Igusa tower associated to G , see [EHL20, Section 2.9.5], [Hid04, Section 8.3.1] or [SU02]. We recall this more formally later when introducing p -adic modular forms with respect to P , see Section 5. This plays a crucial role to p -adically interpolate all operators $u_{w,D_w(j),\kappa}$ as κ varies.

In later section, we also consider the action of the center Z_P of $L_P(\mathbb{Z}_p)$ on $M_\kappa(K_r; R)$. Namely, any $t \in Z_P$ naturally induces a correspondence on Shimura varieties via the double coset operator $U_p(t) := [K_r t K_r] = [t K_r]$. Clearly, this action factors through the center $Z_{P,r} = Z_P/p^r Z_P$ of $L_P(\mathbb{Z}_p/p^r \mathbb{Z}_p)$. As above, we normalize these operators by setting $u_{p,\kappa}(t) := \kappa'(t)U_p(t)$.

2.8.1. *P-ordinary case.* The P -ordinary subspace of a module is later defined as the subspace on which $u_{P,p}$ acts via a generalized eigenvalue which a p -adic unit (when this action is well-defined). This feature can be detected using the following P -ordinary Hecke projector.

Assume the ring R above is also a p -adic ring, i.e. $R = \varprojlim_i R/p^i R$. Then, the action of the limit

$$(57) \quad e_P = e_{P,\kappa} := \varinjlim_n u_{P,p,\kappa}^{n!}$$

on $M_\kappa(K_r; R)$ induced by the action of $u_{P,p,\kappa}$ is well-defined. Since $u_{P,p,\kappa}$ commutes with $G(\mathbb{A}^p)$ and $L_P(\mathbb{Z}_p)$, it stabilizes $S_\kappa(K_r, \tau)$, $M_\kappa(K_r, [\tau]; R)$ and $S_\kappa(K_r, [\tau]; R)$.

Remark 2.33. In later sections, we also consider the case where $R \subset \mathbb{C}$ is a localization of a finite S_0 -algebra at the maximal prime determined by incl_p or the completion of such a ring. In this situation, the limit operator e_P is again well-defined.

It is well-known that the eigenvalues of the generalized eigenspaces for each $u_{w,j,\kappa}$ is a p -adic integer. Hence, by definition, e_P acts as the identity on the generalized eigenspace of $M_\kappa(K_r; R)$ associated to eigenvalues with p -adic valuation 1 and is 0 on all other generalized eigenspaces.

We write

$$M_\kappa^{P\text{-ord}}(K_r; R) := e_{P,\kappa} M_\kappa(K_r; R)$$

and define $S_\kappa^{P\text{-ord}}(K_r; R)$, $M_\kappa^{P\text{-ord}}(K_r, [\tau]; R)$ and $S_\kappa^{P\text{-ord}}(K_r, [\tau]; R)$ similarly.

2.8.2. *P -anti-ordinary case.* An easy computation shows that

$$\langle U_{w, D_w(j), \kappa} \varphi, \varphi' \rangle_{\kappa, K_r} = \langle \varphi, U_{w, D_w(j), \kappa}^- \varphi' \rangle_{\kappa, K_r},$$

for all $\varphi \in S_{\kappa}(K_r; R)$, $\varphi' \in \widehat{S}_{\kappa}(K_r; R)$, $w \in \Sigma_p$ and $1 \leq j \leq D_w(j)$.

Therefore, the kernel of the projection

$$\widehat{S}_{\kappa}(K_r; R) \rightarrow e_{P, \kappa}^- \widehat{S}_{\kappa}(K_r; R)$$

is exactly the annihilator (in $\widehat{S}_{\kappa}(K_r; R)$) of $S_{\kappa}^{P\text{-ord}}(K_r; R)$ via $\langle \cdot, \cdot \rangle_{\kappa, K_r}$. In other words, (49) induces a perfect pairing

$$S_{\kappa}^{P\text{-ord}}(K_r; R) \otimes \widehat{S}_{\kappa}^{P\text{-a.ord}}(K_r; R) \rightarrow R,$$

where

$$\widehat{S}_{\kappa}^{P\text{-a.ord}}(K_r; R) := e_{P, \kappa}^- \widehat{S}_{\kappa}(K_r; R)$$

is the *P -anti-ordinary* subspace of $\widehat{S}_{\kappa}(K_r; R)$. Similarly, one can view

$$\widehat{S}_{\kappa}^{P\text{-a.ord}}(K_r, \tau; R) := e_{P, \kappa}^- \widehat{S}_{\kappa}(K_r, \tau; R)$$

as the R -dual of $S_{\kappa}^{P\text{-ord}}(K_r, \tau; R)$ via $\langle \cdot, \cdot \rangle_{\kappa, K_r, \tau}$.

If R is an $S_r[\tau]$ -algebra, a similar statement holds for $\widehat{S}_{\kappa}^{P\text{-a.ord}}(K_r, [\tau]; R) = e_{P, \kappa}^- \widehat{S}_{\kappa}(K_r, [\tau]; R)$ and $S_{\kappa}^{P\text{-ord}}(K_r, [\tau]; R)$ with respect to the pairing $\langle \cdot, \cdot \rangle_{\kappa, K_r, [\tau]}$.

3. P -(ANTI)-ORDINARY (ANTI)-HOLOMORPHIC AUTOMORPHIC REPRESENTATIONS.

In this section, we frequently use (37) to pass between the language of automorphic forms in $\mathcal{A}_0(G)$ and modular forms as global sections on Shimura varieties. We recall the following convenient notions of holomorphic and anti-holomorphic automorphic representations, following [EHL20, Section 6.5]. We then define P -ordinary and P -anti-ordinary automorphic representations and their P -nebenypus, motivated by Sections 2.5–2.8.

3.1. (Anti-)holomorphic automorphic representations. We continue with the notation of Section 2.6. Recall that we denote the space of cusp forms on G by $\mathcal{A}_0(G)$.

We refer to the irreducible $(\mathfrak{g}, K_h) \times G(\mathbb{A}_f)$ -subrepresentations of $\mathcal{A}_0(G)$ as *cuspidal automorphic representations* of G . In particular, we always assume that a cuspidal automorphic representation π is irreducible.

Furthermore, we write $\pi = \pi_{\infty} \otimes \pi_f$ where π_{∞} is an irreducible (\mathfrak{g}, K_h) -module and π_f is an irreducible $G(\mathbb{A}_f)$ admissible representation.

Let κ be a dominant character of T_{H_0} , as in Section 2.3.1.

Definition 3.1. We say that π is *holomorphic of weight κ* if

$$H^0(\mathfrak{P}_h, K_h; \pi \otimes W_{\kappa}) \neq 0$$

and say that π is *anti-holomorphic of weight κ* if

$$H^d(\mathfrak{P}_h, K_h; \pi \otimes W_{\kappa^D}) \neq 0$$

instead. Clearly, π cannot be both holomorphic and anti-holomorphic (except possibly if $d = 0$).

Equivalently, π is holomorphic of weight κ if and only if

$$H^0(\mathfrak{P}_h, K_h; \pi_\infty \otimes W_\kappa) \neq 0,$$

in which case the latter is 1-dimensional. Similarly, π is anti-holomorphic of weight κ if and only if

$$H^d(\mathfrak{P}_h, K_h; \pi_\infty \otimes W_{\kappa^D}) \neq 0$$

is 1-dimensional over \mathbb{C} .

Remark 3.2. As mentioned in [EHL20, Remark 6.5.2], if π is holomorphic or anti-holomorphic, then π_f is defined over some number field $E(\pi)$. One can choose $E(\pi)$ to be a CM field. See [BHR94] for more details. Enlarging $E(\pi)$ if necessary, we always assume it contains \mathcal{K}' .

3.1.1. *Ramified places away from p .* Let $\pi = \pi_\infty \otimes \pi_f$ be any cuspidal automorphic representation of G . Let $K \subset G(\mathbb{A}_f)$ be any open compact subgroup such that $\pi_f^K \neq 0$. We sometimes say that π_f (or π) has level K in this case.

Let $l \neq p$ be any prime of \mathbb{Q} and consider the set \mathcal{P}_l of all primes of \mathcal{K}^+ above l . Write $\mathcal{P}_l = \mathcal{P}_{l,1} \amalg \mathcal{P}_{l,2}$, where $\mathcal{P}_{l,1}$ is the subset of such primes that split in \mathcal{K} and $\mathcal{P}_{l,2}$ is the complement. Therefore, one naturally has an identification

$$G(\mathbb{Q}_l) = \prod_{v \in \mathcal{P}_{l,1}} \mathrm{GL}_n(\mathcal{K}_v^+) \times G_{l,2},$$

where $G_{l,2}$ is the subgroup of elements $((x_w), t) \in \prod_{w \in \mathcal{P}_{l,2}} \mathrm{GL}_n(\mathcal{K}_w) \times \mathbb{Q}_l^\times$ such that each x_w preserve the Hermitian form on $V \times_{\mathcal{K}} \mathcal{K}_w$ with the same similitude factor t .

Let $S_l = S_l(K_l)$ be the subset of \mathcal{P}_l consisting of all places at which K_l does not contain a hyperspecial subgroup. Let $S_{l,i} = S_l \cap \mathcal{P}_{l,i}$ and define

$$G(\mathbb{Q}_l)^{S_l} = \begin{cases} \prod_{v \in \mathcal{P}_{l,1} \setminus S_{l,1}} \mathrm{GL}_n(\mathcal{K}_v^+) \times G_{l,2}, & \text{if } S_{l,2} = \emptyset \\ \prod_{v \in \mathcal{P}_{l,1} \setminus S_{l,1}} \mathrm{GL}_n(\mathcal{K}_v^+), & \text{otherwise.} \end{cases}$$

In particular, we simply have $G(\mathbb{Q}_l)^{S_l} = G(\mathbb{Q}_l)$ if S_l is empty. Then, let $S = S(K^p)$ be the set of primes $l \neq p$ such that S_l is nonempty. We define

$$(58) \quad G(\mathbb{A}_f^S) = \prod_{l \notin S} G(\mathbb{Q}_l) \times \prod_{l \in S} G(\mathbb{Q}_l)^{S_l},$$

and set $S_p = S_p(K^p) := S \cup \{p\}$ (not to be confused with the ring S_p from Section 2.1.1 which not consider in what follows).

3.1.2. *Spherical vectors.* By definition, for each $l \notin S_p$, π contains a K_l -spherical vector. We fix such a choice $0 \neq \varphi_{l,0} \in \pi_f^{K_l}$ and consider the corresponding factorization

$$(59) \quad \pi_f \xrightarrow{\sim} \widehat{\bigotimes}_l \pi_l,$$

where the restricted tensor product is with respect to our choice of $\varphi_{l,0}$ for each $l \notin S_p$. In particular, π_f^K is identified with

$$(60) \quad \pi_p^{K_p} \otimes \pi_S^{K_S}.$$

Remark 3.3. If π is holomorphic or anti-holomorphic, then we may assume that each $\varphi_{l,0}$ is $E(\pi)$ -rational, see Remark 3.2.

3.1.3. *Contragredient representations and pairings.* Let π^\vee be the contragredient representation of π , and write $\pi^\vee = \pi_\infty^\vee \otimes \pi_f^\vee$ for its decomposition as a $(\mathfrak{g}, K_h) \times G(\mathbb{A}_f)$ -module.

It is well-known that π^\vee is isomorphic to a twist of the complex conjugate $\bar{\pi}$ of π (see (79) for instance). Therefore, π^\vee is again a cuspidal automorphic representation of G .

We identify the tautological pairing $\langle \cdot, \cdot \rangle_\pi : \pi \times \pi^\vee$ of contragredient representation with

$$\langle \varphi, \varphi^\vee \rangle_\pi = \int_{Z \cdot G(\mathbb{Q}) \backslash G(\mathbb{A})} \varphi(g) \varphi^\vee(g) dg, \quad \text{for } \varphi \in \pi, \varphi^\vee \in \pi^\vee,$$

where dg is the Haar measure on $G(\mathbb{A})$ introduced in Section 2.7.1 and Z is the group of real points of the maximal \mathbb{Q} -split subgroup of the center of G .

Suppose that π has level K and let $S_p = S(K^p) \cup \{p\}$ denote the set of ramified places of π , as in Section 3.1.1. Then, both π and π^\vee contain a K_l -spherical vector for each $l \notin S_p$. Fix such vectors $\varphi_{l,0} \in \pi$ and $\varphi_{l,0}^\vee \in \pi^\vee$ for each $l \notin S_p$.

Consider factorization $\pi_f \xrightarrow{\sim} \widehat{\bigotimes}_l \pi_l$ and $\pi_f^\vee \xrightarrow{\sim} \widehat{\bigotimes}_l \pi_l^\vee$ into restricted tensor products over the finite places of \mathbb{Q} , with respect to the vectors $\varphi_{l,0}$ and $\varphi_{l,0}^\vee$, as in (59).

For each place l of \mathbb{Q} , we identify π_l^\vee as the contragredient of π_l . Namely, we fix a $G(\mathbb{Q}_l)$ -equivariant perfect pairing $\langle \cdot, \cdot \rangle_{\pi_l} : \pi_l \otimes \pi_l^\vee \rightarrow \mathbb{C}$. We normalize such pairings so that $\langle \varphi_{l,0}, \varphi_{l,0}^\vee \rangle = 1$ for all finite place $l \notin S_p$.

There exists a constant C (depending on all the choices made above) such that for each pure tensor vectors $\varphi = \otimes_l \varphi_l \in \pi$ and $\varphi^\vee = \otimes_l \varphi_l^\vee \in \pi^\vee$, we have

$$(61) \quad \langle \varphi, \varphi^\vee \rangle_\pi = C \prod_l \langle \varphi_l, \varphi_l^\vee \rangle_{\pi_l},$$

where the product is over all places l of \mathbb{Q} .

3.2. P -(anti)-ordinary automorphic representations. The identifications (5), (6) and (7) induce an isomorphism

$$(62) \quad G(\mathbb{Q}_p) \xrightarrow{\sim} \mathbb{Q}_p^\times \times \prod_{w \in \Sigma_p} G_w ,$$

where $G_w = \mathrm{GL}_n(\mathcal{K}_w)$. Consequently, given any automorphic representation π , its p -factor π_p decomposes as

$$(63) \quad \pi_p \cong \mu_p \otimes \left(\bigotimes_{w \in \Sigma_p} \pi_w \right) ,$$

where μ_p is a character of \mathbb{Q}_p^\times and π_w is an irreducible admissible representation of G_w .

Consider the groups

$$P \xrightarrow{\sim} \prod_{w \in \Sigma_p} P_w \quad ; \quad I_r^0 \xrightarrow{\sim} \mathbb{Z}_p^\times \times \prod_{w \in \Sigma_p} I_{w,r}^0 \quad ; \quad I_r \xrightarrow{\sim} \mathbb{Z}_p^\times \times \prod_{w \in \Sigma_p} I_{w,r}$$

constructed in Section 2.2.2.

Fix a compact open subgroup $K_r = I_r K^p \subset G(\mathbb{A}_f)$ as in Section 2.4 such that $\pi_f^{K_r} \neq 0$, i.e. π_f has level K_r . Then, $\pi_p^{I_r} \neq 0$ and in particular, μ_p is unramified.

3.2.1. P -ordinary case. Assume that π is holomorphic of weight κ . Recall that κ is a character of T_{H_0} identified with a tuple $(\kappa_0, (\kappa_\sigma)_{\sigma \in \Sigma_\mathcal{K}})$ such that $\kappa_0 \in \mathbb{Z}$ and $\kappa_\sigma \in \mathbb{Z}^{b_\sigma}$. Let $\kappa' = (\kappa_{\mathrm{norm}})_p$ be the normalized p -adic weight related to κ as in Section 2.8, see (54).

To lighten the notation, for every $w \in \Sigma_p$, $1 \leq j \leq r_w$, we set

$$k_w(j) := \kappa'(t_{w,D_w(j)}^+) = \kappa'(t_{w,D_w(j)}^-)^{-1} = \left| \kappa'(t_{w,D_w(j)}^+) \right|_p^{-1} ,$$

where $t_{w,D_w(j)}^+, t_{w,D_w(j)}^- \in L_P(\mathbb{Q}_p)$ are introduced at the end of Section 2.2.2, and are both related to the diagonal matrix $t_{w,D_w(j)} \in \mathrm{GL}_n(\mathcal{O}_w)$ from (15).

The normalized Hecke operator $u_{w,D_w(j)}$ defined in (55) naturally acts on $\pi_f^{K_r}$ via the action of $k_w(j)[I_r t_{w,D_w(j)} I_r]$ on $\pi_p^{I_r}$. The factorization in (63) clearly indicates that this corresponds to the action of $k_w(j) U_{w,D_w(j)}^{\mathrm{GL}}$ on $\pi_w^{I_{w,r}}$, where $U_{w,D_w(j)}^{\mathrm{GL}} = [I_{w,r} t_{w,D_w(j)} I_{w,r}]$.

By abusing notation, we denote all of these normalized double coset operators by $u_{w,D_w(j)}$ (or $u_{w,D_w(j),\kappa}$). It is well-known that the generalized eigenvalues of all the operators $u_{w,D_w(j)}$ on $\pi_p^{I_r}$ are p -adically integral. Therefore, the limit in (57) again induces an operator

$$\lim_n \left(\prod_{w \in \Sigma_p} \prod_{j=1}^{r_w} u_{w,D_w(j)} \right)^{n!}$$

on $\pi_p^{I_r}$, which we still denote e_P . It projects $\pi_p^{I_r}$ onto its subspace spanned by generalized eigenspaces associated to generalized eigenvalues that are p -adic units.

Definition 3.4. We say that π as above is P -ordinary of level r if

$$\pi_p^{(P\text{-ord}, r)} := e_{P, \kappa}(\pi_p^{I_r}) \neq 0.$$

Remark 3.5. When $P = B$, a result of Hida (see [Hid98, Corollary 8.3] or [Ehls20, Theorem 6.6.9]) implies that the space of B -ordinary vectors (or simply *ordinary* vectors) is at most 1-dimensional and does not depend on $r \gg 0$. This is no longer true for general parabolic subgroups P . However, Theorem 6.9 yields an analogous result for P -ordinary subspaces.

By working locally at $w \in \Sigma_p$, we see that equivalently, the limit

$$(64) \quad e_{P, w, \kappa} = e_{P, w} := \varinjlim_n \left(\prod_{j=1}^{r_w} u_{w, D_w(j), \kappa} \right)^{n!}.$$

defines an operator on $\pi_w^{I_{w, r}}$. We refer to $e_{P, w}$ as the P_w -ordinary projection operator.

We see that π is P -ordinary if and only if there exists some $r \gg 0$ such that, for all $w \in \Sigma_p$, there exists $0 \neq \phi_w \in \pi_w^{I_{w, r}}$ satisfying $e_{P, w} \phi_w = \phi_w$. Such a vector ϕ_w must then also satisfy $u_{w, D_w(j)} \phi_w = c_{w, D_w(j)} \phi_w$ for some p -adic unit $c_{w, D_w(j)}$.

We say that π_w is P_w -ordinary of level r and that ϕ_w is a P_w -ordinary vector of level r . We let

$$\pi_w^{(P_w\text{-ord}, r)} := e_{P, w, \kappa}(\pi_w^{I_{w, r}})$$

denote the space of all P_w -ordinary vectors of level r .

By definition, π is P -ordinary of level r if and only if μ_p is unramified and π_w is P_w -ordinary of level r for all $w \in \Sigma_p$.

3.2.2. P -anti-ordinary case. For the dual notion, assume π is anti-holomorphic of weight κ . We retain the assumption that $\pi_f^{K_r} \neq 0$.

For each $w \in \Sigma_p$ and $1 \leq j \leq r_w$, the natural action of the operator $u_{w, D_w(j)}^-$ from Section 2.8 on $\pi_f^{K_r}$ factors through the action of $k_w(j)^{-1} [I_r t_{w, D_w(j)}^- I_r]$ on $\pi_p^{I_r}$. Once more, this action is induced by the one of $k_w(j)^{-1} U_{w, D_w(j)}^{\text{GL}, -}$ on $\pi_w^{I_{w, r}}$, where $U_{w, D_w(j)}^{\text{GL}, -} = [I_{w, r} t_{w, D_w(j)}^- I_{w, r}]$. Again, by abuse of notation, we denote all these normalized double coset operators by $u_{w, D_w(j)}^-$.

As in the P -ordinary case, the generalized eigenvalues of $u_{w, D_w(j)}^-$ are p -adic integers, hence the limits

$$(65) \quad e_{P, \kappa}^- = \varinjlim_n \left(\prod_{w \in \Sigma_p} \prod_{j=1}^{r_w} u_{w, D_w(j)}^- \right)^{n!} \quad \text{and} \quad e_{P, w, \kappa}^- = \varinjlim_n \left(\prod_{j=1}^{r_w} u_{w, D_w(j)}^- \right)^{n!}$$

yield well-defined projection operators on $\pi_p^{I_r}$ and $\pi_w^{I_{w, r}}$ respectively.

Definition 3.6. Let π as above be an anti-holomorphic cuspidal automorphic representation of weight κ such that $\pi_f^{K_r} \neq 0$. We say that π as above is *P-anti-ordinary of level r* if $\pi_p^{(P\text{-a.ord}, r)} := e_{P, \kappa}^-(\pi_p^{I_r}) \neq 0$. Similarly, we say that π_w is *P_w-anti-ordinary of level r* if $\pi_w^{(P_w\text{-a.ord}, r)} := e_{P_w, \kappa}^-(\pi_w^{I_w, r}) \neq 0$.

3.3. P-(anti)-weight-level-type. Let π be any cuspidal automorphic representation of G . Assume that π is holomorphic (resp. anti-holomorphic) of weight κ and P -ordinary (resp. P -anti-ordinary) of level r . Let Π_r denote $\pi_p^{(P\text{-ord}, r)}$ (resp. $\pi_p^{(P\text{-a.ord}, r)}$).

Since I_r is normal in the P -Iwahori subgroup $I_r^0 = I_{P, r}^0 \subset G(\mathbb{Z}_p)$ of level r and the matrices $t_{w, D_w(j)}^\pm$, for $w \in \Sigma_p$ and $1 \leq j \leq r_w$, commute with $I_r^0/I_r = L_P(\mathbb{Z}_p/p^r\mathbb{Z}_p)$, we see that the action of I_r^0 on π_p stabilizes Π_r .

In particular, we can decompose Π_r as a direct sum of isotypic components over the (finite-dimensional) irreducible representations of $L_P(\mathbb{Z}_p/p^r\mathbb{Z}_p)$.

Definition 3.7. Let π be any cuspidal automorphic representation of G such that $\pi_f^{K_r} \neq 0$, for some $r \gg 0$. Let τ be some smooth irreducible representation of $L_P(\mathbb{Z}_p)$ factoring through $L_P(\mathbb{Z}_p/p^r\mathbb{Z}_p)$. Let κ be some dominant weight of T_{H_0} .

We say that π has *P-weight-level-type* (κ, K_r, τ) if π is holomorphic of weight κ , π_f has level K_r and is P -ordinary, and the τ -isotypic component $\pi_p^{(P\text{-ord}, r)}[\tau]$ of $\pi_p^{(P\text{-ord}, r)}$ is nonzero. We often say that π has *P-WLT* (κ, K_r, τ) .

For the dual notion, we say that π has *P-anti-weight-level-type* (κ, K_r, τ) if π is anti-holomorphic of weight κ , π_f has level K_r and is P -anti-ordinary, and the τ^\vee -isotypic component $\pi_p^{(P\text{-a.ord}, r)}[\tau^\vee]$ of $\pi_p^{(P\text{-a.ord}, r)}$ is nonzero. We often say that π has *P-anti-WLT* (κ, K_r, τ) .

Remark 3.8. In Definition 2.9, one could replace $I_{P, r}$ with the collection of $g \in G(\mathbb{Z}_p)$ such that $g \bmod p^r$ is in $(\mathbb{Z}_p/p^r\mathbb{Z}_p)^\times \times SP(\mathbb{Z}_p/p^r\mathbb{Z}_p)$. Here, SP is the derived subgroup of P as introduced above Definition 2.9. Let us write the corresponding group by $I_{SP, r}$ momentarily, so that we have $I_{P, r} \subset I_{SP, r} \subset I_{P, r}^0$.

Then, one can define P -ordinary representations of G using $I_{SP, r}$ instead of $I_{P, r}$. By doing so, the space of P -ordinary vectors decomposes a direct sum over all P -nebenotypus of τ that factor through $\det : L_P(\mathbb{Z}_p) \rightarrow \mathbb{Z}_p^\times$. Doing so is obviously less general but has the advantage of simplifying the theory as only characters of $L_P(\mathbb{Z}_p)$ occur as types of P -ordinary vectors. On the other hand, systematically developing the more general theory (with P^u instead of SP) has the advantage that any holomorphic cuspidal representation π of G is trivially $\text{GL}(n)$ -ordinary. We discussed our motivation to study this more general notion in the introduction of this paper.

4. COMPATIBILITY AND COMPARISON BETWEEN PEL DATA.

4.1. Unitary groups for the doubling method. In what follows, we introduce the unitary group opposite to G and briefly review the changes necessary in the previous sections for this group. We then introduce comparison results for cohomology of Shimura varieties and automorphic representations by adapting the material of [Ehls20, Section 6.2] to our situation.

4.1.1. Theory for G_1 and G_2 . Let $\mathcal{P}_1 := \mathcal{P} = (\mathcal{K}, c, \mathcal{O}, L, 2\pi\sqrt{-1}\langle \cdot, \cdot \rangle, h)$ be the PEL datum constructed in Section 2.1. We often write $L_1 := L$, $\langle \cdot, \cdot \rangle_1 := 2\pi\sqrt{-1}\langle \cdot, \cdot \rangle$ and $h_1 := h$.

Define

$$\mathcal{P}_2 = (\mathcal{K}, c, \mathcal{O}, L_2, \langle \cdot, \cdot \rangle_2, h_2) := (\mathcal{K}, c, \mathcal{O}, L, -2\pi\sqrt{-1}\langle \cdot, \cdot \rangle, h(\bar{\cdot})),$$

again a PEL datum of unitary.

The signature of \mathcal{P}_2 at $w \in \Sigma_p$ is $(b_w, a_w) = (a_{\bar{w}}, b_{\bar{w}})$. We typically write G_i for the similitude unitary group corresponding to \mathcal{P}_i when we want to distinguish the underlying PEL datum.

Note that \mathcal{P}_1 and \mathcal{P}_2 are both associated to the same \mathcal{K} -vector space $L \otimes \mathbb{Q}$ but with opposite Hermitian forms. By abusing notation, we denote the vector space associated to \mathcal{P}_1 by V and the one for \mathcal{P}_2 by $-V$.

Consider the natural decomposition $L \otimes \mathbb{Z}_p = L^+ \oplus L^-$ obtained in Section 2.2.1. We now write $L_1^\pm := L^\pm$. Considering the signature of \mathcal{P}_2 , the analogous decomposition $L_2 \otimes \mathbb{Z}_p = L_2^+ \oplus L_2^-$ is given by taking $L_2^\pm = L_1^\mp$.

Obviously, we have $L_2^\pm = \prod_{w|p} L_{2,w}^\pm$, where $L_{2,w}^\pm := L_{1,w}^\mp$. The choice of basis for $L_{1,w}^\pm$ therefore naturally determines a choice of basis for each $L_{2,w}^\pm$ and we can proceed as in Section 2.2.1 for \mathcal{P}_2 to obtain identifications analogous to (6) and (7). However, together with (5) for \mathcal{P}_2 , the corresponding identification of $G_2(\mathbb{Q}_p)$ with $\mathbb{Q}_p^\times \times \prod_{w \in \Sigma_p} \mathrm{GL}_n(\mathcal{O}_w)$ is different than the one for $G_1(\mathbb{Q}_p)$, although there is a natural identification $G_1(\mathbb{Q}_p) = G_2(\mathbb{Q}_p)$.

In general, there is a canonical identification $G_1(\mathbb{A}) = G_2(\mathbb{A})$. Therefore, instead of modifying the identification $\mathrm{GL}_{\mathcal{O}_w}(L_w)$ with $\mathrm{GL}_n(\mathcal{O}_w)$ for \mathcal{P}_1 and \mathcal{P}_2 , we use the same identification twice (i.e. the one for \mathcal{P}_1).

This is harmless for the theory of Section 2.2. The only significant change is that the parabolic subgroup P_2 of G_2/\mathbb{Z}_p corresponding to the partitions \mathbf{d}_w is equal to tP_1 , where $P_1 = P$ is the parabolic subgroup in (12).

In other words, via the identification (7) for \mathcal{P}_1 , the parabolic subgroup $P_{2,w}$ of $\mathrm{GL}_n(\mathcal{O}_w)$ corresponds to ${}^tP_{1,w}$. Therefore, in what follows we always work with $P_w = P_{1,w}$ when consider G_1 and with tP_w when considering G_2 .

Remark 4.1. This leads to an ambiguity in our notation. For instance, we should refer to P -(anti-)ordinary forms on G_2 as tP -(anti-)ordinary forms. We avoid this issue and refer to objects on G_2 as P -(anti-)ordinary.

4.1.2. *Theory for G_3 and G_4 .* For $i = 3, 4$, define similar PEL datum

$$\mathcal{P}_i = (\mathcal{K}, c, \mathcal{O}, L_i, \langle \cdot, \cdot \rangle_i, h_i) \text{ together with } L_i \otimes \mathbb{Z}_p = L_i^+ \oplus L_i^-,$$

again of unitary type in the sense of [EHL20, Section 2.2], where

$$\mathcal{P}_3 := (\mathcal{K} \times \mathcal{K}, c \times c, \mathcal{O} \times \mathcal{O}, L_1 \oplus L_2, \langle \cdot, \cdot \rangle_1 \oplus \langle \cdot, \cdot \rangle_2, h_1 \oplus h_2), L_3^\pm := L_1^\pm \oplus L_2^\pm$$

$$\mathcal{P}_4 := (\mathcal{K}, c, \mathcal{O}, L_3, \langle \cdot, \cdot \rangle_3, h_3), L_4^\pm := L_3^\pm$$

Denote the similitude unitary group in (3) associated to \mathcal{P}_i by G_i . Similarly, let ν_i be the similitude character of G_i and $U_i = \ker \nu_i$.

4.1.3. *Compatibility of level structures.* One readily sees that the reflex field of \mathcal{P}_1 , \mathcal{P}_2 and \mathcal{P}_3 are all equal. However, since \mathcal{P}_4 has signature (n, n) at all archimedean places, its reflex field is \mathbb{Q} .

Therefore, all of the theory introduced in the previous sections can be adapted for G_3 and G_4 over $\mathcal{O}_F \otimes \mathbb{Z}_{(p)}$ and $\mathbb{Z}_{(p)}$, respectively.

Note that some of the notation must be adapted for \mathcal{P}_3 since the PEL datum is associated to two copies of \mathcal{K} instead of a single one. Therefore, all of the associated objects must be adapted to consider two lattices $L_3 = L_1 \oplus L_2$, two vector spaces $V_3 = V_1 \oplus V_2$, and so on, with two idempotent projections e_1 and e_2 relating the objects with ones on \mathcal{P}_1 and \mathcal{P}_2 respectively. The modifications are mostly obvious, hence we omit precise formulation here. For more details, see [EHL20, Section 2].

We have $H_3 = \mathrm{GL}_{(\mathcal{O}_{\mathcal{K}} \times \mathcal{O}_{\mathcal{K}}) \otimes \mathbb{Z}_p}(L_3^+)$ and $H_4 = \mathrm{GL}_{\mathcal{O}_{\mathcal{K}} \otimes \mathbb{Z}_p}(L_4^+)$. Furthermore, the obvious inclusions $G_3 \hookrightarrow G_4$ and $G_3 \hookrightarrow G_1 \times G_2$ induce the canonical inclusions $H_3 \hookrightarrow H_4$ and $H_3 \hookrightarrow H_1 \times H_2$.

The choice of an \mathcal{O}_w -basis of $L_{1,w}^\pm = L_w^\pm$ as in Section 2.2.1 naturally induces a choice of basis of $L_{i,w}^\pm$ for $i = 2, 3$ and 4 as well. In fact, we obtain isomorphisms

$$(66) \quad G_{i/\mathbb{Z}_p} \xrightarrow{\sim} \mathbb{G}_m \times \prod_{w \in \Sigma_p} \begin{cases} \mathrm{GL}_n(\mathcal{O}_w), & \text{if } i = 1, 2, \\ \mathrm{GL}_n(\mathcal{O}_w) \times \mathrm{GL}_n(\mathcal{O}_w), & \text{if } i = 3 \\ \mathrm{GL}_{2n}(\mathcal{O}_w), & \text{if } i = 4 \end{cases}$$

as well as

$$(67) \quad H_{i/\mathbb{Z}_p} \xrightarrow{\sim} \mathbb{G}_m \times \prod_{w|p} \begin{cases} \mathrm{GL}_{a_w}(\mathcal{O}_w), & \text{if } i = 1, \\ \mathrm{GL}_{b_w}(\mathcal{O}_w), & \text{if } i = 2, \\ \mathrm{GL}_{a_w}(\mathcal{O}_w) \times \mathrm{GL}_{b_w}(\mathcal{O}_w), & \text{if } i = 3 \\ \mathrm{GL}_n(\mathcal{O}_w), & \text{if } i = 4 \end{cases}$$

Let $P_1 = P_H \subset H_1 = H$ be the parabolic subgroup introduced in Section 2.2.2 associated to partitions $\mathbf{d}_w = (n_{w,1}, \dots, n_{w,t_w})$ and $\mathbf{d}_{\bar{w}} = (n_{\bar{w},1}, \dots, n_{\bar{w},t_{\bar{w}}})$ of the signature a_w and b_w respectively.

Recall that under the identification $G_1(\mathbb{A}) = G_2(\mathbb{A})$ and our conventions between \mathcal{P}_1 and \mathcal{P}_2 , the corresponding parabolic subgroup of H_2 is $P_2 = {}^t P_H$. Adapting the theory of Section 2.2.2 for G_3 (with the same choice of partitions) amounts

to defining the corresponding parabolic $P_3 \subset H_3$ as the preimage of $P_1 \times P_2$ via $H_3 \hookrightarrow H_1 \times H_2$.

Similarly, consider the two partitions

$$(68) \quad (n_{w,1}, \dots, n_{w,t_w}, n_{\bar{w},1}, \dots, n_{\bar{w},t_{\bar{w}}}) \quad \text{and} \quad (n_{\bar{w},1}, \dots, n_{\bar{w},t_{\bar{w}}}, n_{w,1}, \dots, n_{w,t_w}),$$

viewed as a partition of (n, n) . Let $P_4 \subset H_4$ be the corresponding parabolic, following the approach of Section 2.2.2 for G_4 . Then, the inclusion $G_3 \hookrightarrow G_4$ above induces the canonical inclusion $P_3 \hookrightarrow P_4$.

Let $L_{H,i}$ be the Levi factor of P_i . Then, $L_{H,4} = L_{H,3} = L_{H,1} \times L_{H,2}$. In particular, a P_4 -nebenotypus τ of level r is also a P_3 -nebenotypus of level r . It corresponds to a tensor product $\tau_1 \otimes \tau_2$, where τ_i is a P_i -nebenotypus of level r (for $i = 1$ and 2).

Let P_i is one of those four parabolic subgroup. Let $I_{i,r} := I_{P_i,r}$ and $I_{i,r}^0 := I_{P_i,r}^0$ be the corresponding pro- p P -Iwahoric subgroup and P -Iwahoric subgroup respectively. By abuse of notation, we still use the terminology “ P -Iwahoric” as opposed to “ P_i -Iwahoric”.

Given a compact open subgroup $K_i^p \subset G_i(\mathbb{A}_f^p)$, let $K_{i,r} = I_{i,r} K_i^p$. We denote the moduli space associated to \mathcal{P}_i of level K_i by $M_{i,K_i,r} = M_{K_{i,r}}(\mathcal{P}_i)$ and the corresponding Shimura variety by $_{K_{i,r}}\text{Sh}(G_i)$. If V_i is the vector space associated to the PEL datum \mathcal{P}_i , we sometimes write $_{K_{i,r}}\text{Sh}(V_i)$ instead.

If $K_{3,r} \subset (K_{1,r} \times K_{2,r}) \cap G_3(\mathbb{A}_f)$ and $K_{3,r} \subset K_{4,r} \cap G_3(\mathbb{A}_f)$, there are natural maps

$$(69) \quad i_3 : M_{3,K_{3,r}} \rightarrow M_{4,K_{4,r}} \quad \text{and} \quad i_{1,2} : M_{3,K_{3,r}} \rightarrow M_{1,K_{1,r}} \times M_{2,K_{2,r}}$$

over $S_p = \mathcal{O}_{F,(p)}$. For the exact maps at the level of points of moduli problems, see [EHLS20, (37)–(38)].

All of the above remains compatible if we restrict to Shimura varieties or extend to toroidal compactifications. Furthermore, if $K_{3,r} = (K_{1,r} \times K_{2,r} \cap G_3(\mathbb{A}_f))$ then the Shimura varieties on both sides, as canonical connected components are identifies, hence we obtain an isomorphism

$$(70) \quad i_{1,2} : _{K_{3,r}}\text{Sh}(V_3) \rightarrow _{K_{1,r}}\text{Sh}(V_1) \times _{K_{2,r}}\text{Sh}(V_2)$$

as well as an analogous isomorphism for toroidal compactifications.

4.1.4. Compatibility of canonical bundles. Recall that in Section 2.4.1, we defined a canonical bundle $\mathcal{E} = \mathcal{E}_1$ on the toroidal compactification of the moduli space for \mathcal{P}_1 . Moreover, for all dominant weight κ of the maximal torus $T_{H_{0,1}}$ of $H_{0,1}$, we introduced the associated automorphic vector bundle ω_κ .

Let $V_{1,\mathbb{C}} = L_1 \otimes \mathbb{C}$ with its Hodge decomposition $V_{1,\mathbb{C}} = V_1^{-1,0} \oplus V_1^{0,-1}$ fixed in Section 2.1. The above is associated to a specific choice of $\mathcal{O}_F \otimes \mathbb{Z}_{(p)}$ -module $\Lambda_{0,1} = \Lambda_0$ in the graded module $W_1 = V_{1,\mathbb{C}}/V_1^{0,-1}$. Proceeding as in Section 2.3, we obtain the groups $G_{0,1} = G_0$ and $H_{0,1} = H_0$.

For \mathcal{P}_2 , consider $V_{2,\mathbb{C}} = L_2 \otimes \mathbb{C}$. The corresponding Hodge structure is reversed, i.e. $V_2^{-1,0} = V_1^{0,-1}$ and $V_2^{0,-1} = V_1^{-1,0}$. Therefore, one must choose a module

in $W_2 = V_{2,\mathbb{C}}/V_2^{0,-1} = V_{1,\mathbb{C}}/V_1^{-1,0}$. Using the identifications $\Lambda_1 = \Lambda_{0,1} \oplus \Lambda_{1,0}^\vee$, $V_{1,\mathbb{C}} = V_{2,\mathbb{C}} = \Lambda_1 \otimes \mathbb{C}$ and $\Lambda_{1,0}^\vee \otimes \mathbb{C} = V_1^{-1,0}$, we choose $\Lambda_{2,0}$ to be the image of $\Lambda_{1,0}^\vee$ in W_2 , and $\Lambda_2 = \Lambda_{2,0} \oplus \Lambda_{2,0}^\vee \cong \Lambda_1$.

Similarly, for \mathcal{P}_3 and \mathcal{P}_4 , we pick $\Lambda_{3,0} = \Lambda_{4,0} = \Lambda_{1,0} \oplus \Lambda_{2,0}$ and $\Lambda_3 = \Lambda_4 = \Lambda_1 \oplus \Lambda_2$. These compatible choices induce obvious inclusions

$$(71) \quad H_{3,0} \hookrightarrow H_{4,0} \quad \text{and} \quad H_{3,0} \hookrightarrow H_{1,0} \times H_{2,0}.$$

Let $\pi_i : \mathcal{E}_i \rightarrow \mathcal{M}_{K_i}$ be the corresponding canonical bundle for \mathcal{P}_i . Then, the above also induces natural maps

$$(72) \quad i_3 : \mathcal{E}_3 \rightarrow \mathcal{E}_4 \quad \text{and} \quad i_{1,2} : \mathcal{E}_3 \rightarrow \mathcal{E}_1 \times \mathcal{E}_2,$$

compatible with the maps on moduli spaces in (69). These maps also extend to maps between the $\mathcal{L}_H(\mathbb{Z}_p/p^r\mathbb{Z}_p)$ -torsors $\mathcal{E}_{i,r} \rightarrow \mathcal{E}_i$, for all $r \gg 1$.

The choice of basis over $S_0 = \mathcal{O}_{\mathcal{K}',(p')}$ of $\Lambda_{1,0,\sigma}$ and $\Lambda_{1,0,\sigma}^\vee$, for each $\sigma \in \Sigma_{\mathcal{K}}$ as in Sections 2.3.1, naturally induces bases for $\Lambda_{i,0}$ and $\Lambda_{i,0}^\vee$ for $i = 2, 3$ and 4 as well. We obtain identifications

$$H_{i,0}/S_0 \xrightarrow{\sim} \mathbb{G}_m \times \prod_{\sigma \in \Sigma_{\mathcal{K}}} \begin{cases} \mathrm{GL}_{b_\sigma}(S_0), & \text{if } i = 1, \\ \mathrm{GL}_{a_\sigma}(S_0), & \text{if } i = 2, \\ \mathrm{GL}_{b_\sigma}(S_0) \times \mathrm{GL}_{a_\sigma}(S_0), & \text{if } i = 3, \\ \mathrm{GL}_n(S_0), & \text{if } i = 4, \end{cases}$$

as in (19).

Observe that the definition of H_0 in Section 2.3 yields an obvious identification $H_{1,0} = H_{2,0}$ (by switching the roles of Λ_0 and Λ_0^\vee). With respect to the identifications above, this corresponds to the automorphism

$$(73) \quad (h_0, (h_\sigma)_{\sigma \in \Sigma_{\mathcal{K}}}) \mapsto (h_0, (h_0 {}^t h_{\sigma c}^{-1})_{\sigma \in \Sigma_{\mathcal{K}}})$$

of $H_0 = H_{1,0} = H_{2,0}$ over S_0 .

Furthermore, the embeddings in (71) are the obvious ones with respect to the identifications above. Similar identifications can be made about the Borel subgroups $B_{H_0,i}$, maximal torus $T_{H_0,i}$ and parabolic subgroups $P_{H_0,i}$ introduced in Sections 2.3.1 and 2.3.2.

In particular, $T_{H_0,4} = T_{H_0,3} = T_{H_0,1} \times T_{H_0,2}$ and a dominant weight κ of $T_{H_0,4}$ corresponds to a dominant weight of $T_{H_0,3}$. Similarly, a pair (κ_1, κ_2) consisting of a dominant weight κ_1 for $T_{H_0,1}$ and a dominant weight κ_2 of $T_{H_0,2}$ also corresponds to a dominant weight of $T_{H_0,3}$.

4.1.5. Restriction of algebraic modular forms. Let $r \geq 1$ and let τ be a P_4 -nebenypus (for G_4) of level r . Let R be an $S_0[\tau]$ -algebra. Let $K_{i,r}$ be an open compact subgroup of $G_i(\mathbb{A}_f)$ such that $K_{3,r} \subset K_{4,r} \cap G_3(\mathbb{A}_f)$ and $K_{3,r} \subset (K_{1,r} \times K_{2,r}) \cap G_3(\mathbb{A}_f)$, as in Section 4.1.3.

If κ is an R -valued dominant weight of $T_{H_0,4}$, then pullback i_3 in (72) induces a restriction map

$$(74) \quad \text{Res}_3 = (i_3)^* : M_\kappa(K_{4,r}, \tau; R) \rightarrow M_\kappa(K_{3,r}, \tau; R).$$

Similarly, let κ_1 (resp. κ_2) and τ_1 (resp. τ_2) be an R -valued dominant weight of $T_{H_0,1}$ (resp. $T_{H_0,2}$) and a P_1 -nebenotypus (resp. P_2 -nebenotypus) of level r , respectively. Then, set $\kappa = (\kappa_1, \kappa_2)$ and $\tau = \tau_1 \otimes \tau_2$ be the corresponding weight of $T_{H_0,3}$ and P_3 -nebenotypus. As above, pullback along the second map in (72) induces a restriction

$$(75) \quad \text{Res}_{1,2} = (i_{1,2})^* : M_\kappa(K_{1,r}, \tau_1; R) \otimes M_\kappa(K_{2,r}, \tau_2; R) \rightarrow M_\kappa(K_{3,r}, \tau; R).$$

Naturally, these maps have analogues when considering modular forms without fixed P -nebenotype. In situation of (70), the map $\text{Res}_{1,2}$ is an isomorphism.

4.2. Comparisons between G_1 and G_2 . In this section, we discuss various involutions that allow us to compare spaces of holomorphic forms with spaces of anti-holomorphic forms on both G_1 or G_2 .

Namely, the goal of this section is to explain the functors

$$\begin{array}{ccc} \{\text{holomorphic AR on } G_1\} & \xrightarrow{c_B} & \{\text{anti-holomorphic AR on } G_1\} \\ \downarrow F^\dagger & \begin{array}{c} \nearrow F_\infty \\ \searrow F_\infty \end{array} & \downarrow F^\dagger \\ \{\text{holomorphic AR on } G_2\} & \xrightarrow{c_B} & \{\text{anti-holomorphic AR on } G_2\} \end{array},$$

where ‘‘AR’’ stands for automorphic representations, adapting [EHLS20, Section 6.2] to the P -ordinary setting (notably F^\dagger), as well as the effect of each arrow on weights, levels and types.

4.2.1. The involution c_B on G_1 . Let π be a holomorphic cuspidal automorphic representation for $G = G_1$ of weight κ . All of the following holds for G_2 with the obvious modifications. As explained in [EHLS20, Section 6.2.1], there is a c -semilinear, $G(\mathbb{A}_f)$ -equivariant isomorphism

$$c_B : H^0(\mathfrak{A}_h, K_h; \pi \otimes W_\kappa) \rightarrow H^d(\mathfrak{A}_h, K_h; \bar{\pi} \otimes W_{\kappa^D})$$

induced by the Killing form on \mathfrak{g} and the complex conjugation $\pi \rightarrow \bar{\pi}$, $\varphi \mapsto \bar{\varphi}$ on $\mathcal{A}_0(G)$, where $\bar{\varphi}(g) = \overline{\varphi(g)}$. Equivalently, we have a c -semilinear, $G(\mathbb{A}_f)$ -equivariant isomorphism

$$(76) \quad c_B : H_!^0(\text{Sh}(V), \omega_\kappa) \rightarrow H_!^d(\text{Sh}(V), \omega_{\kappa^D})$$

over \mathbb{C} via (37).

Remark 4.2. From Definition 3.1, the isomorphism c_B shows that an automorphic representation π of G is holomorphic of weight κ if and only if $\bar{\pi}$ is anti-holomorphic of weight κ^D .

4.2.2. *The involution F_∞ .* To compare automorphic representations on G_1 and G_2 , we first compare their associated locally symmetric spaces. Let h (resp. h^c) be the homomorphism associated to G_1 (resp. G_2) inducing a Hodge structure on $V \otimes \mathbb{C}$ and denote the $G(\mathbb{R})$ -conjugacy class of h (resp. h^c) by X_h (resp. X_{h^c}).

Note that the stabilizer $K_h \subset G(\mathbb{R})$ of h is also the stabilizer of h^c . In this section, we write $U_\infty := K_h$ and identify $X = G(\mathbb{R})/U_\infty$ with both X_h and X_{h^c} . However, note that the complex structures induced by h and h^c respectively are opposite. Namely, the pullback of these two complex structures to X are conjugate.

In other words, the natural map $X_h \rightarrow X_{h^c}$ given by $ghg^{-1} \mapsto gh^c g^{-1}$ is anti-holomorphic and provides an anti-holomorphic map $\mathrm{Sh}(V)(\mathbb{C}) \rightarrow \mathrm{Sh}(-V)(\mathbb{C})$.

Remark 4.3. In Section 4.2.4, we study a different map $X_h \rightarrow X_{h^c}$ instead. To help distinguish the two, one does not need to apply complex conjugation anywhere in the definition of F_∞ . This involution is simply a natural consequence of the relation between the complex structures on X_h and X_{h^c} .

Remark 4.4. From now on, we use the notation of Section 4.1. However, we replace the subscripts $i = 1$ and 2 by V and $-V$. In particular, $H_{0,V} := H_{1,0}$ and $H_{0,-V} := H_{2,0}$.

Let $\kappa = (\kappa_0, (\kappa_\sigma))$ be a dominant character of $T_{H_{0,V}}$. Observe that the notion of a *dominant* character is “flipped” as $B_{H_{0,-V}} = {}^t B_{H_{0,V}}$ under the identification (73).

Namely, the weight of $T_{H_{0,-V}}$ given by

$$\kappa^b := (\kappa_0, (\kappa_{\sigma c}))$$

is dominant. To understand the Lie algebra cohomology of the highest weight representation $W_{\kappa^b, -V}$ of $H_{0,-V}$, observe that the Harish-Chandra decomposition of \mathfrak{g} induced by h^c is

$$\mathfrak{g} = \mathfrak{p}_{h^c}^- \oplus \mathfrak{k}_{h^c} \oplus \mathfrak{p}_{h^c}^+,$$

where $\mathfrak{p}_{h^c}^\pm = \mathfrak{p}_h^\mp$ and $\mathfrak{k}_{h^c} = \mathfrak{k}_h$.

Hence, given a $(\mathfrak{g}, K_h) \times G(\mathbb{A}_f)$ -module π , we need to consider

$$(\pi^{\mathfrak{p}_{h^c}^-} \otimes W_{\kappa^b, -V})^{K_{h^c}} = (\pi^{\mathfrak{p}_h^+} \otimes W_{\kappa^b, -V})^{K_h}$$

and understand $W_{\kappa^b, -V}$ as a representation of $H_{0,V}$, via pullback through (73).

In fact, via (73), $W_{\kappa^b, -V}$ is the irreducible highest weight representation of highest weight κ^* for $T_{H_{0,V}}$, where κ^* is as in (43). See [Ehls20, Eq. (121)] for an explicit $H_{0,V}$ -isomorphism $W_{\kappa^b, -V} \xrightarrow{\sim} W_{\kappa^*, V}$.

Furthermore, by definition of κ^D , there is a natural map

$$i_{\kappa^*} : W_{\kappa^*, V} \hookrightarrow \mathrm{Hom}(\wedge^d \mathfrak{p}_h^- \otimes_{\mathbb{C}} \wedge^d \mathfrak{p}_h^+, W_{\kappa^*, V}) = \mathrm{Hom}(\wedge^d \mathfrak{p}_h^-, W_{\kappa^D, V})$$

where the first map is induced by the $H_0(\mathbb{C})$ -equivariant pairing

$$\wedge^d \mathfrak{p}_h^- \otimes_{\mathbb{C}} \wedge^d \mathfrak{p}_h^+ \rightarrow \mathbb{C}$$

obtained from the Killing form on \mathfrak{g} .

Therefore, the above yields

$$(\pi^{\mathfrak{p}_h^-} \otimes W_{\kappa^b, -V})^{K_{h^c}} \xrightarrow{\text{id} \otimes i_{\kappa^*}} \text{Hom}(\wedge^d \mathfrak{p}_h^-, \pi \otimes W_{\kappa^D, V})$$

which induces a $G(\mathbb{A}_f)$ -equivariant isomorphism

$$(77) \quad F_\infty : H^0(\mathfrak{P}_{h^c}, K_{h^c}, \pi \otimes W_{\kappa^b, -V}) \xrightarrow{\sim} H^d(\mathfrak{P}_h, K_h, \pi \otimes W_{\kappa^D, V})$$

over \mathbb{C} .

The case $\pi = \mathcal{A}_0(G)$ yields the $G(\mathbb{A}_f)$ -equivariant isomorphism

$$(78) \quad F_\infty : H_1^0(\text{Sh}(-V), \omega_{\kappa^b, -V}) \xrightarrow{\sim} H_1^d(\text{Sh}(V), \omega_{\kappa^D, V})$$

over \mathbb{C} .

Remark 4.5. Considering the composition of c_B and F_∞ , we see that if π is holomorphic (resp. anti-holomorphic) of weight κ on G_1 , then $\bar{\pi}$ is holomorphic (resp. anti-holomorphic) of weight κ^b on G_2

4.2.3. *The involution $(-)^b$.* Let π be a cuspidal automorphic representation for $G = G_1$. If π is holomorphic of weight κ , then $\xi_{\pi, \infty}(t) = t^{a(\kappa)}$ for all $t \in Z_G(\mathbb{R}) \cong \mathbb{R}^\times$, where ξ_π denote the central character of π .

It follows that

$$\pi \otimes |\xi_\pi \circ \nu|^{-1/2} = \pi \otimes |\nu|^{-a(\kappa)/2}$$

is unitary. Hence, considering its conjugate, we see that $\bar{\pi}$ is isomorphic to

$$(79) \quad \pi^b := \pi^\vee \otimes |\xi_\pi \circ \nu| .$$

The material of Sections 4.2.1–4.2.2 then implies that π^b is anti-holomorphic of weight κ^D for G_1 , or equivalently, holomorphic of weight κ^b for G_2 .

Since π^b is a twist of π^\vee , the pairings $\langle \cdot, \cdot \rangle_\pi$ and $\langle \cdot, \cdot \rangle_{\pi_l}$ from Section 3.1.3 induce pairings $\pi \times \pi^b \rightarrow \mathbb{C}$ and $\pi_l \otimes \pi_l^b \rightarrow \mathbb{C}$, for each place l of \mathbb{Q} , which we again denote $\langle \cdot, \cdot \rangle_\pi$ and $\langle \cdot, \cdot \rangle_{\pi_l}$ respectively.

Remark 4.6. The necessity of working with π^b in this paper is due to the doubling method, see Section 9.1.4, which requires the integration of an Eisenstein series with the product of a *test* vector φ in π and a *test* vector φ^b in a twist of $\pi^b \cong \bar{\pi}$ (or equivalently, π^\vee). It is natural to view the Eisenstein series as a holomorphic modular form on G_3 (or rather the restriction to G_3 of a modular form on G_4) and, dually, π and π^b as anti-holomorphic representations on G_1 and G_2 respectively.

Moreover, the advantage of π^b over $\bar{\pi}$ is that its direct relation with π^\vee facilitates the transition between P -ordinary properties of π and P -anti-ordinary properties of π^b over G_1 , see Lemma 6.10 and Theorem 6.11.

Remark 4.7. Starting in Section 8.2, we assume that π satisfies the multiplicity one hypothesis (see Hypothesis 8.5). In that case, the subspaces of $\mathcal{A}_0(G)$ associated to π^b and $\bar{\pi}$ are in fact equal.

4.2.4. *The involution $(-)^{\dagger}$ for level structures.* We introduce one last involution “ \dagger ”. The main feature is to compare level structures between G_1 and G_2 (and not just weights, as we have done above). Although our approach, arguments and notation follow [EHLS20, Section 6.2.3], the reader should keep in mind that the results presented here generalize *loc. cit.* by considering P -Iwahoric level structures at p for all parabolic subgroups P .

As we are assuming Hypothesis 2.1, there exists a \mathcal{K} -basis \mathcal{B}_h of V that diagonalizes the Hermitian pairing $\langle \cdot, \cdot \rangle_V$ associated to V . Furthermore, if we write \mathcal{B}_h^c for the image of \mathcal{B}_h under complex conjugation, then h takes values in the space of diagonal matrices under the identification

$$\text{End}_{\mathbb{R}}(V \otimes_{\mathcal{K}, \sigma} \mathbb{C}) = \text{Mat}_{2n \times 2n}(\mathbb{R}).$$

induced by the \mathcal{K}^+ -basis $\mathcal{B}_h \cup \mathcal{B}_h^c$ of V , for each $\sigma \in \Sigma$. Clearly, h^c is obtained by conjugating h with the change-of-basis endomorphism interchanging \mathcal{B}_h with \mathcal{B}_h^c .

Let $D = \text{diag}(d_1, \dots, d_n), d_1, \dots, d_n \in \mathcal{K}^+$, be the diagonal Hermitian matrix representing $\langle \cdot, \cdot \rangle_V$ with respect to \mathcal{B}_h .

Let L be the \mathcal{O} -lattice from Section 2.1 associated to \mathcal{P} . As explained in [EHLS20, Section 6.2.3], using Hypothesis 2.2, we can assume that \mathcal{B}_h induces a basis of $L \otimes_{\mathbb{Z}(p)}$ such that D is also the diagonalization of the perfect Hermitian pairing on $L \otimes_{\mathbb{Z}(p)}$ obtained from $\langle \cdot, \cdot \rangle_V$.

The advantage of \mathcal{B}_h is that it provides a holomorphic map $\text{Sh}(V) \rightarrow \text{Sh}(-V)$ (as opposed to the anti-holomorphic one in Section 4.2.2). Indeed, first identify $G_{/\mathbb{Q}}$ as a subgroup of $\text{Res}_{\mathcal{K}/\mathbb{Q}} \text{GL}_n(\mathcal{K})$ using \mathcal{B}_h and consider the automorphism $g \mapsto \bar{g}$ of $G_{\mathbb{Q}}$ induced by the action of c on \mathcal{K} .

Observe that $\bar{g} = IgI$, where $I : V \rightarrow V$ is the \mathcal{K}^+ -involution that interchanges \mathcal{B}_h and \mathcal{B}_h^c by sending any vector $v \in \mathcal{B}_h$ to $v^c \in \mathcal{B}_h^c$ and *vice versa*. Note that I stabilizes $L \otimes_{\mathbb{Z}(p)}$ and its action on $L \otimes_{\mathbb{Z}(p)}$ interchanges L^+ and L^- . Moreover, our explanation above implies that $h^c = \bar{h}$, hence it maps U_{∞} to U_{∞} and yields an automorphism of X .

The composition

$$X_h \xrightarrow{\sim} X \xrightarrow{g \mapsto \bar{g}} X \xrightarrow{\sim} X_{h^c}$$

$$ghg^{-1} \xrightarrow{\hspace{10em}} \bar{g}h^c\bar{g}^{-1}$$

is holomorphic and provides a holomorphic map $\text{Sh}(V)(\mathbb{C}) \rightarrow \text{Sh}(-V)(\mathbb{C})$ as claimed. Given $K^p \subset G(\mathbb{A}_f^p)$ and $K_{r,V} = I_{r,V}K^p$ as in Section 2.5.1, it provides a holomorphic map between ${}_{K_{r,V}}\text{Sh}(V)(\mathbb{C}) \rightarrow {}_{\overline{K_{r,V}}}\text{Sh}(-V)(\mathbb{C})$. However, $\overline{K_{r,V}} \neq I_{r,-V}\overline{K^p}$ or equivalently, $\overline{I_{r,V}} \neq I_{r,-V}$.

To resolve this issue, observe that the basis \mathcal{B}_h of V naturally induces a basis $\mathcal{B}_{h,w}$ of $V_w := V \otimes_{\mathcal{K}} \mathcal{K}_w$ for any $w \in \Sigma_p$. It would be too restrictive to assume that the basis \mathcal{B}^w of V_w induced by the \mathcal{O}_w -bases of L_w^{\pm} , for $w \in \Sigma_p$, leading to the

identifications in (6), is the same as $\mathcal{B}_{h,w}$. In other words, there is no need to assume that all the bases \mathcal{B}^w , as w varies in Σ_p , are all induced by a basis of V .

Instead, consider the identification $\mathrm{GL}_{\mathcal{K}_w}(V_w) = \mathrm{GL}_n(\mathcal{K}_w)$ induced by $\mathcal{B}_{h,w}$ and let $\beta_w \in \mathrm{GL}_n(\mathcal{K}_w)$ be the change-of-basis matrix that maps \mathcal{B}^w to $\mathcal{B}_{h,w}$.

Remark 4.8. This matrix $\beta_w \in \mathrm{GL}_n(\mathcal{K}_w)$ is the inverse of the analogous change-of-basis matrix, also denoted β_w , introduced in [EHLS20, Section 6.2.3].

Let

$$\delta_w = D \cdot {}^t\beta_w \cdot \beta_w \in \mathrm{GL}_n(\mathcal{K}_w)$$

and define $\delta_p = (1, (\delta_w)_{w \in \Sigma_p}) \in \mathbb{Q}_p^\times \times \prod_{w \in \Sigma_p} \mathrm{GL}_n(\mathcal{K}_w) = G(\mathbb{Q}_p)$. This identification is with respect to \mathcal{B}_h (and may be different than the identification obtained from (5) and (6)).

Then, one readily checks that

$$\bar{\delta}_p = \delta_p^{-1} \ ; \ \delta_p^{-1} \overline{G(\mathbb{Z}_p)} \delta_p = G(\mathbb{Z}_p) \ ; \ \delta_p^{-1} \overline{I_{r,V}^0} \delta_p = I_{r,-V}^0 \ ; \ \delta_p^{-1} \overline{I_{r,V}} \delta_p = I_{r,-V}.$$

Therefore, by defining an automorphism $(-)^{\dagger}$ of $G(\mathbb{A})$ as

$$g^{\dagger} := \nu(g)^{-1} \delta_p^{-1} \bar{g} \delta_p,$$

we see that given any $K^p \subset G(\mathbb{A}_f)$, $K = G(\mathbb{Z}_p)K^p$ and $K_{r,V} = I_{r,V}K^p$, we have

$$K^{\dagger} = G(\mathbb{Z}_p) \overline{K^p} \ ; \ (K_{r,V})^{\dagger} = (K^{\dagger})_{r,-V} = K_{r,-V}^{\dagger}.$$

In conclusion, the obvious analogue of [EHLS20, Proposition 6.2.4] also holds in our context (namely, one simply changes the meaning of the level r structure at p from ‘‘Iwahoric’’ to ‘‘ P -Iwahoric’’). In other words, the holomorphic map $K_{r,V} \mathrm{Sh}(V)(\mathbb{C}) \rightarrow K_{r,-V}^{\dagger} \mathrm{Sh}(-V)(\mathbb{C})$ via $g \mapsto \bar{g} \delta_p$ is well-defined over $S_0 = \mathcal{O}_{\mathcal{K}',(p')}$, i.e. it is induced from an isomorphism

$$(80) \quad K_{r,V} \mathrm{Sh}(V) \xrightarrow{\sim} K_{r,-V}^{\dagger} \mathrm{Sh}(-V)$$

over S_0 comparing P -Iwahori level structure on G_1 and G_2 (see Remark 4.1).

To understand this involution in terms of modular forms, consider a dominant character κ of $T_{H_0,V}$. Let $\kappa^{\dagger} = \kappa^b - \underline{a(\kappa)}$, a dominant character of $T_{H_0,-V}$, where $\underline{a(\kappa)}$ is the scalar weight corresponding to the character $\|\nu(-)\|^{a(\kappa)}$.

By definition of κ^b , the natural automorphism $(h_0, (h_{\sigma})) \rightarrow (h_0, (h_{\sigma c}))$ of H_0 induces an isomorphism $W_{\kappa,V} \rightarrow W_{\kappa^b,-V}$. By twisting by $\nu^{-a(\kappa)}$, we obtain an isomorphism $W_{\kappa,V} \xrightarrow{\sim} W_{\kappa^{\dagger},-V}$ given by $\phi \mapsto \phi^{\dagger}$, where

$$\phi^{\dagger}(h_0, (h_{\sigma})) := \phi(h_0, (h_{\sigma c})) h_0^{-a(\kappa)}.$$

One readily sees that this isomorphism is \dagger -equivariant for the action of $K_h = K_{h^c}$. Therefore, the isomorphism $g \mapsto g^{\dagger}$ induces an isomorphism between $\omega_{\kappa,V}$ and the pullback to $\omega_{\kappa^{\dagger},-V}$ from $\mathrm{Sh}(V)$ to $\mathrm{Sh}(-V)$.

In other words, the above induces an isomorphism

$$(81) \quad F^\dagger : H_!^i(\mathrm{Sh}(V), \omega_{\kappa, V}) \xrightarrow{\sim} H_!^i(\mathrm{Sh}(-V), \omega_{\kappa^\dagger, -V}),$$

over \mathbb{C} , that is \dagger -equivariant for the action of $G(\mathbb{A}_f)$.

Remark 4.9. Observe that multiplication by the global section $g \mapsto \|\nu(g)\|^{a(\kappa)}$ yields an isomorphism

$$H_!^0(\mathrm{Sh}(-V), \omega_{\kappa^\flat, -V}) \xrightarrow{\sim} H_!^0(\mathrm{Sh}(-V), \omega_{\kappa^\dagger, -V}),$$

which can be useful to compare the above with the results of Sections 4.2.2–4.2.3.

Using the discussion, (80) and (81) induce an isomorphism

$$(82) \quad F^\dagger : H_!^i({}_{K_r}\mathrm{Sh}(V), \omega_{\kappa, V}) \xrightarrow{\sim} H_!^i({}_{K_r^\dagger}\mathrm{Sh}(-V), \omega_{\kappa^\dagger, -V})$$

over any S_0 -algebra R and any $r \gg 0$.

Similarly, given a P -nebenotypus τ of level r , the map $g \mapsto g^\dagger$ induces an isomorphism between $\omega_{\kappa, r, \tau, V}$ and the pullback of $\omega_{\kappa^\dagger, r, \tau^\vee, -V}$ from ${}_{K_r}\mathrm{Sh}(V)$ to ${}_{K_r^\dagger}\mathrm{Sh}(-V)$. Therefore, we also have an isomorphism

$$F^\dagger : H_!^i({}_{K_r}\mathrm{Sh}(V), \omega_{\kappa, r, \tau, V}) \xrightarrow{\sim} H_!^i({}_{K_r^\dagger}\mathrm{Sh}(-V), \omega_{\kappa^\dagger, r, \tau^\vee, -V})$$

over any $S_0[\tau]$ -algebra R . We now set $\tau^\dagger := \tau^\vee$, motivated by the isomorphism above.

Lastly, to understand this involution in terms of automorphic forms via (37), let π be an arbitrary $(\mathfrak{g}, K_h) \times G(\mathbb{A}_f)$ -subrepresentations of $\mathcal{A}_0(G)$. The map $g \mapsto g^\dagger$ induces a map $\pi \rightarrow \pi^\dagger$, where

$$\pi^\dagger := \{\varphi^\dagger(g) := \varphi(g^\dagger) \mid \varphi \in \pi\} \subset \mathcal{A}_0(G).$$

As explained in [EHLS20, Section 6.2.3], the isomorphism

$$(\pi \otimes W_{\kappa, V})^{K_h} \rightarrow (\pi^\dagger \otimes W_{\kappa^\dagger, -V})^{K_{h^c}},$$

given by $\varphi \otimes \phi \mapsto \varphi^\dagger \otimes \phi^\dagger$ is \dagger -equivariant for the action of $(\mathfrak{g}, K_h) \times G(\mathbb{A}_f)$. Therefore, one obtains

$$F^\dagger : H^i(\mathfrak{P}_h, K_h, \pi \otimes W_{\kappa, V}) \xrightarrow{\sim} H^i(\mathfrak{P}_{h^c}, K_{h^c}, \pi^\dagger \otimes W_{\kappa^\dagger, -V})$$

over \mathbb{C} . The case $\pi = \mathcal{A}_0(G)$ recovers the map (82) over \mathbb{C} .

Remark 4.10. Suppose π is (anti-)holomorphic and P -(anti-)ordinary such that its P -(anti-)WLT (κ, K_r, τ) for G_1 , then π^\dagger is (anti-)holomorphic, P -(anti-)ordinary of P -(anti-)WLT $(\kappa^\dagger, K_r^\dagger, \tau^\dagger)$ for G_2 .

As explained in [EHLS20, Section 6.5.3], if π satisfies the strong multiplicity one hypothesis, then π^\dagger and π^\vee are equal as subspaces of $\mathcal{A}_0(G)$. In that case, one further obtains $\pi^\flat = \pi^\dagger \otimes \|\nu\|^{a(\kappa)}$.

Assume that π is (anti-)holomorphic, P -(anti-)ordinary of P -(anti-)WLT (κ, K_r, τ) for G_1 . Then, setting

$$K_r^b := K_r^\dagger \text{ and } \tau^b := \tau^\dagger = \tau^\vee$$

we have that π^b is (anti-)holomorphic, P -(anti-)ordinary of P -(anti-)WLT $(\kappa^b, K_r^b, \tau^b)$ for G_2 (by definition of κ^b).

5. P -ORDINARY p -ADIC MODULAR FORMS.

Fix a neat open compact subgroup $K^p \subset G(\mathbb{A}_f^p)$. In what follows, we use the notation of Sections 2.4.1 and 2.5.1 freely.

In particular, write $K = G(\mathbb{Z}_p)K^p$ and $K_r = I_r K^p$ for all $r \geq 0$. Furthermore, recall that M_K^{tor} is the smooth toroidal compactification of M_K over $S_p = \mathcal{O}_{F,p}$ (a tower viewed as a single scheme, see 2.1.2), and \mathcal{A} is the universal semiabelian scheme (with extra structure) over M_K^{tor} . The dual semiabelian scheme is denoted \mathcal{A}^\vee and we write ω for the $\mathcal{O}_{M_K^{\text{tor}}}$ -dual of $\text{Lie}_{M_K^{\text{tor}}} \mathcal{A}^\vee$.

In this section, we use the fact that the completion of $\text{incl}_p(S_p)$ is \mathbb{Z}_p to view (compactified) moduli spaces and Shimura varieties of level K (and K_r) over \mathbb{Z}_p .

5.1. Igusa tower.

5.1.1. *Ordinary locus and Igusa cover.* Given $m \geq 1$, let \mathcal{S}_m denote the nonvanishing locus of the Hasse invariant on ${}_K \text{Sh}^{\text{tor}}$ over $\mathbb{Z}_p/p^m \mathbb{Z}_p$. Let \mathcal{S}_m^0 be the open subscheme obtained from the intersection of \mathcal{S}_m and ${}_K \text{Sh}$. Note that \mathcal{S}_1 is dense in the special fiber of ${}_K \text{Sh}^{\text{tor}}$, see [EHLS20, Section 2.8].

Given $r \geq m$, let $\mathcal{T}_{r,m}$ denote the finite étale cover of \mathcal{S}_m such that for any \mathcal{S}_m -scheme S

$$(83) \quad \mathcal{T}_{r,m}(S) = \text{Isom}_S(L^+ \otimes \mu_{p^r}, \mathcal{A}^\vee[p^r]^\circ),$$

where the superscript \circ denotes the identity component and the isomorphisms are of finite flat group schemes over S endowed with $\mathcal{O} \otimes \mathbb{Z}_p$ -actions. One readily sees that $\mathcal{T}_{r,m}$ is a closed subscheme of ${}_K \overline{\text{Sh}}/(\mathbb{Z}_p/p^m \mathbb{Z}_p)$.

Furthermore, $\mathcal{T}_{r,m}/\mathcal{S}_m$ is a Galois cover whose Galois group is canonically isomorphic to $H(\mathbb{Z}_p/p^r \mathbb{Z}_p)$. We refer to $\mathcal{T}_m = \{\mathcal{T}_{r,m}\}_r$ as the *Igusa tower over \mathcal{S}_m* .

Let \mathcal{S} denote the non-vanishing locus in ${}_K \text{Sh}^{\text{tor}}$ of a lift of the Hasse invariant to characteristic 0. This depends on the choice of lift, however its reduction modulo p is isomorphic to \mathcal{S}_1 and the formal completion \mathcal{S}^{ord} of \mathcal{S} along \mathcal{S}_1 does not depend on the choice of lift. Then, the *Igusa tower* $\text{Ig} = \varinjlim_m \varprojlim_r \mathcal{T}_m$ is a pro-étale cover of \mathcal{S}^{ord} with Galois group $H(\mathbb{Z}_p)$. If we want to emphasize the choice of level away from p , we write ${}_K \text{Ig}$ instead of Ig .

By taking pullback of \mathcal{S} via the \mathcal{L}_r -torsor ${}_K \overline{\text{Sh}} \rightarrow {}_K \text{Sh}^{\text{tor}}$, we can similarly define the ordinary locus ${}_K \mathcal{S}$ of ${}_K \overline{\text{Sh}}$. By taking formal completion, we obtain ${}_K \mathcal{S}^{\text{ord}}$. Given any dominant weight κ , restricting a modular form $f \in M_\kappa(K_r, R)$ to this ordinary locus defines an element of $H^0({}_K \mathcal{S}^{\text{ord}}, \omega_{\kappa,r})$.

5.1.2. *Embeddings of Igusa towers.* The above is set with the PEL datum $\mathcal{P} = \mathcal{P}_1$. More generally, for $1 \leq i \leq 4$, fix a neat open compact subgroup $K_i^p \subset G_i(\mathbb{A}_f^p)$ and let $\mathcal{S}_{m,i}$ be the analogue of \mathcal{S}_m associated to the moduli problem associated to \mathcal{P}_i . Let $\mathcal{T}_{r,m,i} \rightarrow \mathcal{S}_{m,i}$ be the corresponding finite étale cover given by (83).

If $K_3^p \subset K_4^p \cap G_3(\mathbb{A}_f^p)$ and $K_3^p \subset (K_1^p \times K_2^p) \cap G_3(\mathbb{A}_f^p)$, the maps from (69) extend to embeddings

$$(84) \quad \mathcal{T}_{r,m,3} \hookrightarrow \mathcal{T}_{r,m,4} \quad \text{and} \quad \mathcal{T}_{r,m,3} \hookrightarrow \mathcal{T}_{r,m,1} \times_{\mathbb{Z}_p} \mathcal{T}_{r,m,2},$$

see [EHLS20, Equations (42)-(43)].

However, as explained in [HLS06, Section 2.1.11] and [EHLS20, Remark 3.4.1], at the level of complex points, the inclusion $_{K_3^p}\mathrm{Ig}_3 \hookrightarrow _{K_4^p}\mathrm{Ig}_4$ induced by the first map above is not a restriction of the natural embedding $i_3 : _{K_{3,r}}\mathrm{Sh}(V_3) \hookrightarrow _{K_{4,r}}\mathrm{Sh}(V_4)$

In fact, the first inclusion in (84) corresponds to the composition of i_3 with the shifted inclusion $G_3(\mathbb{A}) \hookrightarrow G_4(\mathbb{A})$ given by $g \mapsto g \cdot \gamma_p$, where γ_p corresponds to the element of $G_4(\mathbb{A})$ whose component away from p is trivial and whose component at p is $(1, (\gamma_w)_{w \in \Sigma_p}) \in G_4(\mathbb{Q}_p) = \mathbb{G}_m \times \prod_{w \in \Sigma_p} \mathrm{GL}_n(\mathcal{K}_w)$, where

$$\gamma_w = \begin{pmatrix} 1_{a_w} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{b_w} \\ 0 & 0 & 1_{a_w} & 0 \\ 0 & 1_{b_w} & 0 & 0 \end{pmatrix},$$

via the identification (66) for $i = 4$. The reader should keep in mind that this shift by γ_p plays an important role in Sections 10.1.1–10.1.2 for the computation of local zeta integrals at p .

Then, we have an inclusion

$$(85) \quad \gamma_p \circ i_3 : _{K_3^p}\mathrm{Ig}_3 \hookrightarrow _{K_4^p}\mathrm{Ig}_4$$

as described above. On the other hand, we obtain an inclusion

$$(86) \quad i_{1,2} : _{K_3^p}\mathrm{Ig}_3 \hookrightarrow _{K_1^p}\mathrm{Ig}_1 \times _{K_2^p}\mathrm{Ig}_2$$

induced by the second map in (84) without any shifts involved.

5.2. Scalar-valued p -adic modular forms with respect to P . In this section, we introduce a slightly unconventional definition of scalar-valued p -adic modular forms, generalizing the usual notion (see [EHLS20, Section 2.9]). The key idea is to replace the role of the unipotent radical of some standard Borel subgroup with the unipotent radical of the fixed parabolic P , see (87) below. We recover the usual notion when $P = B$ as in Remark 2.8. In Section 5.3, we introduce another notion that allows us to consider vector-valued p -adic modular forms.

In Section 11, the goal is to construct a p -adic family of such scalar-valued p -adic modular forms on G_4 from the Eisenstein series constructed in Section 9.

5.2.1. *Global section over the Igusa tower.* Fix a p -adic ring R , i.e. $R = \varprojlim_m R/p^m R$. Assume that R contains the ring \mathcal{O}' introduced in Section 2.3.3.

For each $r \geq m \geq 0$, let $D_{r,m}$ be the preimage of $D_m = \mathcal{S}_m - \mathcal{S}_m^0$ (with its reduced closed subscheme structure) in $\mathcal{T}_{r,m}$. Then, define

$$\mathcal{V}_{r,m}(R) = H^0(\mathcal{T}_{r,m/R}, \mathcal{O}_{\mathcal{T}_{r,m/R}}) \quad \text{and} \quad \mathcal{V}_{r,m}^{\text{cusp}}(R) = H^0(\mathcal{T}_{r,m/R}, \mathcal{O}_{\mathcal{T}_{r,m/R}}(-D_{r,m})).$$

Clearly, there is a natural action of $H(\mathbb{Z}_p/p^m \mathbb{Z}_p) = \text{Gal}(\mathcal{T}_{r,m}/\mathcal{S}_m)$ on each of these R -modules.

We define the spaces of scalar-valued p -adic modular forms (for G) of level K^p over R as

$$(87) \quad \mathcal{V}(K^p; R) = \varprojlim_m \varinjlim_r \mathcal{V}_{r,m}(R)^{P_H^u(\mathbb{Z}_p)}$$

and its submodule of p -adic cuspidal forms as

$$(88) \quad \mathcal{V}^{\text{cusp}}(K^p; R) = \varprojlim_m \varinjlim_r \mathcal{V}_{r,m}^{\text{cusp}}(R)^{P_H^u(\mathbb{Z}_p)}.$$

Remark 5.1. We sometimes write $\mathcal{V}(G, K^p; R)$ and $\mathcal{V}^{\text{cusp}}(G, K^p; R)$ to emphasize the underlying reductive group if there is any risk of confusion.

Remark 5.2. When $P = B$ as in Remark 2.8, these spaces agree with the usual spaces of p -adic modular and cuspidal forms (see [EHLS20, Section 2.9]).

Naturally, these spaces admit an action by $L_H(\mathbb{Z}_p) = P_H(\mathbb{Z}_p)/P_H^u(\mathbb{Z}_p)$. Consider the maximal torus $T_H(\mathbb{Z}_p) \subset L_H(\mathbb{Z}_p)$ and let κ_p be a p -adic weight of $T_H(\mathbb{Z}_p)$, as in Section 2.3.3.

Let ψ_B denote a $\overline{\mathbb{Q}}_p$ -valued finite-order character of $T_H(\mathbb{Z}_p)$. Let $\mathcal{O}'[\psi_B] \subset \overline{\mathbb{Q}}_p$ denote the smallest ring extension of \mathcal{O}' containing the values of ψ_B .

For any ring R containing $\mathcal{O}'[\psi_B]$, we set

$$(89) \quad \mathcal{V}_{\kappa_p}(K^p, \psi_B; R) := \{f \in \mathcal{V}(K^p; R) \mid t \cdot f = \psi_B(t)\kappa_p(t)f, \forall t \in T_H(\mathbb{Z}_p)\},$$

and we define its submodule $\mathcal{V}_{\kappa_p}^{\text{cusp}}(K_r, \psi_B; R)$ similarly.

5.2.2. *Classical to p -adic modular forms : scalar case.* We now adapt the usual map sending classical forms to p -adic forms, see [EHLS20, Section 2.9.4], to our setup.

For all $n \geq 1$, write $\mu_{p^n} = \text{Spec}(\mathbb{Z}[x, x^{-1}]/(x^{p^n} - 1))$ and identify $\text{Lie}_{\mathbb{Z}_p}(\mu_{p^n})$ with a free \mathbb{Z} -module of rank 1 generated by $x \frac{d}{dx}$. Given any $m \geq 1$ and any $\mathbb{Z}/p^m \mathbb{Z}$ -scheme S , this allows us to view $\text{Lie}_S(\mu_{p^n})$ as the structure sheaf \mathcal{O}_S of S (compatibly as n varies).

Fix a test object $(\underline{A}, \phi) \in \mathcal{T}(S)$ over a p -adic R -algebra S . Here, we write $\phi = (\phi_{n,m})_{n \geq m}$ with $\phi_{n,m} \in \mathcal{T}_{n,m}$ and consider the subsequence $(\phi_{m,m})_m$. For any $1 \leq r \leq m$, the map $\phi_{m,m}$ induces the isomorphism

$$\phi_{m,m,r} : L^+ \otimes \mu_{p^r} \xrightarrow{\sim} \mathcal{A}_{S}^{\vee}[p^r]^{\circ}.$$

Furthermore, using the discussion above, the latter also induces an isomorphism

$$\mathrm{Lie}(\phi_{m,m,r}) : L^+ \otimes \mathcal{O}_S = L^+ \otimes \mathrm{Lies}(\mu_{p^r}) \xrightarrow{\sim} \mathrm{Lies}(\mathcal{A}_S^\vee[p^r]^\circ) = \mathrm{Lies} \mathcal{A}_S^\vee.$$

Therefore, using the identification $\Lambda_0 \otimes \mathbb{Z}_p = L^+$ from Section 2.3.3, we conclude that the tuple

$$(\underline{A}_m, \phi_{m,m,r}, (\mathrm{Lie}(\phi_{m,m,r}))^\vee, \mathrm{id})$$

lies in $\mathcal{E}_r(S)$.

Let κ be any dominant weight of T_{H_0} . For any $r \geq 1$, this yields a map

$$(90) \quad \Omega_{\kappa,r} : M_\kappa(K_r; R) \rightarrow \mathcal{V}(K^p; R).$$

which sends a modular form $f \in M_\kappa(K_r; R)$ to

$$(91) \quad \Omega_{\kappa,r}(f)(\underline{A}, \phi) := \varprojlim_m f(\underline{A}_m, \phi_{m,m,r}, (\mathrm{Lie}(\phi_{m,m,r}))^\vee, \mathrm{id}) \in \varprojlim_m S/p^m S = S.$$

The map $\Omega_{\kappa,r}$ is injective by density of ${}_{K_r}\mathcal{S}/(\mathbb{Z}/p\mathbb{Z})$ in ${}_{K_r}\overline{\mathrm{Sh}}/(\mathbb{Z}/p\mathbb{Z})$. This follows immediately from the fact that \mathcal{S}_1 is dense in ${}_{K_r}\mathrm{Sh}^{\mathrm{tor}}/(\mathbb{Z}/p\mathbb{Z})$. Considering all dominant weights κ , we define

$$\Omega_r := \bigoplus_{\kappa} \Omega_{\kappa,r}.$$

For R sufficiently large, one readily checks that the restriction of $\Omega_{\kappa,r}$ to $M_\kappa(K_r, \psi_B; R)$ factors through the inclusion

$$\mathcal{V}_{\kappa_p}(K_r, \psi_B; R) \hookrightarrow \mathcal{V}(K^p; R),$$

for any κ_p and ψ_B as in (89).

Hence, we obtain a map $\Omega_{\kappa,r} : M_\kappa(K_r, \psi_B; R) \rightarrow \mathcal{V}_{\kappa_p}(K_r, \psi_B; R)$, where κ_p is the p -adic weight associated to κ as in (27).

In fact, the section f on ${}_{K_r}\mathrm{Sh}^{\mathrm{tor}}$ only needs to be defined on the ordinary locus for the formula (91) to be well-defined. In other words, $\Omega_{\kappa,r}$ naturally extends to a map

$$\Omega_{\kappa,r} : H^0({}_{K_r}\mathcal{S}^{\mathrm{ord}}/R, \omega_{\kappa,r}) \rightarrow \mathcal{V}_{\kappa_p}(K_r; R).$$

Conjecture 5.3. *The space*

$$\left(\varinjlim_r \Omega_r \left(\bigoplus_{\kappa} H^0({}_{K_r}\mathcal{S}^{\mathrm{ord}}/R, \omega_{\kappa,r}) \right) \right) [1/p] \cap \mathcal{V}(K^p; R)$$

is p -adically dense in $\mathcal{V}(K^p; R)$.

Remark 5.4. When $P = B$, this density result is a well-known result. See [Hid04, Proposition 8.2, Theorem 8.3] or [EFMV18, Theorem 2.6.1].

5.2.3. *P -ordinary p -adic modular forms : scalar case.* For $w \in \Sigma_p$ and $1 \leq j \leq r_w$, let $t_{w,D_w(j)} = t_{w,D_w(j)}^+ \in G(\mathbb{Q}_p)$ be the matrix introduced in Section 2.2.2, see (15).

It is well-known that the double coset $I_r t_{w,D_w(j)} I_r$ can be written as a disjoint union of right cosets with representatives independent of r (for instance, see the calculations in Section 6.1.1). Note that $\cap_{r \geq 1} I_r^{\text{GL}} = P_H^u(\mathbb{Z}_p)$, hence one can use these same representatives for the double coset $P_H^u(\mathbb{Z}_p) t_{w,D_w(j)} P_H^u(\mathbb{Z}_p)$.

In [Hid04, 8.3.1], Hida demonstrates that $u_{w,D_w(j)} = P_H^u(\mathbb{Z}_p) t_{w,D_w(j)} P_H^u(\mathbb{Z}_p)$ can be interpreted as a correspondence on the Igusa tower. See [EHLS20, Section 2.9.5] as well for further details. This naturally induces an action of $u_{w,D_w(j)}$ on $\mathcal{V}(K^p; R)$ which stabilizes both $\mathcal{V}^{\text{cusp}}(K^p; R)$ and $\mathcal{V}_{\kappa^p}(K^p, \psi_B; R)$. We set

$$u_{P,p} = \prod_{w \in \Sigma_p} \prod_{j=1}^{r_w} u_{w,D_w(j)} \quad \text{and} \quad e_P = e_{P,p} = \varinjlim_n u_{P,p}^n,$$

and define the space of P -ordinary p -adic modular forms and cuspidal forms as $\mathcal{V}^{P\text{-ord}}(K^p; R) := e_P \mathcal{V}^{P\text{-ord}}(K^p; R)$ and $\mathcal{V}^{P\text{-ord,cusp}}(K^p; R) := e_P \mathcal{V}^{P\text{-ord,cusp}}(K^p; R)$,

respectively. We have similar definitions when fixing a p -adic weight κ_p and a character ψ_B .

In Section 11, we use the following conjecture (which is known to hold when $P = B$ as in Remark 2.8) to compare p -adic and classical Eisenstein series.

Conjecture 5.5. *Let κ be a very regular dominant weight and ψ_B be as in (89). The restriction*

$$\Omega_{\kappa,r}^{P\text{-ord}} : S_{\kappa}^{P\text{-ord}}(K_r, \psi_B; R) \rightarrow \mathcal{V}_{\kappa_p}^{P\text{-ord,cusp}}(K_r, \psi_B; R)$$

of $\Omega_{\kappa,r}$ to P -ordinary cusp forms is an isomorphism.

Remark 5.6. Although this is not the focus of this paper, the author expects that one can weaken that assumption that κ is very regular for a condition that is sensitive to our choice of P . In this setting, the assumption of being very regular should be strong enough to hold for all choices of parabolic subgroups P .

5.2.4. *Restrictions of p -adic forms.* Let \mathcal{P}_i be one of the PEL datum introduced in Section 4.1.1–4.1.2, and let (G_i, X_i) be the associated Shimura datum. For $1 \leq i \leq 4$, fix a neat open compact subgroup $K_i^p \subset G_i(\mathbb{A}_f^p)$ and assume that K_3^p is contained in both $(K_1^p \times K_2^p) \cap G_3(\mathbb{A}_f)$ and $K_4^p \cap G_3(\mathbb{A}_f)$. Then, we obtain restriction maps

$$\text{Res}_3 : \mathcal{V}(G_4, K_4^p; R) \rightarrow \mathcal{V}(G_3, K_3^p; R)$$

and

$$\text{Res}_{1,2} : \mathcal{V}(G_1, K_1^p; R) \widehat{\otimes} \mathcal{V}(G_2, K_2^p; R) \rightarrow \mathcal{V}(G_3, K_3^p; R)$$

induced by the embeddings in (84), where $\widehat{\otimes}$ is the complete tensor product of the p -adic ring R . Note that Res_3 is induced by restricting forms along $\gamma_p \circ \iota_3$, see (85).

5.2.5. *Evaluation at ordinary points.* Now, fix $\mathcal{P} = \mathcal{P}_i$ for any $1 \leq i \leq 4$ and set $(G, X) := (G_i, X_i)$. Let $(J'_0, h_0) \rightarrow (G, X)$ be the embedding of Shimura datum, where J'_0 is a torus, from [EHLS20, Section 2.3.2]. In particular, (J'_0, h_0) defines a CM Shimura subvariety of $\text{Sh}(J'_0, h_0)$ of $\text{Sh}(G, X)$. Given a dominant weight κ of T_{H_0} and a level subgroup $K = K_r \subset G(\mathbb{A}_f)$, we obtain a restriction map

$$\text{Res}_{J'_0, h_0} : M_\kappa(G, K_r; R) \rightarrow M_\kappa((J'_0, h_0); R),$$

where the modular forms on (J'_0, h_0) are defined with respect to an appropriate level subgroup.

As in [EHLS20, Section 3.2.4], we say that (J'_0, h_0) is *ordinary* if at the level of points of moduli problems, the image of $\text{Sh}(J'_0, h_0) \rightarrow \text{Sh}(G, X)$ only consists of ordinary abelian varieties (with extra structures). In this case, for all $r \geq m \geq 0$, one can similarly define an Igusa variety $\mathcal{T}_{r,m}(J'_0, h_0)$ as in (83) for (J'_0, h_0) .

The embedding of Shimura datum above similarly induces a map $\mathcal{T}_{r,m}(J'_0, h_0) \rightarrow \mathcal{T}_{r,m}(G, X)$ on Igusa varieties, with the obvious notation. We write $\mathcal{V}_{\kappa_p}((G, X), K^p; R)$ for the space of p -adic modular forms of weight κ_p , level K^p and coefficient R associated to (G, X) . For the analogous space on (J'_0, h_0) , we write $\mathcal{V}_{\kappa_p}((J'_0, h_0); R)$ without specifying any level structure (note that there is no need to specify a parabolic subgroup in (87) as J'_0 is a torus). As above, we obtain a restriction map

$$\text{Res}_{p, J'_0, h_0} : \mathcal{V}_{\kappa_p}((G, X), K^p; R) \rightarrow \mathcal{V}_{\kappa_p}((J'_0, h_0); R).$$

As in Section 5.2.2, we obtain embeddings $\Omega_{\kappa, r}$ for both (G, X) and (J'_0, h_0) . To distinguish both, we write $\Omega_{\kappa, r, G, X}$ (resp. $\Omega_{\kappa, r, J'_0, h_0}$) for the map (90) with respect to G and X (resp. J'_0 and h_0). Therefore, by definition, we obtain the following proposition.

Proposition 5.7. *Using the same notation as above, the following hold :*

(i) *The diagram*

$$\begin{array}{ccc} M_\kappa(G, K_r; R) & \xrightarrow{\Omega_{\kappa, r, G, X}} & \mathcal{V}_{\kappa_p}((G, X), K^p; R) \\ \downarrow \text{Res}_{J'_0, h_0} & & \downarrow \text{Res}_{p, J'_0, h_0} \\ M_\kappa((J'_0, h_0); R) & \xrightarrow{\Omega_{\kappa, r, J'_0, h_0}} & \mathcal{V}_{\kappa_p}((J'_0, h_0); R) \end{array}$$

is commutative

(ii) *Let $f \in \mathcal{V}_{\kappa_p}^{P\text{-ord}}((G, X), K^p; R)$. Suppose that $\text{Res}_{p, J'_0, h_0}(f) = 0$ for every ordinary CM pair (J'_0, h_0) mapping to (G, X) . Then, $f = 0$.*

Proof. The proof is identical to the one of [EHLS20, Proposition 3.2.5]. \square

5.3. **p -adic modular forms valued in locally algebraic representations.** In this section, we introduce a different notion of p -adic modular forms by considering non-trivial vector bundles over the Igusa tower. The goal is to develop the necessary

material to study Hida theory in the context of P -ordinary Hecke algebras acting on P -ordinary automorphic representations.

In Section 8, we use the material discussed here and assume certain results (see Conjectures 8.12 and 8.17), to describe the geometry of P -ordinary Hida families of automorphic representations. Furthermore, in Section 12.1, we again rely on these conjectures to adapt the formalism developed in [Ehls20, Section 7.4] to our situation and construct a p -adic L -function from the Eisenstein measure of Proposition 11.8.

5.3.1. *Locally algebraic coefficient rings.* Fix a weight dominant κ of T_{H_0} and consider the P -parallel lattice $[\kappa]$ passing through κ as in (23). Similarly, fix a P -nebenotypus τ of level r and consider the P -nebenotypus equivalence class $[\tau] = [\tau]_r$ of τ . Fix a p -adic ring R as in Section 5.2 and further assume that R contains $\text{incl}_p(S_r[\tau])$.

Let V_κ and \mathcal{M}_τ be the $L_H(\mathbb{Z}_p)$ -representations over R associated to κ and τ (or equivalently, to $[\kappa]$ and $[\tau]$) respectively, as in Section 2.3.3. We view the R -module $\text{Hom}_R(\mathcal{M}_\tau, V_\kappa)$ as a *locally algebraic* representation of $L_H(\mathbb{Z}_p)$.

Let $[\underline{\kappa}_p, \tau]$ denote the trivial vector bundle over $\mathcal{T}_{r,m/R}$ associated to $\text{Hom}_R(\mathcal{M}_\tau, V_\kappa)$. We include “ $\underline{\kappa}_p$ ” in the notation here instead of κ to emphasize the fact that V_κ is viewed as a representation of $L_H(\mathbb{Z}_p)$ (and not $L_{H_0}(R)$).

Define

$$\mathcal{V}_{r,m}([\underline{\kappa}_p, \tau]; R) = H^0(\mathcal{T}_{r,m/R}, [\underline{\kappa}_p, \tau])$$

and

$$\mathcal{V}_{r,m}^{\text{cusp}}([\underline{\kappa}_p, \tau]; R) = H^0(\mathcal{T}_{r,m/R}, [\underline{\kappa}_p, \tau](-D_{r,m})).$$

Definition 5.8. The space of p -adic modular forms (for G) of level K^p and coefficient $\text{Hom}_R(\mathcal{M}_\tau, V_\kappa)$ over R is defined as

$$\mathcal{V}(K^p, [\underline{\kappa}_p, \tau]; R) = \varprojlim_m \varinjlim_r \mathcal{V}_{r,m}([\underline{\kappa}_p, \tau]; R)^{P_H^u(\mathbb{Z}_p)}$$

and its submodule of p -adic cuspidal forms is defined as

$$\mathcal{V}(K^p, [\underline{\kappa}_p, \tau]; R)^{\text{cusp}} = \varprojlim_m \varinjlim_r \mathcal{V}_{r,m}^{\text{cusp}}([\underline{\kappa}_p, \tau]; R)^{P_H^u(\mathbb{Z}_p)}.$$

Remark 5.9. The space $\mathcal{V}(K^p, [\underline{\kappa}_p, \tau]; R)$ agrees with $\mathcal{V}(K^p; R)$ exactly when $\underline{\kappa}_p$ is a scalar weight and τ is a character.

These spaces are again naturally equipped with an action of $L_H(\mathbb{Z}_p)$ induced by the action of $P_H(\mathbb{Z}_p) \subset H(\mathbb{Z}_p)$ on Igusa towers.

We view a p -adic modular form $f \in \mathcal{V}(K^p, [\underline{\kappa}_p, \tau]; R)$ as vector-valued. More precisely, we view f as a functorial rule such that on each p -adic ring S over R , a test object (\underline{A}, ϕ) is assigned by f to an element of $\text{Hom}_S(\mathcal{M}_{\tau,S}, V_{\kappa,S})$, where $\underline{A} = (\underline{A}_m)_m \in \varprojlim_m \mathcal{S}_m(S)$ and $\phi = (\phi_{r,m}) \in \varinjlim_m \varprojlim_r \mathcal{T}_{r,m}(S)$ with $\phi_{r,m}$ over \underline{A}_m for all $r \gg 0$.

5.3.2. *Classical to p -adic modular forms : locally algebraic case.* One can adapt the material of Section 5.2.2 for vector-valued p -adic modular forms as well. Indeed, define

$$\mathcal{V}_{\kappa_p}(K^p, \tau; R) := \{f \in V(K^p, [\kappa_p, \tau]; R) \mid l \cdot f = ((\tau \otimes \rho_{\kappa_p})(l))(f)\}.$$

Using the fact that $\text{Lie}(\phi \circ l)^\vee = {}^t l^{-1} \circ \text{Lie}(\phi)^\vee$, for all $l \in L_H(\mathbb{Z}_p)$, as well as the relation (25), one readily checks that given $f \in M_\kappa(K_r, \tau; R)$, the formula

$$(92) \quad \Theta_{\kappa, \tau}(f)(\underline{A}, \phi) := \varprojlim_m f(\underline{A}_m, \phi_{m, m, r}, (\text{Lie}(\phi_{m, m, r})^\vee, \text{id}))$$

from (91) similarly yields an injective map

$$\Theta_{\kappa, \tau} : M_\kappa(K_r, \tau; R) \rightarrow \mathcal{V}_{\kappa_p}(K^p, \tau; R).$$

5.3.3. *P -ordinary p -adic modular forms : locally algebraic case.* As in Section 5.2.3, for a p -adic domain R in which p is nonzero, this action stabilizes the image of $\Omega_{\kappa, \tau}$, and given $f \in M_\kappa(K_r, \tau; R)$, we have

$$u_{w, D_w(j)} \Omega_{\kappa, \tau}(f) = \kappa'(t_{w, D_w(j)}) U_{w, D_w(j)} f,$$

where $\kappa' = (\kappa_{\text{norm}})_p$ is as in Section 2.8. In other words, these operators agree with the operators denoted $u_{w, D_w(j)}$, for $w \in \Sigma_p$ and $1 \leq j \leq r_w$, from Section 2.8. In particular, the operator $u_{w, D_w(j)}$ on (vector-valued) p -adic modular forms only depends on κ through the P -parallel lattice $[\kappa]$.

Once more, we set

$$u_{P, p} = \prod_{w \in \Sigma_p} \prod_{j=1}^{r_w} u_{w, D_w(j)} \quad \text{and} \quad e_P = e_{P, p} = \varinjlim_n u_{P, p}^n,$$

as operators on $\mathcal{V}(K^p, [\kappa_p, \tau]; R)$. We define the space of P -ordinary p -adic modular forms with coefficients $\text{Hom}_R(\mathcal{M}_\tau, V_\kappa)$ over R as

$$\mathcal{V}^{P\text{-ord}}(K^p, [\kappa, \tau]; R) := e_P \mathcal{V}^{P\text{-ord}}(K^p, [\kappa, \tau]; R),$$

and we have similar definitions for $\mathcal{V}^{P\text{-ord, cusp}}(K^p, [\kappa, \tau]; R)$, $\mathcal{V}_{\kappa_p}^{P\text{-ord}}(K^p, \tau; R)$ and $\mathcal{V}_{\kappa_p}^{P\text{-ord, cusp}}(K^p, \tau; R)$.

Conjecture 5.10. *Let κ be a very regular dominant weight. Let τ be a P -nebentypus of level $r \geq 1$. The restriction*

$$\Omega_{\kappa, \tau} : S_\kappa^{P\text{-ord}}(K_r, \tau; R) \rightarrow \mathcal{V}_{\kappa_p}^{P\text{-ord, cusp}}(K^p, \tau; R)$$

of $\Omega_{\kappa, \tau}$ to P -ordinary cusp forms is an isomorphism.

5.4. Hecke operators on p -adic modular forms. Given $g \in G(\mathbb{A}_f^p)$, the double coset $T(g) := [KgK]$ natural acts on the space of p -adic modular forms $\mathcal{V}^{\text{cusp}}(K_r; R)$. Namely, one easily adapts (28) for test objects on the Igusa tower instead of the classical Shimura variety.

Given $f \in M_\kappa(K_r; R)$, we obviously have $T(g)\Omega_{\kappa,r}(f) = \Omega_{\kappa,r}(T(g)f)$. Moreover, this extends to define an operator $T(g)$ on the spaces $\mathcal{V}^{\text{cusp}}(K_r; R)$, $\mathcal{V}(K^p, [\kappa_p, \tau]; R)$ and $\mathcal{V}^{\text{cusp}}(K^p, [\kappa_p, \tau]; R)$.

Furthermore, given a matrix t in the center Z_P of $L_H(\mathbb{Z}_p)$ and $f \in M_\kappa(K_r, \tau; R)$,

$$t \cdot \Omega_{\kappa,\tau}(f) = \kappa'(t)\omega_\tau(t)f,$$

where ω_τ is the central character of τ . More generally, $t \cdot \Theta_{\kappa,\tau}(f) = \kappa'(t)(t \cdot f)$, for all $f \in M_\kappa(K_r, [\tau]; R)$.

Namely, we can again view the operator $u_p(t) = u_{p,\kappa}(t)$ introduced in Section 2.8 as an endomorphism of $\mathcal{V}(K^p, [\kappa_p, \tau]; R)$ via the natural action of $P_H^u(\mathbb{Z}_p)tP_H^u(\mathbb{Z}_p) = tP_H^u(\mathbb{Z}_p)$.

In Section 8, we study the Hecke algebras generated by the operators above and the endomorphisms $u_{w,D_w(j)}$, for $w \in \Sigma_p$ and $1 \leq j \leq r_w$. We use the compatibility between these endomorphisms on classical forms and on p -adic forms on several occasions implicitly.

Part II. Families of P -(anti)-ordinary automorphic representations.

6. STRUCTURE AT p OF P -(ANTI)-ORDINARY AUTOMORPHIC REPRESENTATIONS.

The main results of this section are Theorems 6.6 and 6.11. The idea is to describe the space of P -ordinary vectors and P -anti-ordinary vectors via types.

We first study the case of $G = G_1$. Then, taking into account the conventions set in Section 4.1.3, all statements are adapted for G_2 in Sections 6.3 and 6.4.

6.1. P -ordinary theory on G_1 . In what follows, we use the notation of Section 3.2 freely. In particular, we work with a cuspidal automorphic representation π for $G = G_1$ and write $\pi_p = \mu_p \otimes (\otimes_{w \in \Sigma_p} \pi_w)$ for its p -component.

Assume that π is holomorphic and that its weight κ satisfies the inequality :

$$(93) \quad \kappa_{\sigma,b_\sigma} + \kappa_{\sigma c, a_\sigma} \geq n, \forall \sigma \in \Sigma_K.$$

6.1.1. Explicit coset representatives. To clarify arguments in later proofs, we now describe explicit right coset representatives for $U_{w,D_w(j)}^{\text{GL}} = [I_{w,r}t_{w,D_w(j)}I_{w,r}]$. For simplicity, we only compute the right coset representatives when $j \leq t_w$. The same conclusion applies for $j > t_w$ but writing down the matrices is simply more cumbersome. The reader should keep in mind that t_w only denotes an integer while $t_{w,D_w(j)}$ denotes an element of $\text{GL}_n(\mathcal{O}_w)$, see Remark 2.11.

Fix $j \leq t_w$ and write $i = D_w(j)$ (making the dependence on j implicit). Fix a uniformizer $\varpi \in \mathfrak{p}_w$. Given any matrix $X \in I_{w,r}$, write it as

$$X = \begin{pmatrix} A & B \\ \varpi^r C & D \end{pmatrix}$$

where $A \in \mathrm{GL}_i(\mathcal{O}_w)$, $D \in \mathrm{GL}_i(\mathcal{O}_w)$ and $B \in M_{i \times (n-i)}(\mathcal{O}_w)$ and $C \in M_{(n-i) \times i}(\mathcal{O}_w)$.

Fix a set S_w of representatives in \mathcal{O}_w for $\mathcal{O}_w/p\mathcal{O}_w$. Let $B', B'' \in M_{i \times (n-i)}(\mathcal{O}_w)$ be the unique matrices such that B' has entries in S_w and $BD^{-1} = B' + pB''$. Then, we have

$$X = \begin{pmatrix} 1_j & B' \\ 0 & 1_{n-j} \end{pmatrix} \begin{pmatrix} A - \varpi^r B' C & pB'' D \\ \varpi^r C' & D \end{pmatrix} =: X' X''$$

In particular, $t_{w,i}^{-1} X'' t_{w,i}$ is in $I_{w,r}$. Therefore,

$$I_{w,r} t_{w,i} I_{w,r} = \bigsqcup_{x \in M_j} x t_{w,i} I_{w,r}$$

where $M_j \subset \mathrm{GL}_n(\mathcal{K}_w)$ is the subset of matrices $\begin{pmatrix} 1_i & B' \\ 0 & 1_{n-i} \end{pmatrix}$ such that the entries of B' are in S_w .

In particular, this set of representative does not depend on r and one obtains the same result by replacing $I_{w,r}$ with $N_w = \cap_r I_{w,r} = P_w^u(\mathcal{K}_w) \cap \mathrm{GL}_n(\mathcal{O}_w)$. As mentioned above, one readily sees that the calculations above still apply for $t_w < j \leq r_w$.

Let V_w be the \mathcal{K}_w -vector space associated to π_w . By continuity, its N_w -invariant subspace $V_w^{N_w}$ is equal to $\cup_r V_w^{I_{w,r}}$.

Lemma 6.1. *There is a decomposition $V_w^{N_w} = V_{w,\mathrm{inv}}^{N_w} \oplus V_{w,\mathrm{nil}}^{N_w}$ such that, for $1 \leq j \leq r_w$, $U_{w,D_w(j)}^{\mathrm{GL}}$ is invertible on $V_{w,\mathrm{inv}}^{N_w}$ and nilpotent on $V_{w,\mathrm{nil}}^{N_w}$. Moreover, $U_{w,D_w(j)}^{\mathrm{GL}} = I_{w,r} t_{w,D_w(j)} I_{w,r}$ acts as $\delta_{P_w}(t_{D_w(j)})^{-1} t_{D_w(j)}$ on $V_{w,\mathrm{inv}}^{N_w}$.*

Proof. We keep writing $i = D_w(j)$ in this proof and omit the subscript w in what follows.

The first part is a consequence of the explanations in [Hid98, Section 5.2]. Moreover, [Hid98, Proposition 5.1] shows that the natural projection from V to its P -Jacquet module V_P induces an isomorphism $V_{\mathrm{inv}}^N \cong V_P$ that is equivariant for the action of all the U_i^{GL} operators.

From our explicit computations above, it is clear that U_i^{GL} acts on V_P via $|M_j| t_i$, where $|M_j|$ is the cardinality of M_j . To see this, simply note that given any $x \in M_j$, $t_i^{-1} x t_i \in P_w^u(\mathcal{K}_w)$ fixes V_P . Therefore, the result follows since M_j contains exactly $|p|^{-i(n-i)} = \delta_P(t_i)^{-1}$ elements. \square

It is clear from Lemma 6.1 that any P_w -ordinary vector $\phi \in V_w^{N_w}$ lies in $V_{w,\mathrm{inv}}^{N_w}$ and $\pi_w(t_{w,D_w(j)})$ acts on ϕ via multiplication by

$$(94) \quad \kappa'(t_{w,D_w(j)}^{-1}) \delta_{P_w}(t_{w,D_w(j)}) c_{w,D_w(j)},$$

where $c_{w,D_w(j)}$ is its $u_{w,D_w(j)}^{\text{GL}}$ -eigenvalue (a p -adic unit), and $\kappa' = (\kappa_{\text{norm}})_p$ is related to κ as in Section 2.8. In particular, ϕ is a simultaneous eigenvector under the action of π_w for all matrices $t_{w,D_w(j)} \in G(\mathbb{Q}_p)$.

6.1.2. *Bernstein-Zelevinsky geometric lemma for P_w -ordinary representations.* In Section 6.1.3, we obtain results about the structure of the P_w -ordinary subspace of π_w via its relation to its P_w -Jacquet module, see the proof of Lemma 6.1. To understand further the P_w -Jacquet module of π_w , we use a version of the Bernstein-Zelevinsky geometric lemma (see [Ren10, Section VI.5.1] or [Cas95, Theorem 6.3.5]) that is adapted to our setting, see Lemma 6.3. However, we first need to introduce some notation.

Lemma 6.2. *Let π_w be a P_w -ordinary representation of G_w . There exists a parabolic subgroup $Q_w \subset P_w$ of G_w and a supercuspidal representation σ_w of Q such that $\pi_w \subset \iota_{Q_w}^{G_w} \sigma_w$.*

Proof. The following is a minor modification of the proof of a theorem of Jacquet, see [Cas95, Theorem 5.1.2]. We omit the subscript w to lighten the notation.

The fact that π is P -ordinary implies that $\text{r}_P^G \pi \neq 0$. By [Cas95, Theorem 3.3.1], the latter is both admissible and finitely generated so it admits an irreducible admissible quotient ρ as a representation of L .

By Frobenius reciprocity [Cas95, Theorem 2.4.1] and the irreducibility of π , it follows that $\pi \subset \iota_P^G \rho$. Then, it is a theorem of Jacquet [Cas95, Theorem 5.1.2] that there exists a parabolic $Q_L \subset L$ and a supercuspidal representation σ of its Levi factor such that $\rho \subset \iota_{Q_L}^L \sigma$. By transitivity of parabolic induction, the result follows. \square

Fix an embedding $\pi_w \hookrightarrow \iota_{Q_w}^{G_w} \sigma_w$ with the notation as in Lemma 6.2. Let M_w and Q_w^u denote the Levi factor and unipotent radical of Q_w .

Moreover, let B_w denote the Borel subgroup of G_w corresponding to the trivial partitions, as in Remark 2.8. Let T_w denote the Levi factor of B_w . In particular, T_w is the maximal torus of G_w .

Let W be the Weyl group of G_w with respect to (B_w, T_w) and consider

$$W(P_w, Q_w) = \{x \in W \mid x^{-1}(L_w \cap B_w)x \subset B_w, x(M_w \cap B_w)x^{-1} \subset B_w\} .$$

According to [Ren10, Section V.4.7], for each $x \in W(P_w, Q_w)$, $xP_w x^{-1} \cap M_w$ is a parabolic subgroup of M_w with Levi factor equal to $xL_w x^{-1} \cap M_w$. Similarly, the Levi factor of the parabolic subgroup $L_w \cap x^{-1}Q_w x \subset L_w$ is $L_w \cap x^{-1}M_w x$.

Denote the natural *conjugation-by- x* functor that sends a representation of $xLx^{-1} \cap M_w$ to a representation of $L_w \cap x^{-1}M_w x$ by $(\cdot)^x$. Moreover, let $W(L_w, M_w)$ be the subset of $x \in W(P_w, Q_w)$ such that $xL_w x^{-1} \cap M_w = M_w$, and so $L_w \cap x^{-1}M_w x = x^{-1}M_w x$. Note that this does not imply that $L_w \cap x^{-1}Q_w x$ is equal to $x^{-1}Q_w x$ but rather that its Levi subgroup is $x^{-1}M_w x$.

The following is a version of [Cas95, Theorem 6.3.5] that is adapted to our setting and notation.

Lemma 6.3. *Let $Q_w \subset P_w$ denote standard parabolic subgroups of G_w as above and let σ_w be an irreducible supercuspidal representation of M_w .*

There exists a filtration, indexed by $W(L_w, M_w)$, of $r_{P_w}^{G_w} \iota_{Q_w}^{G_w} \sigma_w$ as a representation of L_w such that the subquotient corresponding to $x \in W_{L_w}$ is isomorphic to $\iota_{L_w \cap x^{-1} Q_w x}^{L_w} \sigma_w^x$. One can order the filtration so that subquotient corresponding to $x = 1$ is a subrepresentation.

Proof. As in the previous proofs, we drop the subscript w below.

The Bernstein-Zelevinsky geometric lemma (see [Ren10, Section VI.5.1]) states that there exists a filtration of $r_P^G \iota_Q^G \sigma$ such that the corresponding graded pieces are isomorphic to

$$\iota_{L \cap x^{-1} Q x}^L (r_{x P x^{-1} \cap M}^M \sigma)^x$$

as x runs over all elements of $W(P, Q)$. Moreover, one can order the filtration so that the factor corresponding to σ (i.e. the graded piece corresponding to $x = 1$) is a subrepresentation of $r_P^G \iota_Q^G \sigma$.

Since σ is supercuspidal, the graded piece corresponding to $x \in W(P, Q)$ is nonzero if and only if $x L x^{-1} \cap M = M$, i.e. $x \in W(L, M)$. For such an x , the graded piece is clearly isomorphic to $\iota_{L \cap x^{-1} Q x}^L \sigma^x$. \square

6.1.3. *Structure theorem for P -ordinary representations of G_1 .* For simplicity, we assume that π_p satisfies the following hypothesis :

HYPOTHESIS 6.4. The parabolic subgroup Q_w for π_w from Lemma 6.2 is equal to P_w for all $w \in \Sigma_p$. In particular σ_w is a supercuspidal representation of L_w .

Remark 6.5. This hypothesis is certainly restrictive in our context. For instance, if π_p is B -ordinary, then Lemma 6.2 implies that all local factors π_w lie in a principal series. Furthermore, if π_p is B -ordinary (i.e. *ordinary* in the usual sense) then it follows immediately from our definitions that it is also P -ordinary. Therefore, the case $Q_w \neq P_w$ can certainly occur.

One can argue that this is not a major issue since in the situation above, if π_p is B -ordinary then there is little interest in considering its structure as a P -ordinary representation. One only obtains less information this way. However, if π_p is a general P -ordinary representation whose local factors π_w lie in a principal series, it is not necessarily true that π_p is also B -ordinary. In general, if π_p is P -ordinary and the supercuspidal support of all π_w is Q_w , then π_p might not be Q -ordinary, where $Q = \prod_w Q_w$. Therefore, the hypothesis above restricts us to study certain P -ordinary representations that are not Q -ordinary with respect to any smaller parabolic $B \subset Q \subsetneq P$.

In subsequent work, the author plans to adapt the proof Theorem 6.6 to remove this hypothesis.

Theorem 6.6. *Let π be a holomorphic P -ordinary representation as above satisfying Hypothesis 6.4 such that its weight κ satisfies Inequality (93). Let $\pi_w \subset \iota_{P_w}^{G_w} \sigma_w$ be its component at $w \in \Sigma_p$ as above, a P_w -ordinary representation.*

- (i) *For $r \gg 0$, let $\phi, \phi' \in \pi_w^{I_r}$ be P_w -ordinary vectors. Let φ and φ' be their respective image in $\iota_{P_w}^{G_w} \sigma_w$. If $\phi \neq \phi'$, then $\varphi(1) \neq \varphi'(1)$.*
- (ii) *For $r \gg 0$, let $\phi \in \pi_w^{I_r}$ be a simultaneous eigenvector for the $u_{w, D_w(j)}$ -operators that is not P_w -ordinary. Let φ be its image in $\iota_{P_w}^{G_w} \sigma_w$. Then, $\varphi(1) = 0$.*
- (iii) *Let τ_w be a smooth irreducible representation of $L_w(\mathcal{O}_w)$. Assume there exists an embedding $\tau_w \hookrightarrow \sigma_w$ over $L_w(\mathcal{O}_w)$. Let X_w be the vector space associated to τ_w , viewed as a subspace of the one associated to σ_w .*

Then, given $\alpha \in X_w$, there exists some $r \gg 0$ such that τ_w factors through $L_w(\mathcal{O}_w/\mathfrak{p}_w^r \mathcal{O}_w)$ and some (necessarily unique) P_w -ordinary $\phi_{r, \alpha} \in \pi_w^{I_r}$ such that $\varphi_{r, \alpha}(1) = \alpha$, where $\varphi_{r, \alpha}$ is the image of $\phi_{r, \alpha}$ in $\iota_{P_w}^{G_w} \sigma_w$. Furthermore, the support of $\varphi_{r, \alpha}$ contains $P_w I_{w, r}$. The map $\alpha \mapsto \phi_{r, \alpha}$ yields an embedding of $L_w(\mathcal{O}_w)$ -representations

$$\tau_w \hookrightarrow \pi_w^{(P_w\text{-ord}, r)} .$$

Proof. This proof is inspired by the one of [Ehls20, Lemma 8.3.2] which is itself inspired by arguments in [Hid98, Section 5]. By abuse of notation, we will always write L when we mean $L(\mathcal{K}_w)$. However, we still write $L(\mathcal{O}_w)$ when referring to its maximal compact subgroup. From now on, we omit the subscript w in this proof.

From Lemma 6.1 (and its proof), we know the space of P -ordinary vector is contained in V_{inv}^N and $\text{pr}_P : V \rightarrow V_P$ induces an isomorphism on $V_{\text{inv}}^N \xrightarrow{\sim} V_P$ which is equivariant for the action of $L(\mathcal{O})$ and the $u_{D(j)}^{\text{GL}}$ -operators. Let $s_P : V_P \rightarrow V_{\text{inv}}^N$ denote its inverse.

Consider the natural inclusion $V \hookrightarrow \iota_P^G \sigma$ and the corresponding embedding $V_P \hookrightarrow (\iota_P^G \sigma)_P$ as representations of L , using the fact that the P -Jacquet module functor is exact. Note here that we are using the unnormalized version of the P -Jacquet functor (as opposed to the normalized r_P^G).

Consider the filtration indexed by $W(L, L)$ of $(\iota_P^G \sigma)_P$ from Lemma 6.3. We use a version with unnormalized P -Jacquet functor, hence the graded piece corresponding to $x \in W(L, L)$ is isomorphic to $\sigma^x \delta_P^{1/2}$.

First, we claim that pr_P maps any simultaneous eigenvector for the $u_{D(j)}$ -operators whose eigenvalues are all p -adic units inside the subrepresentation $\sigma \delta_P^{1/2}$ corresponding to $x = 1$.

One readily checks that $x \in W(P, P)$ is in $W(L, L)$ if and only if it simply permutes the $\text{GL}_{n_k}(\mathcal{K}_w)$ -blocks of L of the same size. In particular, exactly one such $x \in W(L, L)$ acts trivially on the center $Z(L)$ of L , namely $x = 1$, while any other $1 \neq x \in W(L, L)$ stabilizes but acts non-trivially on $Z(L)$.

Using the explicit representatives from Section 6.1.1, one readily checks that the operator $u_{D(j)}^{\text{GL}}$ acts on $\sigma^x \delta_P^{1/2}$ via multiplication by

$$\beta_x(s_j) := \kappa'(s_j) \delta_P^{-1/2}(s_j) \omega_\sigma^x(s_j) = |\kappa'(s_j)|_p^{-1} \delta_P^{-1/2}(s_j) \omega_\sigma^x(s_j)$$

where $s_j = t_{D(j)}$, $\omega_\sigma : Z(L) \rightarrow \mathbb{C}^\times$ is the central character of σ , and $\omega_\sigma^x(-) = \omega_\sigma(x(-)x^{-1})$ is the central character of σ^x .

These β_x define unramified characters of $Z(L)$. The P -ordinarity assumption implies that $\beta_1(s_j)$ is a p -adic unit for all $1 \leq j \leq t+r$ and therefore $\beta_1(s)$ is a p -adic unit for all $s \in Z(L)$. We claim that given any $x \in W(L, L)$, the values of β_x on $Z(L)$ are all p -adic units if and only if $x = 1$.

By recalling that δ_P and δ_B agree on $Z(L)$ and proceeding exactly as in the proof of [EHLS20, Lemma 8.3.2], one uses Inequality (93) to show that

$$\theta = |\kappa'|^{-1} \delta_P^{-1/2}$$

is a regular character of $Z(L)$ and β_x satisfies the above property if and only if $\theta^x = \theta$. By regularity, this only occurs when $x = 1$.

The argument above shows that under the natural map

$$(95) \quad V_{\text{inv}}^N \hookrightarrow V \twoheadrightarrow V_P \hookrightarrow (\iota_P^G \sigma)_P,$$

the subspace of P -ordinary vector of V injects into the subrepresentation $\sigma \delta_P^{1/2}$ of $(\iota_P^G \sigma)_P$, as desired.

This map is exactly the composition of $V_{\text{inv}}^N \xrightarrow{\sim} V_P$ with the map $i : V_P \rightarrow \sigma \delta_P^{1/2}$ corresponding under the Frobenius reciprocity to the inclusion $v \mapsto f_v$ of V into $\iota_P^G \sigma$. In other words, this map is $v \mapsto f_v(1)$. Therefore, a P -ordinary vector $v \in V^N$ is uniquely determined by $f_v(1)$. This shows part (i).

For part (ii), pick a simultaneous eigenvector $v \in V_{\text{inv}}^N$ for the $u_{D(j)}^{\text{GL}}$ -operators that is not P -ordinary. Then, as above, the composition $V_{\text{inv}}^N \xrightarrow{\sim} V_P \rightarrow \sigma \delta_P^{1/2}$ sends v to $f_v(1)$. By equivariance of the action of the $u_{D(j)}^{\text{GL}}$ -operators on both sides, we must have $f_v(1) = 0$.

To show part (iii), consider α as an element of the vector space associated to σ , which is also the one associated to $\sigma \delta_P^{1/2} \subset V_P$. Let $\phi = s_P(\alpha) \in V_{\text{inv}}^N$. In particular, $\phi \in \pi^{Lr}$ for some $r \gg 0$. We may assume that r is sufficiently large so that τ factors through $L(\mathcal{O}/\mathfrak{p}^r \mathcal{O})$.

Finally, since pr_P is equivariant under the action of the $u_{D(j)}^{\text{GL}}$ -operators and these act on $\text{pr}_P(\phi) = \alpha$ via multiplication by the p -adic unit $\beta(s_j)$, one concludes that ϕ is P -ordinary. Proceeding as in the proof of part (i), we obtain $\varphi(1) = \text{pr}_P \phi = \alpha$, where $\varphi \in \iota_P^G \sigma$ is the function corresponding to ϕ .

Therefore, $\phi_{r,\alpha} := \phi$ is the desired vector, necessarily unique by part (i). The last statement holds because s_P is $L(\mathcal{O}_w)$ -equivariant. \square

Remark 6.7. As a consequence of the proof for part (i) above, we see that π_w is P_w -ordinary (of level $r \gg 0$) if and only if

$$(96) \quad \beta(s) = |\kappa'(s)|_p^{-1} \delta_{P_w}^{-1/2}(s) \omega_\sigma(s)$$

is a p -adic unit for all $s \in Z(L_w(\mathcal{K}_w))$. In other words, not all supercuspidal representation σ_w can occur. Furthermore, when π_w is P_w -ordinary (of level $r \gg 0$), then all P_w -ordinary vectors share the same $u_{w, D_w(j), \kappa}^{\text{GL}}$ -eigenvalue, namely $\beta(t_{w, D_w(j)})$.

Remark 6.8. We now view τ_w as a representation of $I_{w,r}^0$ via the identity $I_{w,r}^0/I_{w,r} = L_w(\mathcal{O}_w/\mathfrak{p}_w^r \mathcal{O}_w)$. Clearly, the embedding constructed in Theorem 6.6 (iii) is an embedding of $I_{w,r}^0$ -representations.

This shows π_w contains a cover of τ_w from L_w to $I_{w,r}^0$, in the sense of [BK98, BK99], in its subspace $\pi_{w,r}^{P_w\text{-ord}}[\tau_w]$ of P_w -ordinary vectors of type τ_w . However, we do not use this point of view explicitly in this paper.

Recall that in Section 1.2.3, we fixed an ‘‘SZ-types’’ τ_w , namely a smooth irreducible representation of $L_w(\mathcal{O}_w)$ such that $\sigma_w|_{L_w(\mathcal{O}_w)}$ contains τ_w with multiplicity one. Such a representation exists but is not necessarily unique.

We sometimes refer to τ_w as the SZ-type of π_w . Let τ be the representation of $L_P(\mathbb{Z}_p)$ corresponding to $\otimes_{w \in \Sigma_p} \tau_w$ under the natural identification $L_P = \prod_{w \in \Sigma_p} L_w$ induced by (62). We refer to τ as the (fixed choice of) SZ-type of π_p .

Theorem 6.9. *Let π be a holomorphic P -ordinary representation as above such that its weight κ satisfies Inequality (93). Let τ be the SZ-type of π_p . Then,*

$$\text{Hom}_{L_P(\mathbb{Z}_p)}(\tau, \pi_p^{(P\text{-ord}, r)})$$

is 1-dimensional for all $r \gg 0$. In other words, π is of P -WLT (κ, K_r, τ) for all $r \gg 0$, the space $\pi_p^{(P\text{-ord}, \tau)} := \pi_p^{(P\text{-ord}, r)}[\tau]$ of P -ordinary vectors of type τ is independent of $r \gg 0$ and

$$\dim \left(\pi_p^{(P\text{-ord}, \tau)} \right) = \dim \tau .$$

Proof. Fix $w \in \Sigma_p$ and consider $\pi_w^{(P_w\text{-ord}, r)} = e_w \pi_w^{I_{w,r}}$ for $r \gg 0$. By Theorem 6.6 (iii), there is a natural isomorphism

$$\text{Hom}_{L_w(\mathcal{O}_w)}(\tau_w, \sigma_w) = \text{Hom}_{L_w(\mathcal{O}_w)}(\tau_w, \pi_w^{(P_w\text{-ord}, r)}[\tau_w]) ,$$

where τ_w is any smooth irreducible representation of $L_w(\mathcal{O}_w)$. The result follows by applying the above to $\tau_w = \tau_w$. \square

6.2. P -anti-ordinary theory on G_1 . Let π be an anti-holomorphic cuspidal representation on $G = G_1$ of weight κ such that $\pi_f^{K_f} \neq 0$. Recall that π is P -anti-ordinary of level r if π_w is P_w -anti-ordinary of level r , for all $w \in \Sigma_p$.

Recall that according to our conventions set in Section 1, given any representation ρ , we denote its contragredient representation by ρ^\vee .

Lemma 6.10. *The representation π_w is P_w -anti-ordinary of level $r \geq 0$ if and only if π_w^\vee is P_w -ordinary of level r . In that case, π_w is P -anti-ordinary of all level $r \gg 0$.*

Proof. This is a simple generalization of [Ehls20, Lemma 8.3.6 (i)]. The proof goes through verbatim by replacing the pro- p Iwahori subgroup (also denoted $I_{w,r}$) by $I_{P_w,w,r}$ and only considering the Hecke operators $u_{w,D_w(j)}^{\text{GL},-}$ and $u_{w,D_w(j)}^{\text{GL}}$, for $1 \leq j \leq r_w$. The key part is that all these operators commute with one another. \square

6.2.1. Conventions on contragredient pairings. Let σ_w be an admissible irreducible supercuspidal representation of $L_w(\mathcal{K}_w)$. Its contragredient σ_w^\vee is again an admissible irreducible supercuspidal representation of $L_w(\mathcal{K}_w)$.

Let $\langle \cdot, \cdot \rangle_{\sigma_w} : \sigma_w \times \sigma_w^\vee \rightarrow \mathbb{C}$ be the tautological pairing on a pair of contragredient representations. Define

$$\begin{aligned} \langle \cdot, \cdot \rangle_w &: \iota_{P_w}^{G_w} \sigma_w \times \iota_{P_w}^{G_w} \sigma_w^\vee \rightarrow \mathbb{C} \\ \langle \varphi, \varphi^\vee \rangle_w &= \int_{G_w(\mathcal{O}_w)} \langle \varphi(k), \varphi^\vee(k) \rangle_{\sigma_w} dk, \end{aligned}$$

a perfect $G_w(\mathcal{K}_w)$ -equivariant pairing. Here dk is the Haar measure on $G_w(\mathcal{O}_w)$ that such that $\text{Vol}(G_w(\mathcal{O}_w)) = 1$ with respect to dk . Then $\langle \cdot, \cdot \rangle_w$ naturally identifies $\iota_{P_w}^{G_w} \sigma_w^\vee$ as the contragredient of $\iota_{P_w}^{G_w} \sigma_w$.

Let π_w be the constituent at $w \in \Sigma_p$ of π_p as above. From now on, we assume π_w is the unique irreducible quotient $\iota_{P_w}^{G_w} \sigma_w \twoheadrightarrow \pi_w$. Equivalently, π_w^\vee is the unique irreducible subrepresentation $\pi_w^\vee \hookrightarrow \iota_{P_w}^{G_w} \sigma_w^\vee$, see Remark 6.5. If one restricts the second argument of $\langle \cdot, \cdot \rangle_w$ to π_w^\vee , then the first argument factors through π_w . In other words, $\langle \cdot, \cdot \rangle_w$ induces the tautological pairing $\langle \cdot, \cdot \rangle_{\pi_w} : \pi_w \times \pi_w^\vee \rightarrow \mathbb{C}$ and

$$\langle \phi, \phi^\vee \rangle_{\pi_w} = \int_{G_w(\mathcal{O}_w)} \langle \varphi(k), \varphi^\vee(k) \rangle_{\sigma_w} dk, \quad \forall \phi \in \pi_w, \phi^\vee \in \pi_w^\vee,$$

where φ is any lift of ϕ and φ^\vee is the image of ϕ^\vee .

Let (τ_w, X_w) be the SZ-type of σ_w , a representation of $L_w(\mathcal{O}_w)$. Then, its contragredient (τ_w^\vee, X_w^\vee) is the SZ-type of σ_w^\vee . One can find $L_w(\mathcal{O}_w)$ -embeddings $\tau_w \hookrightarrow \sigma_w$ and $\tau_w^\vee \hookrightarrow \sigma_w^\vee$ (both unique up to scalar) such that for all $\alpha \in X_w$, $\alpha^\vee \in X_w^\vee$,

$$\langle \alpha, \alpha^\vee \rangle_{\sigma_w} = \langle \alpha, \alpha^\vee \rangle_{\tau_w}.$$

More generally, upon restriction of σ_w and σ_w^\vee to representations of $L_w(\mathcal{O}_w)$, there are direct sum decompositions

$$\sigma_w = \bigoplus_{\tau_w} \sigma_w[\tau_w] \quad \text{and} \quad \sigma_w^\vee = \bigoplus_{\tau_w} \sigma_w^\vee[\tau_w]$$

where τ_w runs over all smooth irreducible representations of $L_w(\mathcal{O}_w)$ and the square brackets $[\cdot]$ denote isotypic subspaces. The restriction of $\langle \cdot, \cdot \rangle_{\sigma_w}$ to $\sigma_w[\tau_w] \times \sigma_w^\vee[\tau_w^\vee]$ is identically zero if $\tau_w \not\cong \tau_w^\vee$. On the other hand, its restriction to $\sigma_w[\tau_w] \times \sigma_w^\vee[\tau_w^\vee]$ is a perfect $L_w(\mathcal{O}_w)$ -invariant pairings.

6.2.2. *Structure theorem for P -anti-ordinary representations of G_1 .*

Theorem 6.11. *Let π be an anti-holomorphic P -anti-ordinary representation such that its weight κ satisfies Inequality (93). Let $w \in \Sigma_p$ and π_w be a constituent of π , a P_w -anti-ordinary representation of level $r \gg 0$. Assume π_w is the unique irreducible quotient $\iota_{P_w}^{G_w} \sigma_w \twoheadrightarrow \pi_w$ for some supercuspidal σ_w and let (τ_w, X_w) be the SZ-type of σ_w .*

Given any $\alpha \in X_w$, let $\varphi_{w,r}^{P_w\text{-a.ord}} \in \iota_{P_w}^{G_w} \sigma_w$ be the unique vector with support $P_w I_{w,r}$ such that $\varphi_{w,r}^{P_w\text{-a.ord}}(1) = \alpha$ and $\varphi_{w,r}^{P_w\text{-a.ord}}$ is fixed by $I_{w,r}$.

Then, the image $\phi_{w,r}^{P_w\text{-a.ord}} \in \pi_w^{I_{w,r}}$ of $\varphi_{w,r}^{P_w\text{-a.ord}}$ is P_w -anti-ordinary of level $r \gg 0$. Furthermore, it satisfies :

- (i) Let $\phi^\vee \in \pi_w^{\vee, I_{w,r}}$ and denote its image in $\iota_{P_w}^{G_w} \sigma_w$ by φ^\vee . Then,

$$\langle \phi_{w,r}^{P_w\text{-a.ord}}, \phi^\vee \rangle_{\pi_w} = \text{Vol}(I_{w,r}^0) \langle \alpha, \varphi^\vee(1) \rangle_{\sigma_w}$$

In particular, $\langle \phi_{w,r}^{P_w\text{-a.ord}}, \phi^\vee \rangle_{\pi_w} \neq 0$ if and only if ϕ^\vee is P_w -ordinary and the component of $\varphi^\vee(1)$ in $\sigma_w^\vee[\tau_w^\vee]$ is non-zero.

- (ii) The vector $\phi_{w,r}^{P_w\text{-a.ord}}$ lies in the τ_w -isotypic space of $\pi_w^{I_{w,r}}$. Moreover, any other P_w -anti-ordinary vector of type τ_w is obtained as above for some other choice $\alpha' \in X_w$.
- (iii) One can pick different choices of α for each $r' \geq r$ so that

$$(97) \quad \sum_{\gamma \in I_{w,r}/(I_{w,r}^0 \cap I_{w,r'})} \pi_w(\gamma) \phi_{w,r'}^{P_w\text{-a.ord}} = \phi_{w,r}^{P_w\text{-a.ord}}$$

Proof. Write $\phi_{w,r}$ and $\varphi_{w,r}$ instead of $\phi_{w,r}^{P_w\text{-a.ord}}$ and $\varphi_{w,r}^{P_w\text{-a.ord}}$ respectively. We first show that property (i) holds. By Lemma 6.10, π_w^\vee is P_w -ordinary of level r . Write

$$\pi_w^{\vee, I_{w,r}} = \bigoplus_{a=1}^A V_a,$$

where each V_a is a simultaneous generalized eigenspace for the Hecke operators $u_{w, D_w(j)}^{\text{GL}}$.

From the proof of Theorem 6.6 and the remark that follows, exactly one V_a has generalized eigenvalues that are all p -adic units. We may assume that this holds true for V_1 . The exact eigenvalue of $u_{w, D_w(j)}^{\text{GL}}$ is given by Equation (96), denote it $\beta_{w, D_w(j)}$. For $1 < a \leq A$, at least one generalized eigenvalue for V_a is not a p -adic unit.

Given $\phi^\vee \in \pi_w^{\vee, I_{w,r}}$, write it as a sum

$$\phi^\vee = \sum_{a=1}^A \phi_a^\vee,$$

with $\phi_a^\vee \in V_a$. Let φ_a^\vee denote the images of ϕ_a^\vee in $\iota_{P_w}^{G_w} \sigma_w^\vee$. Then,

$$\langle \phi_{w,r}, \phi^\vee \rangle_{\pi_w} = \sum_{a=1}^A \langle \phi_{w,r}, \phi_a^\vee \rangle_{\pi_w} = \sum_{a=1}^A \int_{G_w(\mathcal{O}_w)} \langle \varphi_{w,r}(k), \varphi_a^\vee(k) \rangle_{\sigma_w} dk$$

Recall that the support of $\varphi_{w,r}$ is $P_w I_{w,r}$. Also, the intersection of $P_w I_{w,r}$ with $G_w(\mathcal{O}_w)$ is equal to $I_{w,r}^0$ and by Theorem 6.6 (ii), $\varphi_a^\vee(I_{w,r}^0) = 0$ for all $a \neq 1$. Therefore,

$$\langle \phi_{w,r}, \phi^\vee \rangle_{\pi_w} = \int_{I_{w,r}^0} \langle \varphi_{w,r}(k), \varphi_1^\vee(k) \rangle_{\sigma_w} dk$$

Since $I_{w,r}^0 = L_w(\mathcal{O}_w) I_{w,r}$ and $\varphi_{w,r}^{P_w\text{-a.ord}}$, φ_1^\vee are both fixed by $I_{w,r}$, one obtains

$$\langle \phi_{w,r}, \phi^\vee \rangle_{\pi_w} = \int_{I_{w,r}^0} \langle \varphi_{w,r}(1), \varphi_1^\vee(1) \rangle_{\sigma_w} dk = \text{Vol}(I_{w,r}^0) \langle \alpha, \varphi_1^\vee(1) \rangle_{\sigma_w} .$$

The desired relation holds by noting that $\varphi_1^\vee(1) = \varphi^\vee(1)$. The second part of (i) follows immediately from the discussion about isotypic subspaces at the end of Section 6.2.1.

As a consequence of property (i), we immediately obtain $\langle \phi_{w,r}, V_a \rangle_{\pi_w} = 0$ for all $a > 1$. Furthermore, for all $\phi^\vee \in V_1$, we have

$$\langle u_{w,D_w(j)}^{\text{GL},-} \phi_{w,r}, \phi^\vee \rangle_{\pi_w} = \langle \phi_{w,r}, u_{w,D_w(j)}^{\text{GL}} \phi^\vee \rangle_{\pi_w} = \beta_{w,D_w(j)} \langle \phi_{w,r}, \phi^\vee \rangle_{\pi_w} .$$

By combining these two facts, we obtain

$$\langle u_{w,D_w(j)}^{\text{GL},-} \phi_{w,r}, \phi^\vee \rangle_{\pi_w} = \beta_{w,D_w(j)} \langle \phi_{w,r}, \phi^\vee \rangle_{\pi_w} .$$

for all ϕ^\vee in $\pi_w^{\vee, I_{w,r}}$. In other words, $\phi_{w,r}$ is P_w -anti-ordinary.

Furthermore, note that the argument above implies that the subspace of P_w -anti-ordinary vectors of type τ_w in $\pi_w^{\vee, I_{w,r}}$ is dual to the subspace of P_w -ordinary vectors of type τ_w^\vee . From Theorem 6.6 (iii), they both have dimension $\dim \tau_w = \dim \tau_w^\vee$.

In particular, the space generated by the action of $L_w(\mathcal{O}_w/\mathfrak{p}_w^r \mathcal{O}_w)$ on $\phi_{w,r}$, which is of dimension $\dim \tau_w$, is exactly the subspace of P_w -anti-ordinary vectors of type τ_w . Therefore, any other P_w -anti-ordinary vector $\phi'_{w,r}$ of type τ_w is equal to $\pi_w(l) \phi_{w,r}$, for some $l \in L_w(\mathcal{O}_w/\mathfrak{p}_w^r \mathcal{O}_w)$. One readily sees that it obtained by picking $\alpha' = \tau_w(l) \alpha$ in X_w instead of α . This proves the second sentence of part (ii).

Finally, part (iii) follows immediately from the fact that the analogous properties hold for $\varphi_{w,r}$. \square

Keeping the assumption and notation of Theorem 6.11, fix a vector $\alpha \in X_w$. From Lemma 6.10, we know $\pi_w^\vee \hookrightarrow \iota_{P_w}^{G_w} \sigma_w^\vee$ is P_w -ordinary.

Let (τ_w^\vee, X^\vee) be the SZ-type of π_w^\vee and fix any $\alpha^\vee \in X^\vee$ such that $\langle \alpha, \alpha^\vee \rangle_{\tau_w} = 1$. Let $\phi_{w,r}^{\vee, P_w\text{-ord}}$ be the P_w -ordinary vector associated to α^\vee obtained from Theorem 6.6 (iii).

In fact, as r increases, one may pick compatible choices of α so that (97) holds and compatible choices of α^\vee such that $\langle \alpha, \alpha^\vee \rangle_{\tau_w} = 1$ for all $r \gg 0$. Then, as a consequence of Theorem 6.11 (i),

$$\frac{\langle \phi_{w,r}^{P_w\text{-a.ord}}, \phi_{w,r}^{\vee, P_w\text{-ord}} \rangle_w}{\text{Vol}(I_{w,r}^0)} = \langle \alpha, \alpha^\vee \rangle_{\sigma_w} = \langle \alpha, \alpha^\vee \rangle_{\tau_w} = 1$$

is independent of $r \gg 0$.

Furthermore, one readily obtains a result analogous to Theorem 6.9 from Theorem 6.11. Namely, let $\tau = \bigotimes_{w \in \Sigma_p} \tau_w$ be the SZ-type of π_p , using the identification (62).

Corollary 6.12. *Let π be an anti-holomorphic cuspidal representation of G of weight κ . Suppose κ satisfies Inequality (93). Then, π is P -anti-ordinary if and only if π^\vee is P -ordinary.*

In that case, assume π^\vee satisfies Hypothesis 6.4 and let $\tau = \bigotimes_{w \in \Sigma_p} \tau_w$ be the SZ-type of π . There exists a unique (up to the action of $L_P(\mathbb{Z}_p)$) P -anti-ordinary vector $\phi_r^{P\text{-a.ord}}$ of level r and type τ in $\pi_p^{I_{P,r}}$. Furthermore, for each $w \in \Sigma_p$, there exists P_w -ordinary vectors $\phi_{w,r}^{P_w\text{-a.ord}}$ of level r and type τ_w as in Theorem 6.11 such that, under the identification $\pi_p = \mu_p \otimes \bigotimes_{w \in \Sigma_p} \pi_w$, we have $\phi_r^{P\text{-a.ord}} = \bigotimes_{w \in \Sigma_p} \phi_{w,r}^{P_w\text{-a.ord}}$.

6.3. P -ordinary theory on G_2 . In this section, we proceed to compare the theory of P -(anti)-ordinary representations on G_1 and G_2 , where G_i is the unitary group associated to the PEL datum \mathcal{P}_i introduced in Section 4.1.1. We add a subscript V (resp. $-V$) in our notation whenever we want to emphasize that we are working with G_1 (resp. G_2).

6.3.1. Comparison between representations of G_1 and G_2 . Recall that there is a canonical identification $G_1(\mathbb{A}) = G_2(\mathbb{A})$. Furthermore, the identification from isomorphism (5) remains the same for both G_1 and G_2 . However, the opposite choices of $\mathcal{O}_K \otimes \mathbb{Z}_p$ -lattices $L_1^\pm = L_2^\mp$ introduce many changes in the notation.

For instance, under the identification $G_1(\mathbb{A}) = G_2(\mathbb{A})$, the group $H_{0,-V} = H_0$ for G_1 corresponds to $H_{0,-V}$ (by switching the roles of Λ_0 and Λ_0^\vee .) However, the identification from isomorphism (18) interchanges the role of $\sigma \in \Sigma_K$ and σc (where c denotes complex conjugation).

Given a dominant weight κ of $T_1 := T_{H_{0,V}}$, it is identified with a tuple $(\kappa_0, (\kappa_\sigma)_\sigma)$ where $\kappa_0 \in \mathbb{Z}$ and $\kappa_\sigma \in \mathbb{Z}^{b_\sigma}$. The torus $T_2 := T_{H_{0,-V}}$ is equal to T_1 but the corresponding isomorphism (18) for G_2 identifies κ with $(\kappa_0, (\kappa_{\sigma c})_\sigma)$. We denote the latter by κ^b . In particular, $\kappa_{\sigma c} \in \mathbb{Z}^{a_\sigma} = \mathbb{Z}^{b_{\sigma c}}$ and κ^b is dominant with respect to $B_{H_{0,-V}}^{\text{op}}$.

As explained in [EHLS20, Sections 6.2.1-6.2.2], if π is a cuspidal (anti-)holomorphic automorphic representation for G_1 of weight κ , then $\pi^b = \pi^\vee \otimes ||\nu||^{a(\kappa)}$ (as in Section 4.2.3) is naturally a cuspidal (anti-)holomorphic automorphic representation for G_2 of weight κ^b .

Furthermore, by choosing the same partitions \mathbf{d}_w introduced in Section 2.2.2, the parabolic subgroup $P_w \subset \mathrm{GL}_n(\mathcal{O}_w)$ for G_1 corresponding to $w \in \Sigma_p$ is replaced by the opposite parabolic subgroups, which in our case is simply its transpose ${}^tP_w \subset \mathrm{GL}_n(\mathcal{O}_w)$, when working with G_2 . Similarly, P is replaced by tP and the (resp. pro- p) P -Iwahori subgroup of level r is replaced by the (resp. pro- p) tP -Iwahori subgroup of level r .

In particular, if $\pi_p \cong \mu_p \otimes \bigotimes_{w \in \Sigma_p} \pi_w$ is the identification obtained from (62) for G_1 , the corresponding factorization on G_2 induces

$$\pi_p^{\flat} \cong \mu_p^{\flat} \otimes \bigotimes_{w \in \Sigma_p} \pi_w^{\flat},$$

where $\pi_w^{\flat} = \pi_w^{\vee}$ and $\mu_p^{\flat} = \mu_p^{-1} |\nu|_p^{\alpha(\kappa)}$, by definition of π^{\flat} .

6.3.2. Structure theorem for P -ordinary representations of G_2 . The discussion above shows that π_w is P_w -ordinary of level $r \gg 0$ (for G_1) if and only if π_w^{\flat} is P_w -ordinary of level $r \gg 0$ (for G_2). As explained in Section 4.1.3, adapting the definitions for P -ordinary theory from G_1 to G_2 requires to change P_w for tP_w and the double coset operators $U_{w,j}^{\mathrm{GL}}$ for $U_{w,j}^{\flat, \mathrm{GL}} = {}^tI_{w,r} t_{w,j}^{-1} {}^tI_{w,r}$. The analogue of Theorem 6.6 is the following.

Lemma 6.13. *Let π be an holomorphic cuspidal representation of G_1 . Suppose its weight κ satisfies Inequality (93). Assume that π_w is P_w -ordinary of level $r \gg 0$ (for G_1) and that it is the unique irreducible subrepresentation of $\iota_{P_w}^{G_w} \sigma_w$ for some irreducible supercuspidal σ_w . Let (τ_w, X_w) be the SW-type of π_w .*

- (i) *The unique irreducible quotient of $\iota_{P_w}^{G_w} \sigma_w^{\vee}$ is isomorphic to π_w^{\flat} .*
- (ii) *Let $(\tau_w^{\vee}, X_w^{\vee})$ be the contragredient of (τ_w, X_w) , the BK-type of σ_w^{\vee} . Consider X_w^{\vee} as a subspace of the vector space associated to σ_w^{\vee} , via a fix embedding (unique up to scalar) $\tau_w^{\vee} \hookrightarrow \sigma_w^{\vee}$.*

For any $\alpha^{\vee} \in X_w^{\vee}$, let $\varphi_w^{\flat} \in \iota_{P_w}^{G_w} \sigma_w^{\vee}$ be the unique function with support $P_w {}^tI_{w,r}$ (for all $r \gg 0$) such that $\varphi_w^{\flat}(1) = \alpha^{\vee}$ and φ_w^{\flat} is fixed by ${}^tI_{w,r}$ (for all $r \gg 0$). Let ϕ_w^{\flat} denote its image in π_w^{\flat} .

Then, ϕ_w^{\flat} is P_w -ordinary of type $\tau_w^{\vee} = \tau_w^{\flat}$ of level $r \gg 0$. In particular, π_w^{\flat} is P_w -ordinary of level $r \gg 0$ (for G_2).

This induces a natural isomorphism between τ_w^{\flat} and the subspace of π_w^{\flat} of P_w -ordinary vectors of type τ_w^{\flat} of level $r \gg 0$. The latter is independent of $r \gg 0$ and has dimension $\dim \tau_w^{\flat} = \dim \tau_w$.

Proof. Consider the composition of $\pi_w \hookrightarrow \iota_{P_w}^{G_w} \sigma_w$ with the map (of vector spaces)

$$\begin{aligned} \iota_{P_w}^{G_w} \sigma_w &\rightarrow \iota_{{}^tP_w}^{G_w} \sigma_w^{\vee} \\ \phi &\mapsto \phi^{\vee}(g) := \phi({}^tg^{-1}) \end{aligned}$$

Its image is $\pi_w^b = \pi_w^\vee$ and the above realizes π_w^b as the unique irreducible subrepresentation of $\iota_{P_w}^{G_w} \sigma_w^\vee$. In particular, all the consequences of Theorem 6.6 hold for π_w^b by replacing P_w by tP_w and σ_w by σ_w^\vee .

Given $\alpha^\vee \in X_w^\vee$ as above, let $\phi_w^\vee \in \pi_w^{\vee, I_{w,r}}$ and $\varphi_w^\vee \in \iota_{P_w}^{G_w} \sigma_w^\vee$ be the vectors obtained from Theorem 6.6 (iii) associated to α^\vee . In particular, ϕ_w^\vee is a P_w -ordinary vector of type τ_w^\vee and the subspace generated by the action of $L_w(\mathcal{O}_w)$ on ϕ_w^\vee is exactly the space of all P_w -ordinary vectors of type τ_w^\vee . In particular, the latter is independent of $r \gg 0$ and isomorphic to τ_w^\vee as a representation of $L_w(\mathcal{O}_w)$.

Lastly, consider the standard intertwining operator $\iota_{P_w}^{G_w} \sigma_w^\vee \rightarrow \iota_{P_w}^{G_w} \sigma_w^\vee$. Its image is π_w^b , hence identifies π_w^b as the unique irreducible quotient of $\iota_{P_w}^{G_w} \sigma_w^\vee$.

One readily checks that under this intertwining operator, the vector $\varphi_w^b \in \iota_{P_w}^{G_w} \sigma_w^\vee$ described in the statement of the lemma maps to φ_w^\vee . Therefore, the desired properties for ϕ_w^b can be verified through φ_w^\vee . \square

6.4. P -anti-ordinary theory on G_2 . Going back to the discussion of Section 6.3, we know that π_w is P_w -anti-ordinary of level $r \gg 0$ (for G_1) if and only if π_w^b is P_w -anti-ordinary of level $r \gg 0$ (for G_2). As explained in Section 4.1.3, adapting the definitions for P -anti-ordinary theory from G_1 to G_2 requires to change P_w for tP_w and the double coset operators $U_{w,j}^{\text{GL},-}$ for $U_{w,j}^{b,\text{GL},-} = {}^tI_{w,r} t_{w,j} {}^tI_{w,r}$. The analogue of Theorem 6.11 is the following.

Lemma 6.14. *Let π be an anti-holomorphic cuspidal representation of G_1 . Suppose its weight κ satisfies Inequality (93). Assume that π_w is P_w -anti-ordinary of level $r \gg 0$ (for G_1) and that it is the unique irreducible quotient of $\iota_{P_w}^{G_w} \sigma_w$ for some irreducible supercuspidal σ_w . Let (τ_w, X_w) be the SZ-type of π_w .*

- (i) *The unique irreducible subrepresentation of $\iota_{P_w}^{G_w} \sigma_w^\vee$ is isomorphic to π_w^b .*
- (ii) *Let (τ_w^\vee, X_w^\vee) be the contragredient of (τ_w, X_w) , the SZ-type of σ_w^\vee . Consider X_w^\vee as a subspace of the vector space associated to σ_w^\vee , via a fix embedding (unique up to scalar) $\tau_w^\vee \hookrightarrow \sigma_w^\vee$.*

For each $r \gg 0$ and $\alpha \in X_w^\vee$, there exists some unique P_w -anti-ordinary $\phi_{w,r}^b \in \pi_w^{b, {}^tI_r}$ of type τ_w^\vee and level r such that $\phi_{w,r}^b(1) = \alpha$, where $\phi_{w,r}^b$ is the image of $\phi_{w,r}^\vee$ in $\iota_{P_w}^{G_w} \sigma_w^\vee$, and the support of $\phi_{w,r}^b$ contains $P_w {}^tI_{w,r}$. In particular, π_w^b is P_w -anti-ordinary of level $r \gg 0$ (for G_2).

- (iii) *For $r' > r \gg 0$, one can choose $\alpha, \alpha' \in X_w^\vee$ such that the vectors $\phi_{w,r}^b$ and $\phi_{w,r'}^b$ corresponding to α and α' respectively satisfy*

$$\sum_{\gamma \in {}^tI_{w,r} / ({}^tI_{w,r'}^0 \cap {}^tI_{w,r})} \pi_w^b(\gamma) \phi_{w,r'}^b = \phi_{w,r}^b$$

Proof. As in the proof of Lemma 6.13, the map

$$\begin{aligned} \iota_{P_w}^{G_w} \sigma_w^\vee &\rightarrow \iota_{P_w}^{G_w} \sigma_w \\ \phi &\mapsto \phi^\vee(g) := \phi({}^t g^{-1}), \end{aligned}$$

realizes $\pi_w^b = \pi_w^\vee$ as the unique irreducible quotient of $\iota_{P_w}^{G_w} \sigma_w^\vee$.

In particular, all the consequences of Theorem 6.11 hold for π_w^b by replacing P_w by ${}^t P_w$ and σ_w by σ_w^\vee . Given $\alpha \in X_w^\vee$ as above, let $\varphi'_{w,r} \in \iota_{P_w}^{G_w} \sigma_w^\vee$ be the vectors obtained from Theorem 6.11 associated to α .

Furthermore, consider the standard intertwining operator $\iota_{P_w}^{G_w} \sigma_w^\vee \xrightarrow{\sim} \iota_{P_w}^{G_w} \sigma_w^\vee$. Its image is both the unique irreducible quotient of $\iota_{P_w}^{G_w} \sigma_w^\vee$, namely π_w^b , and the unique irreducible subrepresentation of $\iota_{P_w}^{G_w} \sigma_w^\vee$. This proves part (i).

To conclude, let $\phi_{w,r}^b$ (resp. $\varphi_{w,r}^b$) be the image of $\varphi'_{w,r}$ in π_w^\vee (resp. $\iota_{P_w}^{G_w} \sigma_w^\vee$) via this intertwining operator. The fact that $\phi_{w,r}^b$ is ${}^t P_w$ -anti-ordinary of type τ_w^\vee and level r follows from Theorem 6.11 (ii). Similarly, part (iii) follows from Theorem 6.11 (iii) (upon making the appropriate adjustments between G_1 and G_2). The properties of $\varphi'_{w,r}$ are obtained from an easy computation using the definition of $\varphi'_{w,r}$ and the exact formula for the intertwining operator above. \square

7. EXPLICIT CHOICE OF P -(ANTI)-ORDINARY VECTORS.

In what follows, we freely use the notation from Sections 3.1.1 and 3.1.3. In particular, let $\pi = \pi_\infty \otimes \pi_f$ be a cuspidal automorphic representation for G_1 of level $K \subset G_1(\mathbb{A}_f)$ and unramified away from $S = S(K^p)$ and p , and let π^\vee denote its contragredient.

The goal of this section is to single out a set of *test vectors* in a P -anti-ordinary anti-holomorphic cuspidal automorphic representation π on G_1 . Our strategy is to construct local test vectors $\varphi_l \in \pi_l$ for all places l of \mathbb{Q} and consider $\varphi = \otimes_l \varphi_l \in \pi$ via (59).

Then, we use the involutions in Section 4.2 to obtain a compatible space of test vectors for π^b on G_2 . Recall that π^b is defined as a twist of π^\vee , hence it suffices specify a space of test vectors in π^\vee .

Throughout this section, we assume that π is anti-holomorphic of a certain weight κ , hence π_f (and π_f^b) is defined over some number field $E(\pi)$, see Remark 3.2. Recall that we always assume that $E(\pi)$ contains \mathcal{K}' . We further assume that π (resp. π^b) is P -anti-ordinary of level $r \gg 0$ for G_1 (resp. for G_2).

We work with the $G(\mathbb{Q}_l)$ -equivariant perfect pairing $\langle \cdot, \cdot \rangle_{\pi_l} : \pi_l \times \pi_l^\vee \rightarrow \mathbb{C}$, for each place $l \leq \infty$ of \mathbb{Q} , as in Section 3.1.3.

Remark 7.1. Using the involution F_∞ from Section 4.2.2, this leads to test vectors for holomorphic, P -ordinary cuspidal automorphic representations.

7.1. Local test vectors at places away from p and ∞ .

7.1.1. *Local test vectors at unramified places.* For each finite prime $l \notin S \cup \{p\}$, we fix $E(\pi)$ -rational K_l -spherical vectors $\varphi_{l,0} \in \pi_l$ and $\varphi_{l,0}^\vee \in \pi_l^\vee$ such that $\langle \varphi_{l,0}, \varphi_{l,0}^\vee \rangle_{\pi_l} = 1$, as in Section 3.1.3.

7.1.2. *Local test vectors at ramified places.* On the other hand, the choice of local test vectors at $l \in S$ is non-canonical. We adapt the same conventions as in [EHLS20, Section 4.2.2].

Given $l \in S$, fix an arbitrary irreducible $U_1(\mathbb{Q}_l)$ -subrepresentation $\underline{\pi}_l$ of π_l . The dual $\underline{\pi}_l^\vee$ of $\underline{\pi}_l$ occurs as an irreducible $U_1(\mathbb{Q}_l)$ -subrepresentation of π_l^\vee . Furthermore, the bilinear $\langle \cdot, \cdot \rangle_{\pi_l} : \pi_l \times \pi_l^\vee \rightarrow \mathbb{C}$ induces a perfect $U_1(\mathbb{Q}_l)$ -equivariant pairing between $\underline{\pi}_l$ and $\underline{\pi}_l^\vee$, again denoted $\langle \cdot, \cdot \rangle_{\pi_l}$.

Since U_1 is the restriction of scalar of a reductive group from \mathcal{K}^+ to \mathbb{Q} , we have $U_1(\mathbb{Q}_l) = \prod_{v|l} U_{1,v}$, where the product is over the places v of \mathcal{K}^+ above l and $U_{1,v}$ is the set of \mathcal{K}_v^+ -points of a unitary group over \mathcal{K}^+ . Similarly, we obtain

$$(98) \quad \underline{\pi}_l \cong \bigotimes_{v|l} \underline{\pi}_v \quad \text{and} \quad \underline{\pi}_l^\vee \cong \bigotimes_{v|l} \underline{\pi}_v^\vee$$

for irreducible admissible representations $\underline{\pi}_v$ and $\underline{\pi}_v^\vee$ of $U_{1,v}$.

Naturally, there are $U_{1,v}$ -equivariant perfect pairings $\langle \cdot, \cdot \rangle_{\pi_v} : \underline{\pi}_v \times \underline{\pi}_v^\vee \rightarrow \mathbb{C}$, identifying $\underline{\pi}_v^\vee$ as the contragredient of $\underline{\pi}_v$, such that $\langle \cdot, \cdot \rangle_{\pi_l} = \prod_{v|l} \langle \cdot, \cdot \rangle_{\pi_v}$.

Fix *any* nonzero vectors $\varphi_v \in \underline{\pi}_v$ and $\varphi_v^\vee \in \underline{\pi}_v^\vee$ such that $\langle \varphi_v, \varphi_v^\vee \rangle_{\pi_v} = 1$. Our choice of local test vectors $\varphi_l \in \pi_l$ and $\varphi_l^\vee \in \pi_l^\vee$ are

$$\varphi_l := \otimes_{v|l} \varphi_v \quad \text{and} \quad \varphi_l^\vee := \otimes_{v|l} \varphi_v^\vee,$$

via (98). In Section 8.4.4, we restrict our attention slightly and choose an integral structure for such local test vectors.

Remark 7.2. In Section 9.4, we use this naive choice of test vectors at $l \in S$ suffices to obtain non-zero constant local zeta integrals, essentially volume factors, that are insensitive to the variation of π in a p -adic family. This approach is standard in the literature, see [EHLS20, Section 4.2.2].

7.2. **Local test vectors at p .** In this section, we choose test vectors at p following the strategy developed in [EHLS20, Section 4.3.4]. However, to generalize their results, we need to work out various extra details due to the fact that spaces of P -ordinary vectors are not 1-dimensional in general, see Theorem 6.11 and Lemma 6.14. The theory of types of P -(anti)-ordinary vectors here is used as a substitute for the lack of “ordinary nebentypus”, see [EHLS20, Section 6.6.6], in the general P -(anti)-ordinary setting.

7.2.1. *Local representations over CM type at p .* Let $w \in \Sigma_p$ and set $G_w := \mathrm{GL}_n(\mathcal{K}_w)$. As in Section 3.2, the isomorphisms (5) and (7) induce an identification $G(\mathbb{Q}_p) =$

$\mathbb{Q}_p^\times \times \prod_{w \in \Sigma_p} G_w$ as well as an isomorphism

$$(99) \quad \pi_p \cong \mu_p \otimes \left(\bigotimes_{w \in \Sigma_p} \pi_w \right),$$

where μ_p is some character of \mathbb{Q}_p and π_w is an irreducible admissible representations of G_w . Since π is P -anti-ordinary of level $r \gg 0$, we know μ_p is unramified and

$$\pi_p^{I_{P,r}} \cong \bigotimes_{w \in \Sigma_p} \pi_w^{I_{w,r}} \neq 0.$$

Similarly, for the contragredient π^\vee of π , we have

$$(100) \quad \pi_p^\vee \cong \mu_p^{-1} \otimes \left(\bigotimes_{w \in \Sigma_p} \pi_w^\vee \right)$$

and $(\pi_w^\vee)^{I_{w,r}} \neq 0$ for each $w \in \Sigma_p$. Note that $\pi_w^\vee = \pi_w^b$ for each $w \in \Sigma_p$.

7.2.2. Compatibility of parabolic subgroups. For each $w \in \Sigma_p$ and integer $d \geq 1$, let $G_w(d)$ denote the algebraic group $\mathrm{GL}(d)$ over $\mathcal{O}_w = \mathcal{O}_{\mathcal{K}_w}$. However, when $d = n$, we still write G_w instead of $G_w(n)$. Let (a_w, b_w) be the signature at $w \in \Sigma_p$ associated to the PEL datum $\mathcal{P} = \mathcal{P}_1$, as in Section 2.1.

Proceeding as in Section 2.2.2, let $P_{a_w} \subset G_w(a_w)$, $P_{b_w} \subset G_w(b_w)$ and $P_{a_w, b_w} \subset G_w$ be the standard upper triangular parabolic subgroups associated to partitions

$$\mathbf{d}_{a_w} = (n_{w,1}, \dots, n_{w,t_w}) \quad ; \quad \mathbf{d}_{b_w} = (n_{w,t_w+1}, \dots, n_{w,r_w}) \quad ; \quad \mathbf{d}_w = (a_w, b_w)$$

of a_w , b_w and n , respectively. We also work with the parabolic subgroup $P_w \subset G_w$ constructed in (11). Note that $P_w \subset P_{a_w, b_w} \subset G_w$.

For any one of these parabolic subgroup P_\bullet , let L_\bullet denote its standard Levi subgroup consisting of block-diagonal matrices (corresponding to the decomposition defining P_\bullet). Similarly, consider the pro- p Iwahori subgroup $I_{\bullet,r}$ of level r associated to P_\bullet consisting of invertible matrices g (of the appropriate size) over \mathcal{O}_w such that $g \bmod \mathfrak{p}_w^r$ is in $P_\bullet^u(\mathcal{O}_w/\mathfrak{p}_w^r \mathcal{O}_w)$, see Definition 2.9 and (14).

Let $K_\bullet = L_\bullet(\mathcal{O}_w)$ and $I_{\bullet,r}^0 = K_\bullet I_{\bullet,r}$. Setting $K_{w,j} = \mathrm{GL}_{n_{w,j}}(\mathcal{O}_w)$, we have

$$K_{a_w} = \prod_{j=1}^{t_w} K_{w,j} \quad ; \quad K_{b_w} = \prod_{j=t_w+1}^{r_w} K_{w,j} \quad ; \quad K_w = K_{a_w} \times K_{b_w},$$

where the products take place in $G_w(a_w)$, $G_w(b_w)$ and G_w , respectively.

7.2.3. Compatibility of local representations. Since π is P -anti-ordinary, we may assume (see Lemma 6.2, Lemma 6.10 and Section 6.2.1) that there exists an admissible irreducible representation σ_w of L_w such that π_w is the unique irreducible quotient of $\iota_{P_w}^{G_w} \sigma_w$. Equivalently, π_w^\vee is the unique irreducible subrepresentation of $\iota_{P_w}^{G_w} \sigma_w^\vee$.

Remark 7.3. We do not assume that σ_w is supercuspidal.

Write $\sigma_w = \boxtimes_{j=1}^{r_w} \sigma_{w,j}$ and consider the representations

$$\sigma_{a_w} = \boxtimes_{j=1}^{t_w} \sigma_{w,j} \quad ; \quad \sigma_{b_w} = \boxtimes_{j=t_w+1}^{r_w} \sigma_{w,j}$$

of L_{a_w} and L_{b_w} . Let π_{a_w} and π_{b_w} be the unique irreducible quotients

$$(101) \quad \iota_{P_{a_w}}^{G_w(a_w)} \sigma_{a_w} \twoheadrightarrow \pi_{a_w} \quad \text{and} \quad \iota_{P_{b_w}^{\text{opp}}}^{G_w(b_w)} \sigma_{b_w} \twoheadrightarrow \pi_{b_w} ,$$

and set $\pi_{a_w, b_w} := \pi_{a_w} \boxtimes \pi_{b_w}$. Under the canonical isomorphism

$$(102) \quad \iota_{P_w}^{G_w} \sigma_w \xrightarrow{\sim} \iota_{P_{a_w, b_w}}^{G_w} \left(\iota_{P_{a_w} \times P_{b_w}^{\text{opp}}}^{G_w(a_w) \times G_w(b_w)} \sigma_{a_w} \boxtimes \sigma_{b_w} \right) ,$$

given by $\phi \mapsto (g \mapsto (h \mapsto \phi(hg)))$, π_w is the unique irreducible quotient

$$(103) \quad \iota_{P_{a_w, b_w}}^{G_w} (\pi_{a_w, b_w}) \twoheadrightarrow \pi_w .$$

7.2.4. Conventions on local pairings above p . In this section, we refine the conventions on pairings set in Section 6.2.1 to local places above p . This follows the approach of [Ehls20, Section 4.3.3].

Let $\langle \cdot, \cdot \rangle_{\sigma_{w,j}}$ be the tautological pairing between $\sigma_{w,j}$ and its contragredient $\sigma_{w,j}^\vee$. Then, define $(\cdot, \cdot)_{a_w} = \otimes_{i=1}^{t_w} \langle \cdot, \cdot \rangle_{\sigma_{w,j}}$ so that

$$\begin{aligned} \langle \cdot, \cdot \rangle_{a_w} &: \left(\iota_{P_{a_w}}^{G_w(a_w)} \sigma_{a_w} \right) \times \left(\iota_{P_{a_w}}^{G_w(a_w)} \sigma_{a_w}^\vee \right) \rightarrow \mathbb{C} \\ \langle \varphi, \varphi^\vee \rangle_{a_w} &= \int_{K_{a_w}} (\varphi(k), \varphi^\vee(k))_{a_w} dk \end{aligned}$$

is the perfect $G_w(a_w)$ -invariant pairing that identify the above pair as contragredient representations. A similar logic applies for $(\cdot, \cdot)_{b_w} = \otimes_{i=t_w+1}^{r_w} \langle \cdot, \cdot \rangle_{\sigma_{w,j}}$ and

$$\begin{aligned} \langle \cdot, \cdot \rangle_{b_w} &: \left(\iota_{P_{b_w}^{\text{opp}}}^{G_w(b_w)} \sigma_{b_w} \right) \times \left(\iota_{P_{b_w}^{\text{opp}}}^{G_w(b_w)} \sigma_{b_w}^\vee \right) \rightarrow \mathbb{C} \\ \langle \varphi, \varphi^\vee \rangle_{b_w} &= \int_{K_{b_w}} (\varphi(k), \varphi^\vee(k))_{b_w} dk . \end{aligned}$$

Taking the dual of the surjections in Equation (101) yields injections

$$(104) \quad \pi_{a_w}^\vee \hookrightarrow \iota_{P_{a_w}}^{G_w(a_w)} \sigma_{a_w}^\vee \quad \text{and} \quad \pi_{b_w}^\vee \hookrightarrow \iota_{P_{b_w}^{\text{opp}}}^{G_w(b_w)} \sigma_{b_w}^\vee$$

and restricting the second argument of $\langle \cdot, \cdot \rangle_{a_w}$ to $\pi_{a_w}^\vee$ makes the first argument of the pairing factor through π_{a_w} . It is identified with the tautological pairing $\langle \cdot, \cdot \rangle_{\pi_{a_w}} : \pi_{a_w} \times \pi_{a_w}^\vee \rightarrow \mathbb{C}$. Again, a similar logic applies for $\langle \cdot, \cdot \rangle_{\pi_{b_w}} : \pi_{b_w} \times \pi_{b_w}^\vee \rightarrow \mathbb{C}$.

Let $(\cdot, \cdot)_w = \langle \cdot, \cdot \rangle_{\pi_{a_w}} \otimes \langle \cdot, \cdot \rangle_{\pi_{b_w}}$. As above, it determines a pairing

$$\langle \cdot, \cdot \rangle_w : \iota_{P_{a_w, b_w}}^{G_w} (\pi_{a_w, b_w}) \times \iota_{P_{a_w, b_w}}^{G_w} (\pi_{a_w, b_w}^\vee) \rightarrow \mathbb{C}$$

as well as a pairing $\langle \cdot, \cdot \rangle_w : \pi_w \times \pi_w^\vee \rightarrow \mathbb{C}$, using the dual $\pi_w^\vee \hookrightarrow \iota_{P_{a_w, b_w}}^{G_w} (\pi_{a_w, b_w}^\vee)$ induced from Equation (103).

Remark 7.4. One may normalize these pairings so that $\langle \cdot, \cdot \rangle_{\pi_p} = \prod_{w \in \Sigma_p} \langle \cdot, \cdot \rangle_{\pi_w}$.

For any $\phi \in \pi_w$, $\phi^\vee \in \pi_w^\vee$, if φ is a lift of ϕ and φ^\vee is the image of ϕ^\vee , then

$$(105) \quad \langle \phi, \phi^\vee \rangle_{\pi_w} = \int_{\mathrm{GL}_n(\mathcal{O}_w)} (\varphi(k), \varphi^\vee(k))_w dk$$

7.2.5. Compatibility of test vectors. For each $1 \leq j \leq r_w$, let $\tau_{w,j}$ be a smooth (finite-dimensional) irreducible representation of $K_{w,j}$. We assume that r is large enough so that $\tau_{w,j}$ factors through $\mathrm{GL}_{n_{w,j}}(\mathcal{O}_w/\mathfrak{p}_w^r \mathcal{O}_w)$. Assume there exists an embedding $\alpha_{w,j}$ of $\tau_{w,j}$ in the restriction of $\sigma_{w,j}$ as a representation of $K_{w,j}$.

Let $\alpha_{a_w} : \tau_{a_w} \rightarrow \sigma_{a_w}$ and $\alpha_{b_w} : \tau_{b_w} \rightarrow \sigma_{b_w}$ be the corresponding embeddings over K_{a_w} and K_{b_w} respectively, where

$$\tau_{a_w} = \boxtimes_{j=1}^{t_w} \tau_{w,j} \quad ; \quad \tau_{b_w} = \boxtimes_{j=t_w+1}^{r_w} \tau_{w,j} .$$

Remark 7.5. Implicitly, we think of τ_{a_w} as the SZ-type of σ_{a_w} , in the sense Section 1.2.3. In that case, there exists a unique such embedding $\alpha_{w,j}$ (up to scalar) and Theorem 6.11 is concerned about constructing a canonical lift of α_{a_w} to an embedding of τ_{a_w} into $\pi_{a_w}^{(P_w\text{-a.ord}, r)}$. For now, Theorem 6.11 only deals with σ_{a_w} supercuspidal. However, in the following we proceed as if this theorem held for arbitrary admissible σ_{a_w} . In other words, we conjecture that we can omit the supercuspidal hypothesis 6.4 and proceed without comments. Note that similar statements can be made about τ_{b_w} and $\tau_w := \tau_{a_w} \boxtimes \tau_{b_w}$.

For each $j = 1, \dots, r_w$, fix a vector $\phi_{w,j}$ in the image of $\alpha_{w,j}$ and consider

$$(106) \quad \phi_{a_w}^0 := \bigotimes_{j=1}^{t_w} \phi_{w,j} \quad ; \quad \phi_{b_w}^0 := \bigotimes_{j=t_w+1}^{r_w} \phi_{w,j}$$

as vectors in the image of α_{a_w} and α_{b_w} respectively.

Remark 7.6. In Section 11, given local representations $\tau_{w,j}$ and $\sigma_{w,j}$ as above, we work with such local vectors with respect $\tau_{w,j} \otimes \psi_{w,j}$ and $\sigma_{w,j} \otimes \psi_{w,j}$, where $\psi_{w,j}$ is a finite-order character of $K_{w,j}$ (viewing $\tau_{w,j}$ as fixed and $\psi_{w,j}$ as varying). We always assume that the corresponding test vectors in the image of $\alpha_{w,j} \otimes \mathrm{id}$ are $\phi_{w,j} \otimes 1$, i.e. essentially the ‘‘same’’ local vectors. See Remark 1.5.

Let $\varphi_{a_w} \in \iota_{P_{a_w}}^{G_w(a_w)} \sigma_{a_w}$ be the unique function fixed by $I_{a_w,r}$ that has support $P_{a_w} I_{a_w,r}$ and

$$(107) \quad \varphi_{a_w}(\gamma) = \sigma_{a_w}(\gamma) \phi_{a_w}^0 = \tau_{a_w}(\gamma) \phi_{a_w}^0 ,$$

for all $\gamma \in I_{a_w,r}^0$. Denote its image in π_{a_w} by ϕ_{a_w} .

Remark 7.7. Here, we implicitly identify τ_{a_w} with its image in σ_{a_w} and as a representation of $I_{a_w,r}^0$ that factors through $I_{a_w,r}^0/I_{a_w,r} \cong L_{a_w}(\mathcal{O}_w/\mathfrak{p}_w^r \mathcal{O}_w)$. In what follows, we similarly identify τ_{b_w} (resp. $\tau_{a_w}^\vee, \tau_{b_w}^\vee$) with its cover as a representation of ${}^t I_{b_w,r}^0$ (resp. ${}^t I_{a_w,r}^0, I_{b_w,r}^0$) contained in σ_{b_w} (resp. $\sigma_{a_w}^\vee, \sigma_{b_w}^\vee$).

Let $\varphi_{b_w} \in \iota_{P_{b_w}^{\text{op}}}^{G_w(b_w)} \sigma_{b_w}$ be the unique function whose support is $P_{b_w}^{\text{op}} {}^t I_{b_w, r}$ such that

$$(108) \quad \varphi_{b_w}(\gamma) = \tau_{b_w}(\gamma) \phi_{b_w}^0,$$

for all $\gamma \in {}^t I_{b_w, r}^0$. Let ϕ_{b_w} denote its image in π_{b_w} .

Lastly, consider the unique function $\varphi_w \in \iota_{P_w}^{G_w} \sigma_w$ fixed by $I_{w, r}$ whose support is $P_w I_{w, r}$ and

$$(109) \quad \varphi_w(\gamma) = \tau_w(\gamma) (\phi_{a_w}^0 \otimes \phi_{b_w}^0),$$

for all $\gamma \in I_{w, r}^0$, where $\tau_w = \tau_{a_w} \boxtimes \tau_{b_w}$. Here, we view τ_w as a $I_{w, r}^0$ -subrepresentation of σ_w , see Remark 7.7.

For our purposes, it is more convenient to work with the vector corresponding to φ_w via the map $\iota_{P_w}^{G_w} \sigma_w \rightarrow \iota_{P_{a_w, b_w}}^{G_w} \pi_{a_w, b_w}$ induced by the maps in (101) and (102). We denote this image by φ_w again, which should not cause any confusion since we will only ever work with φ_w in $\iota_{P_{a_w, b_w}}^{G_w} \pi_{a_w, b_w}$ from now on.

One easily checks that the support of φ_w is $P_{a_w, b_w} I_{w, r}$ and

$$\varphi_w(\gamma) = \tau_w(\gamma) (\phi_{a_w} \otimes \phi_{b_w}),$$

for all $\gamma \in I_{w, r}^0$. Let ϕ_w be the image of φ_w in π_w .

Remark 7.8. If σ_w is supercuspidal, for each $w \in \Sigma_p$, then ϕ_{a_w} (resp. ϕ_{b_w} , ϕ_w) is a P_{a_w} -anti-ordinary (resp. ${}^t P_{b_w}$ -anti-ordinary, P_w -anti-ordinary) vector of level r and type τ_{a_w} (resp. τ_{b_w} , τ_w) as in 6.11.

We now proceed similarly by constructing explicit vectors related to the contra-redient representations. Since $\sigma_{w, j}$ is admissible, for $j = 1, \dots, r_w$, we also have an embedding $\alpha_{w, j}^\vee : \tau_{w, j}^\vee \rightarrow \sigma_{w, j}^\vee$ of $K_{w, j}$ -representations. We identify the natural contra-redient pairing on $\tau_{w, j} \times \tau_{w, j}^\vee$ with the restriction of $\langle \cdot, \cdot \rangle_{\sigma_{w, j}}$ via their fixed embedding in $\sigma_{w, j} \times \sigma_{w, j}^\vee$.

Remark 7.9. If $\tau_{w, j}$ is the SZ-type of $\sigma_{w, j}$ as in Remark 7.5, then $\tau_{w, j}^\vee$ is also the SZ-type of $\sigma_{w, j}^\vee$. In that case, such maps $\alpha_{w, j}^\vee$ again exist and are unique up to scalar.

Fix a vector $\phi_{w, j}^\vee \in \sigma_{w, j}^\vee$ in the image of $\alpha_{w, j}^\vee$ such that $\langle \phi_{w, j}, \phi_{w, j}^\vee \rangle_{\sigma_{w, j}} = 1$ and define

$$(110) \quad \phi_{a_w}^{\vee, 0} := \bigotimes_{j=1}^{t_w} \phi_{w, j}^\vee \quad ; \quad \phi_{b_w}^{\vee, 0} := \bigotimes_{j=t_w+1}^{r_w} \phi_{w, j}^\vee$$

as vectors in $\sigma_{a_w}^\vee$ and $\sigma_{b_w}^\vee$ respectively.

Remark 7.10. As in Remark 7.6, if we replace $\tau_{w, j}$ by $\tau_{w, j} \otimes \psi$, for some finite-order character $\psi_{w, j}$ of $K_{w, j}$, then we always assume that the corresponding choice

of local vector in the image of $\alpha_{w,j}^\vee \otimes \text{id}$ is $\phi_{w,j}^\vee \otimes 1$. Once again, see Remark 1.5 for further details.

Assume there exists a vector $\phi_{a_w}^\vee$ in $\pi_{a_w}^\vee$ fixed by ${}^t I_{a_w,r}$ such that the support of its image $\varphi_{a_w}^\vee$ in ${}^{\iota_{P_{a_w}^{G_w(a_w)}}} \sigma_{a_w}^\vee$ contains $P_{a_w} {}^t I_{a_w,r}$ and that

$$(111) \quad \varphi_{a_w}^\vee(\gamma) = \tau_{a_w}^\vee(\gamma) \phi_{a_w}^{\vee,0}, \quad \forall \gamma \in {}^t I_{a_w,r}^0.$$

Similarly, assume there exists a vector $\phi_{b_w}^\vee$ in $\pi_{b_w}^\vee$ fixed by $I_{b_w,r}$ such that the support of its image $\varphi_{b_w}^\vee$ in ${}^{\iota_{P_{b_w}^{G_w(b_w)}}} \sigma_{b_w}^\vee$ contains $P_{b_w} I_{b_w,r}$ and that

$$(112) \quad \varphi_{b_w}^\vee(\gamma) = \tau_{b_w}^\vee(\gamma) \phi_{b_w}^{\vee,0}, \quad \forall \gamma \in I_{b_w,r}^0.$$

Lastly, assume there exists a vector ϕ_w^\vee in π_w^\vee fixed by ${}^t I_{w,r}$ such that the support of its image φ_w^\vee in ${}^{\iota_{P_{a_w,b_w}^{G_w}}} \pi_{a_w,b_w}^\vee$ contains $P_w {}^t I_{w,r}$ and that

$$(113) \quad \varphi_w^\vee(\gamma) = \tau_w^\vee(\gamma) (\phi_{a_w}^\vee \otimes \phi_{b_w}^\vee), \quad \forall \gamma \in {}^t I_{w,r}^0.$$

Remark 7.11. As in Remark 7.8, assume that σ_w is supercuspidal for each $w \in \Sigma_p$. In that case, Lemma 6.14 proves the existence of the vectors $\phi_{a_w}^\vee$, $\phi_{b_w}^\vee$ and ϕ_w^\vee . In the last case, we are implicitly using the isomorphism (102) to compare *loc. cit.* with our notation here.

In particular, in that case $\phi_{a_w}^\vee$ (resp. $\phi_{b_w}^\vee$, ϕ_w^\vee) is ${}^t P_{a_w}$ -anti-ordinary (resp. P_{b_w} -anti-ordinary, ${}^t P_w$ -anti-ordinary) of type $\tau_{a_w}^\vee$ (resp. $\tau_{b_w}^\vee$, τ_w^\vee) in the sense of Section 6.4.

7.2.6. *Choice of P -anti-ordinary test vectors (and twists).* Our choice of test vectors at p is

$$(114) \quad \varphi_p = 1 \otimes \left(\bigotimes_{w \in \Sigma_p} \phi_w \right) \in \pi_p \quad \text{and} \quad \varphi_p^\vee = 1 \otimes \left(\bigotimes_{w \in \Sigma_p} \phi_w^\vee \right) \in \pi_p^\vee,$$

via (99) and (100). By definition of π^b , φ_p^\vee naturally corresponds to some $\varphi_p^b \in \pi_p^\vee$.

Observe that for each $w \in \Sigma_p$, the construction of ϕ_w not only depends on the choice of SZ-type $\tau_{w,j}$ of $\sigma_{w,j}$ but also on the choice of nonzero vectors $v_{w,j} \in \tau_{w,v}$, see (106).

Let π' be some other anti-holomorphic P -anti-ordinary automorphic representation of G_1 , with SZ-type τ' at p . Let $\sigma'_{w,j}$ and $\tau'_{w,j}$ be the analogues for π' of $\sigma_{w,j}$ and $\tau_{w,j}$ for π , as in Section 7.2.3.

If $\tau' = \tau \otimes \psi$ for some character ψ of $L(\mathbb{Z}_p)$, for instance if $\sigma_{w,j} = \sigma'_{w,j} \otimes \psi_{w,j}$ for some unramified character $\psi_{w,j}$ of $\text{GL}_{n_{w,j}}(K_w)$ (see conventions set in Section 1.2.3), then the vectors spaces for $\tau_{w,j}$ and $\tau'_{w,j}$ are canonically identified.

We always assume that the vector φ'_p for π'_p is obtained from the same choices of vectors $v_{w,j}$ in this situation. We impose a similar convention for the dual vectors $\varphi_p^\vee \in \pi_p^\vee$ and $\varphi'_p{}^\vee \in \pi'_p{}^\vee$.

7.2.7. *Inner products between test vectors.* Observe that the intersection of the support of φ_w with $\mathrm{GL}_n(\mathcal{O}_w)$ is $P_{a_w, b_w} I_{w, r} \cap \mathrm{GL}_n(\mathcal{O}_w) = I_{a_w, b_w, r}^0$. Therefore,

$$\langle \phi_w, \phi_w^\vee \rangle_{\pi_w} = \int_{I_{a_w, b_w, r}^0} (\varphi_w(k), \varphi_w^\vee(k))_{a_w, b_w} d^\times k ,$$

Write any $k \in I_{a_w, b_w, r}^0$ as

$$k = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}$$

where $A \in \mathrm{GL}_{a_w}(\mathcal{O}_w)$, $D \in \mathrm{GL}_{b_w}(\mathcal{O}_w)$, $B \in M_{a_w \times b_w}(\mathcal{O}_w)$ and $C \in \mathfrak{p}_w^r M_{b_w \times a_w}(\mathcal{O}_w)$.

Since $\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}$ is in P_{a_w, b_w} and $\begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}$ is in both $I_{w, r}$ and ${}^t I_{w, r}$, we see that

$$\varphi_w(k) = \varphi_w \left(\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right) = \pi_{a_w}(A) \phi_{a_w} \otimes \pi_{b_w}(D) \phi_{b_w}$$

and

$$\varphi_w^\vee(k) = \varphi_w^\vee \left(\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right) = \pi_{a_w}^\vee(A) \phi_{a_w}^\vee \otimes \pi_{b_w}^\vee(D) \phi_{b_w}^\vee ,$$

so we obtain

$$(115) \quad \langle \phi_w, \phi_w^\vee \rangle_{\pi_w} = \mathrm{Vol}(I_{a_w, b_w, r}^0) \langle \phi_{a_w}, \phi_{a_w}^\vee \rangle_{\pi_{a_w}} \langle \phi_{b_w}, \phi_{b_w}^\vee \rangle_{\pi_{b_w}} .$$

Similar arguments yield

$$(116) \quad \langle \phi_{a_w}, \phi_{a_w}^\vee \rangle_{\pi_{a_w}} = \mathrm{Vol}(I_{P_{a_w, r}}^0) (\phi_{a_w}^0, \phi_{a_w}^{\vee, 0})_{a_w} = \mathrm{Vol}(I_{P_{a_w, r}}^0)$$

and

$$(117) \quad \langle \phi_{b_w}, \phi_{b_w}^\vee \rangle_{\pi_{b_w}} = \mathrm{Vol}(I_{P_{b_w, r}}^0) (\phi_{b_w}^0, \phi_{b_w}^{\vee, 0})_{b_w} = \mathrm{Vol}(I_{P_{b_w, r}}^0) ,$$

using the fact that $(\phi_{w, j}^0, \phi_{w, j}^{\vee, 0}) = 1$ for each $1 \leq j \leq r_w$. Ultimately, we obtain

$$(118) \quad \langle \phi_w, \phi_w^\vee \rangle_{\pi_w} = \mathrm{Vol}(I_{a_w, b_w, r}^0) \mathrm{Vol}(I_{P_{a_w, r}}^0) \mathrm{Vol}(I_{P_{b_w, r}}^0) = \mathrm{Vol}(I_{w, r}^0) ,$$

which in particular is nonzero.

7.3. Local test vectors at ∞ . In this section, we choose local test vectors for π_∞ and π_∞^\vee . This material is well-established in the literature. The author redirects the reader to [Ehls20, Section 4.4] for ample details.

7.3.1. *Anti-holomorphic modules for G_1 .* First consider $G^* = R_{\mathcal{K}/\mathbb{Q}} \mathrm{GU}^+(V, \langle \cdot, \cdot \rangle)$, where $\mathrm{GU}^+(V, \langle \cdot, \cdot \rangle)$ is the full unitary group associated to $\mathcal{P} = \mathcal{P}_1$. We have

$$G^*(\mathbb{R}) = \prod_{\sigma \in \Sigma} G_\sigma ,$$

where $G_\sigma = \mathrm{GU}^+(V)_{\mathcal{K}_\sigma} \simeq \mathrm{GU}^+(a_\sigma, b_\sigma)$. Here, we implicitly use the identification between $\Sigma_{\mathcal{K}^+}$ and Σ . Note that $G(\mathbb{R})$ consists of the subgroup of elements $(g_\sigma)_{\sigma \in \Sigma}$ for which the similitude factors $v(g_\sigma)$ are independent of $\sigma \in \Sigma$.

We view the map h introduced in Section 2.1 associated to \mathcal{P} as a homomorphism

$$h = \prod_{\sigma \in \Sigma} h_{\sigma} : R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m/\mathbb{C}) \rightarrow G_{\mathbb{R}}^*$$

whose image is contained in $G_{\mathbb{R}}$. Note that $\mathfrak{g} = \mathrm{Lie}(G(\mathbb{R}))_{\mathbb{C}} = \mathrm{Lie}(G^*(\mathbb{R}))_{\mathbb{C}} = \oplus_{\sigma} \mathfrak{g}_{\sigma}$, where $\mathfrak{g}_{\sigma} = \mathrm{Lie}(G_{\sigma})$.

Let $U_{\infty} = C(\mathbb{R}) \subset G(\mathbb{R})$ be the stabilizer of h via conjugation, as in Section 2.6.1. Then, π_{∞} and π_{∞}^{\vee} are both irreducible $(\mathfrak{g}, U_{\infty})$ -modules.

For each $\sigma \in \Sigma_{\mathcal{K}^+}$, let $U_{\sigma} = U_{\infty} \cap G_{\sigma}$ and let $K_{\sigma}^{\circ} \subset U_{\sigma}$ be its maximal compact subgroup. One readily checks that K_{σ} is isomorphic to $U(a_{\sigma}) \times U(b_{\sigma})$.

As in Section 2.6.3, the Harish-Chandra decomposition for \mathfrak{g}_{σ} is

$$\mathfrak{g}_{\sigma} = \mathfrak{p}_{\sigma}^{-} \oplus \mathfrak{k}_{\sigma} \oplus \mathfrak{p}_{\sigma}^{+},$$

where $\mathfrak{p}_{\sigma}^{\pm}$ is the (± 1) -eigenspace of $\mathrm{ad} h_{\sigma}(\sqrt{-1})$ on \mathfrak{g}_{σ} and \mathfrak{k}_{σ} is the 0-eigenspace. In particular, $\mathfrak{k}_{\sigma} = \mathrm{Lie}(U_{\sigma}) = \mathfrak{z}_{\sigma} \oplus \mathrm{Lie}(K_{\sigma}^{\circ})$, where \mathfrak{z}_{σ} is the \mathbb{R} -split center of \mathfrak{g}_{σ} .

Recall that we assume that h is *standard*, see Hypothesis 2.1. This implies the above decomposition is rational over $\sigma(\mathcal{K}) \subset \mathbb{C}$.

Furthermore, the fact that h is standard is equivalent to the existence of a specific maximal rational torus T of G such that h factors through $T \hookrightarrow G$. See [EHL20, Section 2.3.2] for further details and the exact construction of T (denoted $J_0^{(n)}$). In particular, (T, h) is a Shimura datum.

We decompose π_{∞} and π_{∞}^{\vee} as

$$\pi_{\infty} = \bigotimes_{\sigma \in \Sigma} \pi_{\sigma} \quad \text{and} \quad \pi_{\infty}^{\vee} = \bigotimes_{\sigma \in \Sigma} \pi_{\sigma}^{\vee},$$

for contragredient pairs of irreducible $(\mathfrak{g}_{\sigma}, U_{\sigma})$ -modules π_{σ} and π_{σ}^{\vee} .

The fact that π_{∞} is anti-holomorphic (for G_1) of weight $\kappa = (\kappa_0, (\kappa_{\sigma}))$ implies that for each $\sigma \in \Sigma$,

$$(119) \quad \pi_{\sigma} \cong U(\mathfrak{g}_{\sigma}) \bigotimes_{U(\mathfrak{k}_{\sigma} \oplus \mathfrak{p}_{\sigma}^{+})} W_{\kappa_{\sigma}} =: \mathbb{D}_c(\kappa_{\sigma}),$$

where $U(-)$ is the universal enveloping algebra functor, and $W_{\kappa_{\sigma}}$ is the irreducible representation of U_{σ} of highest weight κ_{σ} .

7.3.2. Anti-holomorphic modules for G_2 . If we consider π^{\flat} as a representation of G_2 instead of π as a representation of G_1 , all of the theory above remains the same. However the roles of $\mathfrak{p}_{\sigma}^{+}$ and $\mathfrak{p}_{\sigma}^{-}$ are reversed.

One therefore obtains the isomorphism

$$(120) \quad \pi_{\sigma}^{\flat} \cong U(\mathfrak{g}_{\sigma}) \bigotimes_{U(\mathfrak{k}_{\sigma} \oplus \mathfrak{p}_{\sigma}^{-})} W_{\kappa_{\sigma}^{\flat}} = \mathbb{D}_c(\kappa_{\sigma}^{\flat}),$$

and it follows from our discussion in Section 4.2.2 that $\mathbb{D}(\kappa_{\sigma}) := \mathbb{D}_c(\kappa_{\sigma})^{\vee} \cong \mathbb{D}_c(\kappa_{\sigma}^{\flat})$ as representations of $U(a_{\sigma}, b_{\sigma})$.

Furthermore, $\mathbb{D}_c(\kappa_\sigma)$ is isomorphic to the complex conjugate of $\mathbb{D}_c(\kappa_\sigma^b)$ with respect to the \mathbb{R} -structure on \mathfrak{g}_σ .

Remark 7.12. It is well-known that when the weight κ satisfies Inequality (93), i.e. κ is *strongly positive*, then the modules $\mathbb{D}_c(\kappa_\sigma)$ and $\mathbb{D}(\kappa_\sigma)$ are the *anti-holomorphic* and *holomorphic discrete series* representations of G_σ respectively. See [Ehls20, Section 4.4.1].

7.3.3. Choice of anti-holomorphic test vectors. One refers to the subspaces $1 \otimes W_{\kappa_\sigma}$ and $1 \otimes W_{\kappa_\sigma^b}$ as the *minimal U_σ -type* of $\mathbb{D}_c(\kappa_\sigma)$ and $\mathbb{D}_c(\kappa_\sigma^b)$ respectively.

For each σ , let $\varphi_{\kappa_\sigma, -} \in \pi_\sigma$ be a lowest-weight vector in the minimal U_σ -type of $\mathbb{D}_c(\kappa_\sigma)$ and $\varphi_{\kappa_\sigma^b, -}^b \in \pi_\sigma^b$ be a lowest-weight vector in the minimal U_σ -type of $\mathbb{D}_c(\kappa_\sigma^b)$, both unique up to scalar. We normalize them so that $\langle \varphi_{\kappa_\sigma, -}, \varphi_{\kappa_\sigma^b, -}^b \rangle_\sigma = 1$.

Recall that we assume that the homomorphism h associated to the PEL data \mathcal{P}_1 is *standard*. In particular, $\varphi_{\kappa_\sigma, -}$ (resp. $\varphi_{\kappa_\sigma^b, -}^b$) is an eigenvector for T_σ of weight $-\kappa$ (resp. $-\kappa_\sigma^b$). Here, $T_\sigma \subset G_\sigma$ is the σ -component of $T(\mathbb{R})$.

Our choice of local test vectors φ_∞ and φ_∞^b are

$$(121) \quad \varphi_\infty = \otimes_\sigma \varphi_{\kappa_\sigma, -} \quad \text{and} \quad \varphi_\infty^b = \otimes_\sigma \varphi_{\kappa_\sigma^b, -}^b.$$

8. P -(ANTI-)ORDINARY HIDA FAMILIES.

8.1. Hecke algebras for modular forms with respect to P .

8.1.1. (Anti-)holomorphic Hecke algebras. We now construct the Hecke algebra of level $K_r = I_r K^p$ generated by Hecke operators at unramified places and at p . Let $S = S(K^p)$ as in Section 3.1.1.

Let R be a p -adic algebra over $S_0 = \mathcal{O}_{K', (p')}$, as in Section 5.2. Let $\mathbf{T}_{K_r, \kappa, R}$ denote the R -subalgebra of $\text{End}_{\mathbb{C}}(S_\kappa(K_r; \mathbb{C}))$ generated by the operators

- (i) $T(g) = T_r(g)$, for all $g \in G(\mathbb{A}_f^{S, p})$,
- (ii) $u_{w, D_w(j)} = u_{w, D_w(j), \kappa}$, for all $w \in \Sigma_p$, $1 \leq j \leq r_w$, and
- (iii) $u_p(t) = u_{p, \kappa}(t)$, for all $t \in Z_P$.

In particular, $\mathbf{T}_{K_r, \kappa, R}$ is an algebra over $R[Z_P]$, where Z_P is the center of L_P . In fact, setting $Z_{P, r} = Z_P / (1 + p^r Z_P)$, then it is equivalently an algebra over $R[Z_{P, r}]$.

If R is also an $S_0[\tau]$ -algebra, we define $\mathbf{T}_{K_r, \kappa, \tau, R}$ as the quotient algebra obtained by restricting each operator to an endomorphism of $S_\kappa(K_r, \tau; \mathbb{C})$. Finally, if R is also an $S_r[\tau]$ -algebra, we define $\mathbf{T}_{K_r, \kappa, [\tau], R}$ as the quotient algebra obtained upon restriction to $S_\kappa(K_r, [\tau]; \mathbb{C})$, where we recall that $[\tau] = [\tau]_r$ denotes the equivalence class of τ as a P -nebentypus of level r .

If $R = S_0$, $S_0[\tau]$ or $S_r[\tau]$, we omit R from the notation. Moreover, if r is clear from the context or does not affect the argument, we omit K_r from the notation and simply write $\mathbf{T}_{\kappa, \tau}$ or $\mathbf{T}_{\kappa, [\tau]}$.

Similarly, we define the Hecke algebra $\mathbf{T}_{K_r, \kappa, R}^d$ as we constructed $\mathbf{T}_{K_r, \kappa, R}$ above but we replace $S_\kappa(K_r; \mathbb{C})$ by $\widehat{S}_\kappa(K_r; \mathbb{C})$, and each $u_{w, D_w(j)}$ by $u_{w, D_w(j)}^-$, for all $w \in \Sigma_p$ and $1 \leq j \leq r_w$. Lastly, we define $\mathbf{T}_{K_r, \kappa, \tau, R}^d$ (resp. $\mathbf{T}_{K_r, \kappa, [\tau], R}^d$) analogously as a subalgebra of $\text{End}_{\mathbb{C}}(\widehat{S}_\kappa(K_r, \tau; \mathbb{C}))$ (resp. $\text{End}_{\mathbb{C}}(\widehat{S}_\kappa(K_r, [\tau]; \mathbb{C}))$).

For each of these Hecke algebras $\mathbf{T}_\bullet^?$, we write $\mathbf{T}_\bullet^{?, p}$ for the subalgebra generated by the operators $T_r(g)$, $g \in G(\mathbb{A}_f^S)$, i.e. by omitting the Hecke operators at p .

Remark 8.1. Implicitly, all of the above is stated for $G = G_1$. The definitions for $G = G_2$ are identical, considering the conventions set in Section 4.2. If we want to distinguish the two situations, we write $T_{V, \bullet}^?$ for G_1 and $T_{-V, \bullet}^?$ for G_2 .

8.1.2. *Hecke equivariance.* Let $\varphi \in H_!^0(K_r, \text{Sh}(V), \omega_\kappa)$ and $\varphi' \in H_!^d(K_r, \text{Sh}(V), \omega_{\kappa^D})$. By definition of (49), one readily checks that

$$\langle T(g)\varphi, \varphi' \rangle_{\kappa, K_r} = \langle \varphi, T(g)^d \varphi' \rangle_{\kappa, K_r}$$

and

$$\langle u_{w, D_w(j), \kappa} \varphi, \varphi' \rangle_{\kappa, K_r} = \langle \varphi, u_{w, D_w(j), \kappa^D}^- \varphi' \rangle_{\kappa, K_r},$$

for all $g \in G(\mathbb{A}_f^{S, p})$ and $w \in \Sigma_p$, $1 \leq j \leq r_w$, where $T(g)^d := \|\nu(g)\|^{a(\kappa)} T(g^{-1})$.

Similarly, using notation from Section 4.2.4 and the isomorphism in Remark 4.9, if $\varphi^b = F^\dagger(\varphi) \in S_{\kappa^b}(-V, K_r^b; R)$, we have

$$T(g)^b \varphi^b = F^\dagger(T(g)\varphi); \quad u_{w, D_w(j), \kappa}^b \varphi^b = F^\dagger(u_{w, D_w(j), \kappa} \varphi),$$

where $T(g)^b := T(g^\dagger) = T(\bar{g})$ and $u_{w, D_w(j), \kappa}^b := u_{w, n, \kappa^b}^{-1} u_{w, n - D_w(j), \kappa^b}$.

We obtain the next result as a consequence.

Lemma 8.2. *Let $R \subset \mathbb{C}$ be any subring.*

(i) *The map $\mathbf{T}_{K_r, \kappa, R} \rightarrow \mathbf{T}_{K_r, \kappa^D, R}^d$ induced by*

$$T(g) \mapsto T(g)^d \quad \text{and} \quad u_{w, D_w(j), \kappa} \mapsto u_{w, D_w(j), \kappa^D}^-,$$

is an isomorphism.

(ii) *The map in (i) induces an isomorphism $\mathbf{T}_{K_r, \kappa, \tau, R} \xrightarrow{\sim} \mathbf{T}_{K_r, \kappa^D, \tau^\vee, R}^d$.*

(iii) *The map $\mathbf{T}_{V, K_r, \kappa, R} \rightarrow \mathbf{T}_{-V, K_r^b, \kappa^b, R}$ induced by*

$$T(g) \mapsto T(g)^b \quad \text{and} \quad u_{w, D_w(j), \kappa} \mapsto u_{w, D_w(j), \kappa}^b$$

is an isomorphism.

(iv) *The map in (iii) induces an isomorphism $\mathbf{T}_{V, K_r, \kappa, \tau, R} \xrightarrow{\sim} \mathbf{T}_{-V, K_r^b, \kappa^b, \tau^b, R}$.*

We use the isomorphisms of Lemma 8.2 to view $\widehat{S}_\kappa(K_r; R)$ and $S_{\kappa^b}(-V, K_r^b; R)$ (resp. $\widehat{S}_\kappa(K_r, \tau; R)$ and $S_{\kappa^b}(-V, K_r^b, \tau^b; R)$) as modules over $\mathbf{T}_{K_r, \kappa, R}$ (resp. $\mathbf{T}_{K_r, \kappa, \tau, R}$).

8.1.3. P -(anti-)ordinary Hecke algebras. Let $\mathbf{T}_{K_r, \kappa, R}^{P\text{-ord}} := e_\kappa \mathbf{T}_{K_r, \kappa, R}$ and $\mathbf{T}_{K_r, \kappa, R}^{P\text{-a.ord}} := e_{\bar{\kappa}} \mathbf{T}_{K_r, \kappa, R}$. We define $\mathbf{T}_{K_r, \kappa, \tau, R}^{P\text{-ord}}$, $\mathbf{T}_{K_r, \kappa, [\tau], R}^{P\text{-ord}}$, $\mathbf{T}_{K_r, \kappa, \tau, R}^{P\text{-a.ord}}$ and $\mathbf{T}_{K_r, \kappa, [\tau], R}^{P\text{-a.ord}}$ similarly.

Lemma 8.3. *The isomorphism of Lemma 8.2 (i) induces an isomorphism*

$$\mathbf{T}_{K_r, \kappa, R}^{P\text{-ord}} \xrightarrow{\sim} \mathbf{T}_{K_r, \kappa^D, R}^{d, P\text{-a.ord}}.$$

Similarly, The isomorphism of Lemma 8.2 (iii) induces an isomorphism

$$\mathbf{T}_{V, K_r, \kappa, R}^{P\text{-ord}} \xrightarrow{\sim} \mathbf{T}_{-V, K_r^b, \kappa^b, R}^{P\text{-ord}}.$$

Remark 8.4. Similar isomorphisms exist for Hecke algebras associated to a (class of) type but we omit the explicit statement.

Consequently, $\widehat{S}_\kappa^{P\text{-a.ord}}(K_r; R)$ (resp. $\widehat{S}_\kappa^{P\text{-a.ord}}(K_r, \tau; R)$, $\widehat{S}_\kappa^{P\text{-a.ord}}(K_r, [\tau]; R)$) has a natural structure as a module over $\mathbf{T}_{K_r, \kappa, R}^{P\text{-ord}}$ (resp. $\mathbf{T}_{K_r, \kappa, \tau, R}^{P\text{-ord}}$, $\mathbf{T}_{K_r, \kappa, [\tau], R}^{P\text{-ord}}$).

8.2. Lattices of holomorphic P -ordinary forms. Let π be a holomorphic cuspidal automorphic representation on G of weight κ and level $K = K_r = K_{P,r} = I_{P,r} K^P$, for some $r \geq 0$. In what follows, we use the notation of Section 3.1 without comments.

In particular, we identify $(\pi^{p,S})^{K^{p,S}}$ as a 1-dimensional \mathbb{C} -vector space with a natural $E(\pi)$ -rational structure, see Remark 3.3.

Furthermore, we fix a choice of a highest weight vector φ_∞ in π_∞ . By definition of the weight of π , this is equivalent to the choice of a nonzero vector in the 1-dimensional \mathbb{C} -vector space

$$H^0(\mathfrak{P}_h, K_h; \pi_\infty \otimes W_\kappa).$$

Using the above and (38), we obtain an embedding

$$\pi_p^{I_{P,r}} \otimes \pi_S^{K^S} \hookrightarrow S_\kappa(K_r; \mathbb{C}),$$

over \mathbb{C} , which is equivariant for the action of $\mathbf{T}_{K_r, \kappa}^p$.

Let λ_π^p be the character via which $\mathbf{T}_{K_r, \kappa}^p$ acts on π^{K_r} , namely its action on $(\pi^{p,S})^{K^{p,S}}$. Then the embedding above factors through

$$j_\pi : \pi_p^{I_{P,r}} \otimes \pi_S^{K^S} \hookrightarrow S_\kappa(K_r; \mathbb{C})(\pi),$$

where $S_\kappa(K_r; \mathbb{C})(\pi)$ denotes the λ_π^p -isotypic component of $S_\kappa(K_r; \mathbb{C})$.

For the remainder of this article, we assume the following :

HYPOTHESIS 8.5 (Multiplicity one for π). For any holomorphic cuspidal automorphic representation π' of weight κ such that $(\pi'_f)^{K_r} \neq 0$, if $\pi' \neq \pi$, then $\lambda_{\pi'}^p \neq \lambda_\pi^p$.

Remark 8.6. This is the same multiplicity one hypothesis as [EHL20, Hypothesis 6.6.4]. See the comments below *loc. cit* to see the limitations of this hypothesis and the cases where it is known to hold.

Lemma 8.7. *Let π , κ and $K_r = K_{P,r}$ be as above. Assume that π satisfies Hypothesis 8.5. Then, the embedding j_π is an isomorphism.*

To study this isomorphism further, assume that π is P -ordinary. Let (τ, \mathcal{M}_τ) is the SZ-type of π_p , as in Section 6.1.3. We assume r is large enough so that τ is a P -nebenotypus of level r .

It follows from Remark 6.7 that the Hecke operator $u_{w, D_w(j)} = u_{w, D_w(j), \kappa}$, for $w \in \Sigma_p$ and $1 \leq j \leq r_w$, acts as a scalar on $\pi_p^{(P\text{-ord}, r)}[\tau]$, independent of $r \gg 0$ and our choice of SZ-type from Section 1.2.3. Hence, the character λ_π^p extends uniquely to a character λ_π of $\mathbf{T}_{K_r, \kappa}$ corresponding to its action on $\pi_p^{(P\text{-ord}, r)} \otimes \pi_S^{K_S}$, and λ_π factors through $\mathbf{T}_{K_r, \kappa, \tau, R}^{P\text{-ord}}$.

Let $E(\lambda_\pi)$ denote the smallest extension of $E(\pi)$ also containing the values of λ_π . Let $R(\lambda_\pi)$ denote the localization of the ring of integers of $E(\lambda_\pi)$ at the maximal ideal determined by incl_p . One readily sees that λ_π is $R(\lambda_\pi)$ -valued.

We always denote the residue field of $R(\lambda_\pi)$ by $k(\pi)$, the reduction of λ_π in $k(\pi)$ by $\bar{\lambda}_\pi$, and the p -adic completion of $R(\lambda_\pi)$ by \mathcal{O}_π . In particular, we view $\bar{\lambda}_\pi$ as being valued in a fixed algebraic closure of $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.

Let

$$\varphi^\circ = \left(\bigotimes_{l \notin S} \varphi_{l,0} \right) \otimes \varphi_S^\circ \otimes \varphi_\infty \otimes \varphi_p \in H^0(\mathfrak{P}_h, K_h; \pi^{K_r} \otimes W_\kappa),$$

where each local factor is a test vector for π chosen as in Section 7. In particular, $\varphi_p = \varphi_{p, \iota, v} := \iota(v)$ depends on the choice of an \mathcal{L}_r -embedding $\iota : \tau \hookrightarrow \pi_p^{(P\text{-ord}, r)}$ and a nonzero vector $v \in \mathcal{M}_\tau$.

From (37), one readily sees that the (canonical) choice of test vectors away from $S \cup \{p\}$ induces a map

$$(122) \quad \pi_p^{I_r} \otimes \pi_S^{K_S} \rightarrow S_\kappa(K_r; \mathbb{C}),$$

that is equivariant under the action of $\mathbf{T}_{K_r, \kappa, \tau}^p$.

The above can be improved via (40) to incorporate our choice of φ_p as follows. Following our discussion from Section 2.6.2, we can tensor this map by \mathcal{M}_τ^\vee (and apply P -ordinary projections), to obtain a map

$$(123) \quad \text{Hom}_{\mathcal{L}_r}(\tau, \pi_p^{(P\text{-ord}, r)}) \otimes \pi_S^{K_S} \hookrightarrow S_\kappa^{P\text{-ord}}(K_r, \tau; \mathbb{C})$$

that is equivariant under the action of $\mathbf{T}_{K_r, \kappa, \tau}$. Let f° and F° denote the image of $\varphi_p \otimes \varphi_S^\circ$ and $\iota \otimes \varphi_S^\circ$ via (122) and (123) respectively. Then, one readily sees that $F^\circ(v) = f^\circ$.

By definition of λ_π , we in fact have a $\mathbf{T}_{K_r, \kappa, \tau}$ -equivariant embedding

$$j_\pi : \text{Hom}_{\mathcal{L}_r}(\tau, \pi_p^{(P\text{-ord}, r)}) \otimes \pi_S^{K_S} \hookrightarrow S_\kappa^{P\text{-ord}}(K_r, \tau; E(\lambda_\pi))[\lambda_\pi] \otimes_{E(\lambda_\pi)} \mathbb{C},$$

where $[\lambda_\pi]$ indicates the λ_π -isotypic component.

By Theorem 6.9, the space $\mathrm{Hom}_{\mathcal{L}_r}(\tau, \pi_p^{(P\text{-ord}, r)})$ is 1-dimensional and ι corresponds to a basis element. Therefore, the following is an immediate consequence of the above together with Lemma 8.7.

Proposition 8.8. *Let π , κ and K_r be as in Lemma 8.7. Let τ and ι be as above. Suppose that π satisfies Hypothesis 8.5.*

Let $R \subset \mathbb{C}$ be the localization of a finite extension of $R(\lambda_\pi)$ at the prime determined by incl_p or the p -adic completion of such a ring. Let $E = R[1/p]$. Then, j_π induces an isomorphism between

$$\mathrm{Hom}_{\mathcal{L}_r}(\tau, \pi_p^{(P\text{-ord}, r)}) \otimes \pi_S^{K_S} = \pi_S^{K_S}.$$

and $S_\kappa^{P\text{-ord}}(K_r, \tau; E)[\lambda_\pi] \otimes_E \mathbb{C}$.

Furthermore, let $\mathfrak{m}_\pi \subset \mathbf{T}_{K_r, \kappa, \tau, R}$ be the kernel of the reduction of λ_π modulo the maximal ideal of R . Let $S_\kappa^{P\text{-ord}}(K_r, \tau; R)_\pi$ denote the localization of $S_\kappa^{P\text{-ord}}(K_r, \tau; R)$ at the maximal ideal \mathfrak{m}_π , and set

$$\begin{aligned} S_\kappa^{P\text{-ord}}(K_r, \tau; R)[\pi] &:= S_\kappa^{P\text{-ord}}(K_r, [\tau]; R)_\pi \cap S_\kappa^{P\text{-ord}}(K_r, \tau; E)[\lambda_\pi] \\ &= S_\kappa^{P\text{-ord}}(K_r, \tau; R)_\pi \cap S_\kappa^{P\text{-ord}}(K_r, \tau; E)[\lambda_\pi]. \end{aligned}$$

Then, j_π identifies $S_\kappa^{P\text{-ord}}(K_r, \tau; R)[\pi]$ with an R -lattice in $\pi_S^{K_S}$.

To finish this section, we also identify $S_\kappa^{P\text{-ord}}(K_r, [\tau]; R)_\pi$ as a lattice in a space of automorphic forms. To do so, we consider congruence between automorphic forms modulo p .

Namely, define the set $\mathcal{S}(\pi, \kappa, K_r, [\tau])$ as the collection of P -ordinary holomorphic cuspidal automorphic representation π' of P -WLT (κ, K_r, τ') such that $[\tau]_r = [\tau']_r$ and $\bar{\lambda}_\pi = \bar{\lambda}_{\pi'}$. Here, τ' is again chosen to be the SZ-type of π' .

In particular, for any $\pi' \in \mathcal{S}(\pi, \kappa, K_r, \tau)$, both λ_π and $\lambda'_{\pi'}$ both factor through characters of $\mathbf{T}_{K_r, \kappa, [\tau], R}$ (for some sufficiently large ring R), and $\mathfrak{m}_\pi = \mathfrak{m}_{\pi'}$.

Proposition 8.9. *With notation as in Proposition 8.8, $S_\kappa^{P\text{-ord}}(K_r, [\tau]; R)_\pi$ is identified with an R -lattice in*

$$\bigoplus_{\pi' \in \mathcal{S}(\pi, \kappa, K_r, [\tau])} \mathrm{Hom}_{\mathcal{L}_r}(\tau', \pi_p^{(P\text{-ord}, r)}) \oplus (\pi'_S)^{K_S} = \bigoplus_{\pi' \in \mathcal{S}(\pi, \kappa, K_r, [\tau])} (\pi'_S)^{K_S}$$

via the map $\bigoplus_{\pi'} j_{\pi'}$.

8.3. Lattices of anti-holomorphic P -anti-ordinary forms. We now adjust the theory above in the anti-holomorphic case for π^b on G , where π is as in the previous section. We keep our assumption that π satisfies Hypothesis 8.5, hence $\pi^b = \bar{\pi}$ as subspaces of $\mathcal{A}_0(G)$, see Remark 4.7.

8.3.1. *Lattices in π^b .* Let

$$\varphi^{b, \circ} = \left(\bigotimes_{l \notin S} \varphi_{l, 0}^b \right) \otimes \varphi_S^{b, \circ} \otimes \varphi_\infty^b \otimes \varphi_p^b \in H^d(\mathfrak{B}_h, K_h; \pi^{b, K_r} \otimes W_{\kappa^D}),$$

where each local factor is a test vector for π^b chosen as in Section 7. Again, $\varphi_p^b = \iota^\vee(v^\vee)$ depends on the choice of an \mathcal{L}_r -embedding $\iota : \tau^\vee \hookrightarrow (\pi_p^b)^{(P\text{-a.ord}, r)}$ and a nonzero vector $v^\vee \in \mathcal{M}_\tau^\vee$.

Similar to the holomorphic case, after fixing a basis of the 1-dimensional complex vector space $H^d(\mathfrak{P}_h, K_h; \pi^b \otimes W_{\kappa^D})$ and unramified local vectors, we obtain an embedding

$$\pi_p^{b, I_r} \otimes \pi_S^{b, K_S} \hookrightarrow H_{\kappa^D}^d(K_r, \mathbb{C}) = H_{\kappa^D}^d(K_r, \text{Sh}(V), \omega_{\kappa^D}),$$

via the identification (37).

Assume π is P -ordinary with SZ-type τ , or equivalently, that π^b is P -anti-ordinary with SZ-type τ^b . Then, $\mathbf{T}_{K_r, \kappa^D, \tau^b}^d$ acts on $(\pi_p^b)^{(P\text{-a.ord}, r)}[\tau^b] \otimes \pi_S^{b, K_S}$ via some character λ_π^b . We use Lemma 8.2 to view λ_π^b as a character of $\mathbf{T}_{K_r, \kappa, \tau}$.

Remark 8.10. It follows from the definition of the isomorphism in Lemma 8.2 (ii) that $\lambda_\pi^b = \lambda_\pi$ as characters of $\mathbf{T}_{K_r, \kappa, \tau}$. In particular, the ring $R(\lambda_\pi)$ and its p -adic completion \mathcal{O}_π defined in the previous section are the same when working with π or π^b . Furthermore, the kernel of λ_π^b is again the maximal \mathfrak{m}_π of $\mathbf{T}_{K_r, \kappa, \tau}$.

Again, the map above further induces an embedding

$$j_{\pi^b} : \text{Hom}_{\mathcal{L}_r}(\tau^b, \pi_p^{b, (P\text{-a.ord}, r)}) \otimes \pi_S^{b, K_S} \hookrightarrow \widehat{S}_\kappa(K_r, \tau; E(\lambda_\pi))[\lambda_\pi] \otimes_{E(\lambda_\pi)} \mathbb{C},$$

using the identification (40), that is $\mathbf{T}_{K_r, \kappa, \tau}$ -equivariant. From Corollary 6.12, we know $\text{Hom}_{\mathcal{L}_r}(\tau^b, \pi_p^{b, (P\text{-a.ord}, r)})$ is 1-dimensional and ι^\vee corresponds to a basis element.

Let $R \subset \mathbb{C}$ be any ring as in Prop 8.8, and let $E = R[1/p]$. Given any $\mathbf{T}_{K_r, \kappa, \tau, R}$ -module M , we again denote its λ_π -isotypic (or equivalently, λ_π^b -isotypic) component by $M[\lambda_\pi]$ and its localization at \mathfrak{m}_π by M_π . Moreover, we define

$$\begin{aligned} \widehat{S}_\kappa^{P\text{-a.ord}}(K_r, \tau; R)[\pi] &:= \widehat{S}_\kappa^{P\text{-a.ord}}(K_r, [\tau]; R)_\pi \cap \widehat{S}_\kappa^{P\text{-a.ord}}(K_r, \tau; E)[\lambda_\pi] \\ &= \widehat{S}_\kappa^{P\text{-a.ord}}(K_r, \tau; R)_\pi \cap \widehat{S}_\kappa^{P\text{-ord}}(K_r, \tau; E)[\lambda_\pi] \end{aligned}$$

Lemma 8.11. *Let π be as above, R , and E be as above. Assume π satisfies Hypothesis 8.5. Then,*

(i) *The embedding j_π^b induces an isomorphism*

$$\pi_S^{b, K_S} \xrightarrow{\sim} \widehat{S}_\kappa(K_r, \tau; E)[\lambda_\pi] \otimes_E \mathbb{C}.$$

(ii) *The isomorphism from part (i) identifies $\widehat{S}_\kappa^{P\text{-a.ord}}(K_r, \tau; R)[\pi]$ with an R -lattice in π_S^{b, K_S} . Similarly, $\widehat{S}_\kappa^{P\text{-a.ord}}(K_r, [\tau]; R)_\pi$ with an R -lattices in*

$$\bigoplus_{\pi' \in \mathcal{S}(\pi, \kappa, K_r, [\tau])} (\pi_S^{\prime, b})^{K_S}$$

via the map $\bigoplus_{\pi'} j_{\pi'}^b$.

(iii) *The pairing $\langle \cdot, \cdot \rangle_{\kappa, K_r, \tau}$ induces perfect $\mathbf{T}_{K_r, \kappa, \tau, R}^{P\text{-ord}}$ -equivariant pairings*

$$S_\kappa^{P\text{-ord}}(K_r, \tau; R)[\pi] \otimes \widehat{S}_\kappa^{P\text{-a.ord}}(K_r, \tau; R)[\pi] \rightarrow R$$

and the pairing $\langle \cdot, \cdot \rangle_{\kappa, K_r, [\tau]}$ induces perfect $\mathbf{T}_{K_r, \kappa, [\tau], R}^{P\text{-ord}}$ -equivariant pairings

$$S_{\kappa}^{P\text{-ord}}(K_r, [\tau]; R)_{\pi} \otimes \widehat{S}_{\kappa}^{P\text{-a.ord}}(K_r, [\tau]; R)_{\pi} \rightarrow R$$

8.4. Big Hecke algebra and P -anti-ordinary Hida families.

8.4.1. *Independence of weights.* Let R , κ , τ and K_r be as above. Consider the algebra

$$\varprojlim_r \mathbf{T}_{K_r, \kappa, [\tau], R}^{P\text{-ord}}$$

over $\Lambda_R := R[[Z_P]]$. It follows from the discussion at the end of Section 5.4 that this algebra can be viewed as a subquotient of $\text{End}_R(\mathcal{V}^{P\text{-ord}}(K^p, [\kappa_p, \tau], R))$.

Conjecture 8.12. *Let κ_1 and κ_2 be two dominant characters such that $[\kappa_1] = [\kappa_2]$. There is a canonical isomorphism*

$$\varprojlim_r \mathbf{T}_{K_r, \kappa_1, [\tau], R}^{P\text{-ord}} \xrightarrow{\sim} \varprojlim_r \mathbf{T}_{K_r, \kappa_2, [\tau], R}^{P\text{-ord}}.$$

From now on, we assume that this conjecture holds without comments. Furthermore, we write $\mathbf{T}_{K^p, [\kappa, \tau], R}^{P\text{-ord}}$ instead of $\varprojlim_r \mathbf{T}_{K_r, \kappa, [\tau], R}^{P\text{-ord}}$ to emphasize the fact that this algebra (conjecturally) only depends on the set $[\kappa]$ of dominant weights obtained as P -parallel shifts of κ .

Remark 8.13. When $P = B$ as in Remark 2.8, this result holds and is due to Hida, see [Ehls20, Theorem 7.1.1].

Recall that the normalized Serre pairing is stable under the trace map, see (51) and (52). In particular, for τ of level $r' > r \gg 0$, we have a map

$$\text{tr}_{K_r/K_{r'}} : \widehat{S}_{\kappa}(K_{r'}, [\tau]; R) \rightarrow \widehat{S}_{\kappa}(K_r, [\tau]; R).$$

Therefore, the above induces natural maps

$$\mathbf{T}_{K_{r'}, \kappa^D, [\tau^{\vee}], R}^{d, P\text{-a.ord}} \rightarrow \mathbf{T}_{K_r, \kappa^D, [\tau^{\vee}], R}^{d, P\text{-a.ord}}$$

that are compatible with the isomorphisms of Lemma 8.3 and the maps

$$\mathbf{T}_{K_{r'}, \kappa, [\tau], R}^{P\text{-ord}} \rightarrow \mathbf{T}_{K_r, \kappa, [\tau], R}^{P\text{-ord}}.$$

In other words, the algebra

$$\mathbf{T}_{K^p, [\kappa^D, \tau^{\vee}], R}^{d, P\text{-a.ord}} := \varprojlim_r \mathbf{T}_{K_r, \kappa^D, [\tau^{\vee}], R}^{d, P\text{-a.ord}}$$

is well-defined and isomorphic to $\mathbf{T}_{K^p, [\kappa, \tau], R}^{P\text{-ord}}$ via Lemma 8.3. In particular, Conjecture 8.12 implies a similar independence of weight for $\mathbf{T}_{K^p, [\kappa^D, \tau^{\vee}], R}^{d, P\text{-a.ord}}$.

8.4.2. *Classical points of P -anti-ordinary families.* Let Z_P° denote the maximal pro- p -subgroup of Z_P . There exists a finite group $\Delta_P \subset Z_P$ of order prime-to- p such that $Z_P = \Delta_P \times Z_P^\circ$.

For a p -adic ring R as in the previous section, let $\Lambda_R^\circ \subset \Lambda_R$ be the complete group algebra associated to Z_P° over R . We refer to $\mathcal{W} = \text{Spec } \Lambda_R^\circ$ as the *weight space* over R (associated to the parabolic P). The *weight map* is the structure homomorphism $\Omega : \Lambda_R^\circ \rightarrow \mathbf{T}_{K^p, [\kappa, \tau], R}^{P\text{-ord}}$ sending $t \mapsto u_p(t)$ for all $t \in Z_P^\circ$.

Let κ , K_r and τ be as in the previous sections. Let κ_p be the p -adic weight corresponding to κ , viewed as an algebraic character of $T_H(\mathbb{Z}_p)$. Denote its restriction to a character of Z_P by κ_p again. Furthermore, let ω_τ denote the central character of τ , a finite order character of Z_P .

Definition 8.14. We say that a homomorphism $\Lambda_R^\circ \rightarrow R$ is *arithmetic* if it is induced by an R -valued character of Z_P of the form $\kappa_p \cdot \omega_\tau$ for some κ and τ as above. We sometimes say that $\kappa_p \cdot \omega_\tau$ is an *arithmetic character* of Z_P .

Definition 8.15. Let $\lambda : \mathbf{T}_{K^p, [\kappa_p, \tau], R}^{P\text{-ord}} \rightarrow R^\times$ be a continuous character. We say that λ is *arithmetic* if its composition $\lambda \circ \Omega : \Lambda_R^\circ \rightarrow R$ with the weight map is arithmetic.

If we fix ‘‘base points’’ κ and τ of $[\kappa]$ and $[\tau]$ respectively, note that any arithmetic character of Λ_R° corresponds to a product of an algebraic character $(\kappa + \theta)_p$ and a finite-order character $\omega_{\tau \otimes \psi}$, for some P -parallel weight θ and some finite-order character ψ of $L_H(\mathbb{Z}_p)$. Recall that we use additive notation for the binary operation on the set of algebraic weights.

Furthermore, one readily sees that for all P -anti-ordinary automorphic representation π , the associated character λ_π constructed in Section 8.3.1 is arithmetic, valued in \mathcal{O}_π and factors through $(\mathbf{T}_{K^p, [\kappa, \tau], \mathcal{O}_\pi}^{P\text{-ord}})_{\mathfrak{m}_\pi}$.

Definition 8.16. We say that an arithmetic character λ is *classical* if it arises as $\lambda = \lambda_\pi$ for some π as above. If π is of P -anti-WLT (κ, K_r, τ) , we say λ has weight κ , level $r \gg 0$ and P -nebentypus τ .

For any tame character ϵ of Z_P , we write $\Lambda_{R, \epsilon}$ (resp. $\Lambda_{R, \epsilon}^\circ$) for the localization of Λ_R (resp. Λ_R°) at the maximal ideal of Λ_R (resp. Λ_R°) defined by ϵ . Note that the quotient map $\Lambda_\pi \rightarrow \Lambda_\pi / (\mathfrak{m}_\pi \cap \Lambda_\pi)$ is the homomorphism induced by some tame character of Z_P .

Conjecture 8.17. *Let R , κ , τ and K_r be as above, and assume Conjecture 8.12.*

- (i) *For each tame character ϵ , the localization $\mathbf{T}_{K^p, [\kappa, \tau], R, \epsilon}^{P\text{-ord}}$ of the Hecke algebra $\mathbf{T}_{K^p, [\kappa, \tau], R}^{P\text{-ord}}$ at the maximal ideal defined by ϵ is finite free over $\Lambda_{R, \epsilon}^\circ$.*
- (ii) *Let κ be a P -very regular weight and let κ_p be the corresponding p -adic weight of $T_H(\mathbb{Z}_p)$, as in (26). Let I_κ be the kernel of the homomorphism $\Lambda_P^\circ \rightarrow R \subset \mathbb{C}_p$ induced by the restriction of κ_p to Z_P° . Then, the natural*

homomorphism

$$\mathbf{T}_{K_p, [\kappa, \tau], R}^{P\text{-ord}} \otimes \Lambda_R^\circ / I_\kappa \rightarrow \mathbf{T}_{K_r, \kappa, [\tau], R}^{P\text{-ord}}$$

is an isomorphism.

Remark 8.18. Again, when $P = B$ as in Remark 2.8, this result holds and is due to Hida, see [Ehls20, Theorem 7.2.1].

Now, let π be cuspidal automorphic representation of $G = G_1$. Assume that π is anti-holomorphic and P -anti-ordinary of anti- P -WLT (κ, K_r, τ) . In particular, π^b is holomorphic P -ordinary on G_1 , or equivalently, anti-holomorphic P -anti-ordinary on G_2 . In what follows, we work with $R = \mathcal{O}_\pi$ and set $\Lambda_\pi := \Lambda_{\mathcal{O}_\pi} = \mathcal{O}_\pi[[Z_P]]$, $\Lambda_\pi^\circ := \Lambda_{\mathcal{O}_\pi}^\circ = \mathcal{O}_\pi[[Z_P^\circ]]$.

Let λ_π be the classical character, see Definition 8.16, of the Λ_π -algebra $\mathbf{T}_{K^p, [\kappa, \tau], \mathcal{O}_\pi}$ associated to π as in Section 8.3.

Denote the localization of $\mathbf{T}_{K^p, [\kappa, \tau], \mathcal{O}_\pi}^{P\text{-ord}}$ at \mathfrak{m}_π by $\mathbb{T} = \mathbb{T}_\pi$. Similarly, denote the localization of $\mathbf{T}_{K_r, \kappa, [\tau], \mathcal{O}_\pi}^{P\text{-ord}}$ at \mathfrak{m}_π by $\mathbb{T}_{K_r, \kappa, [\tau], \mathcal{O}_\pi}$. We do not include the superscript “ P -ord” in the notation of the localized Hecke algebras $\mathbb{T}_?$ as we do not ever consider such localization of “non- P -ordinary” Hecke algebras in what follows.

Proposition 8.19. Assume Conjectures 8.12 and 8.17. Then,

- (i) The Hecke algebra \mathbb{T} is finite, free over Λ_π° .
- (ii) Let κ be a very regular weight and let κ_p be the corresponding p -adic weight of $T_H(\mathbb{Z}_p)$, as in (26). Let I_κ be the kernel of the homomorphism $\Lambda_P^\circ \rightarrow \mathcal{O}_\pi \subset \mathbb{C}_p$ induced by the restriction of κ_p to Z_P° . Then, the natural homomorphism

$$\mathbb{T} \otimes \Lambda_R^\circ / I_\kappa \rightarrow \mathbb{T}_{K_r, \kappa, [\tau], \mathcal{O}_\pi}$$

is an isomorphism.

Definition 8.20. The representation π , or more precisely the homomorphism λ_π , corresponds to an \mathcal{O}_π -valued point $\text{Spec } \mathbb{T}_\pi$. We refer to \mathbb{T}_π as a P -anti-ordinary Hida family associated to π .

Remark 8.21. Note that this Hida family is not an irreducible component of \mathbb{T}_π . The p -adic L -function constructed in Section 12 is well-defined on all of $\text{Spec } \mathbb{T}_\pi$. This (connected) space is implicitly a branch corresponding to our choice of SZ-type τ associated to π . However, the choice of τ does not affect the p -adic interpolation formula of the p -adic L -function constructed in this paper. Namely, the reader should note that expression at the end of Theorem 12.6 does not depend on τ .

Let π' be an anti-holomorphic, P -anti-ordinary cuspidal automorphic representation of $G = G_1$ of anti- P -WLT $(\kappa', K_{r'}, \tau')$. Assume that $[\kappa'] = [\kappa]$ and $[\tau'] = [\tau]$.

The canonical isomorphism provided by Conjecture 8.12 identifies the maximal ideal $\mathfrak{m}_{\pi'}$ associated to π' as a maximal ideal of $\mathbb{T}_{K^p, [\kappa, \tau], R}$, for some \mathcal{O}_π -algebra R . This allows us to generalize the set $\mathcal{S}(K_r, \kappa, [\tau], \pi)$ defined at the end of Section 8.2

for other levels and weights, i.e. let

$$(124) \quad \mathcal{S}(K_{r'}, \kappa', [\tau], \pi) := \{\pi' \text{ as above such that } \mathfrak{m}_{\pi'} = \mathfrak{m}_\pi\}.$$

Similarly, let

$$(125) \quad \mathcal{S}(K^p, \pi) = \mathcal{S}(K^p, [\kappa], [\tau], \pi) := \bigcup_{r \geq 1} \bigcup_{\kappa' \in [\kappa]} \mathcal{S}(K_{r'}, \kappa', [\tau], \pi),$$

hence a classical character corresponds to a point $\lambda = \lambda_{\pi'}$ of $\text{Spec } \mathbb{T}_\pi$, for some representation $\pi' \in \mathcal{S}(K^p, \pi)$.

The image of $\pi' \in \mathcal{S}(K^p, \pi)$ in \mathcal{W} is $\omega_{\tau \otimes \psi} \cdot (\kappa_p + \theta_p)$. Implicitly, in what follows, we view π as a choice of “base point” and π' as a “shift” from π by $\psi \cdot \theta_p$.

Remark 8.22. Naturally, our constructions in the following sections do not depend on the choice of a base point. However, this perspective of “shifting” (or “twisting”) π by $\psi \cdot \theta_p$ is useful to understand the construction of the P -ordinary Eisenstein measure, see Proposition 11.8.

8.4.3. *P -anti-ordinary vectors and minimal ramification.* In what follows, we view

$$\widehat{S}_\kappa^{P\text{-a.ord}}(K_r, [\tau]; \mathcal{O}_\pi)_\pi := \text{Hom}_{\mathcal{O}_\pi}(S_\kappa^{P\text{-a.ord}}(K_r, [\tau]; \mathcal{O}_\pi), \mathcal{O}_\pi)_{\mathfrak{m}_\pi}$$

as a module over $\mathbb{T}_{K_r, \kappa, [\tau], \mathcal{O}_\pi}$. Similarly, we view

$$\widehat{S}_\kappa^{P\text{-a.ord}}(K^p, [\tau]; \mathcal{O}_\pi)_\pi := \varprojlim_r \widehat{S}_\kappa^{P\text{-ord}}(K_r, [\tau]; \mathcal{O}_\pi)_\pi$$

as a \mathbb{T} -module.

HYPOTHESIS 8.23 (Gorenstein Hypothesis). Let $\widehat{\mathbb{T}}$ denote the Λ_π° -dual of \mathbb{T} .

- (i) The \mathbb{T} -module $\widehat{\mathbb{T}}$ is free of rank one. Fix an isomorphism $G_\pi : \mathbb{T} \xrightarrow{\sim} \widehat{\mathbb{T}}$ of \mathbb{T} -modules.
- (ii) The \mathbb{T} -module $\widehat{S}_\kappa^{P\text{-a.ord}}(K^p, [\tau]; \mathcal{O}_\pi)_\pi$ is finite, free.

From now on, we always assume that the Gorenstein hypothesis above holds. We fix any \mathbb{T} -basis of $\widehat{S}_\kappa^{P\text{-a.ord}}(K^p, [\tau]; \mathcal{O}_\pi)_\pi$ and let \widehat{I}_π denote the \mathcal{O}_π -lattice spanned by this basis. In particular, we have an isomorphism

$$\mathbb{T} \otimes_{\mathcal{O}_\pi} \widehat{I}_\pi \xrightarrow{\sim} \widehat{S}_\kappa^{P\text{-a.ord}}(K^p, [\tau]; \mathcal{O}_\pi)_\pi.$$

Assume that the weight κ of π is very regular. Then, the vertical control theorem Proposition 8.19 (ii) implies that taking tensor with $\Lambda_\pi^\circ / I_\kappa$ on both sides yields an isomorphism

$$(126) \quad \mathbb{T}_{K_r, \kappa, [\tau], \mathcal{O}_\pi} \otimes \widehat{I}_\pi \xrightarrow{\sim} \widehat{S}_\kappa^{P\text{-a.ord}}(K_r, [\tau]; \mathcal{O}_\pi)_\pi.$$

Similarly, the λ_π -isotypic component $\mathbb{T}_{K_r, \kappa, [\tau], \mathcal{O}_\pi}[\lambda_\pi] = \mathbf{T}_{K_r, \kappa, \tau, \mathcal{O}_\pi}^{P\text{-ord}}[\lambda_\pi]$ is free of rank 1 over \mathcal{O}_π , by the multiplicity one hypothesis 8.5. Hence, the identification (126) also induces an isomorphism

$$\widehat{I}_\pi \xrightarrow{\sim} (\mathbb{T}_{K_r, \kappa, [\tau], \mathcal{O}_\pi} \otimes \widehat{I}_\pi)[\lambda_\pi] \xrightarrow{\sim} \widehat{S}_\kappa^{P\text{-a.ord}}(K_r, [\tau]; \mathcal{O}_\pi)[\lambda_\pi],$$

and note that the last term is equal to $\widehat{S}_\kappa^{P\text{-a.ord}}(K_r, \tau; \mathcal{O}_\pi)[\lambda_\pi]$.

Therefore, the isomorphism j_π^b from Lemma 8.11 (i) induces an embedding
(127)

$$\widehat{I}_\pi \xrightarrow{\sim} \widehat{S}_\kappa^{P\text{-a.ord}}(K_r, \tau; \mathcal{O}_\pi)[\lambda_\pi] \xrightarrow{(j_\pi^b)^{-1}} \text{Hom}_{\mathcal{L}_r}(\tau^b, \pi_p^{b, (P\text{-a.ord}, r)}) \otimes \pi_S^{b, K_S} = \pi_S^{b, K_S}.$$

For the dual picture, we map both sides of (126) to their quotients modulo $\ker(\lambda_\pi)$ and obtain

$$\widehat{I}_\pi \xrightarrow{\sim} (\mathbb{T}_{K_r, \kappa, [\tau], \mathcal{O}_\pi} / \ker(\lambda_\pi)) \otimes \widehat{I}_\pi \xrightarrow{\sim} \text{Hom}_{\mathcal{O}_\pi}(S_\kappa^{P\text{-ord}}(K_r, \tau; \mathcal{O}_\pi)[\lambda_\pi], \mathcal{O}_\pi).$$

Define I_π as the \mathcal{O}_π -dual of \widehat{I}_π . Then the above, together with the isomorphism j_π from Lemma 8.7 (i), induces an embedding

$$(128) \quad I_\pi \xrightarrow{\sim} S_\kappa^{P\text{-ord}}(K_r, \tau; \mathcal{O}_\pi)[\lambda_\pi] \xrightarrow{j_\pi^{-1}} \text{Hom}_{\mathcal{L}_r}(\tau, \pi_p^{(P\text{-ord}, r)}) \otimes \pi_S^{K_S} = \pi_S^{K_S}.$$

Remark 8.24. Note that I_π and \widehat{I}_π only depend on \mathfrak{m}_π . Therefore, the embedding (127) (resp. (128)) identifies a lattice of P -anti-ordinary (resp. P -ordinary) anti-holomorphic (resp. holomorphic) automorphic forms shared by all $(\pi')^b$ (resp. π') such that $\mathfrak{m}_\pi = \mathfrak{m}_{\pi'}$ as a maximal ideal of $\mathbf{T}_{K^p, [\kappa, \tau], \mathcal{O}_\pi}^{P\text{-ord}}$.

In other words, the embedding (127) (resp. (128)) obtained by assuming the Gorenstein Hypothesis 8.23 implies that the \mathbb{C} -dimension of local representations at ramified places of all $(\pi')^b$ (resp. π') as above is constant. This can therefore be viewed as a certain *minimality hypothesis* on the behavior over the ramified places of the P -ordinary Hida family associated to π .

The discussion above, together with Remark 8.24, proves the following proposition (the analogue of [EHLS20, Proposition 7.3.5] in the context of P -ordinary representations).

Proposition 8.25. *Let π , r , κ and τ be as above. Let $\pi' \in \mathcal{S}(\pi, \kappa', K_{r'}, [\tau])$, for some $r' \geq 1$ and some very regular weight κ' such that $[\kappa] = [\kappa']$.*

There is an isomorphism of $\mathbb{T}_{K_{r'}, \kappa', [\tau], \mathcal{O}_\pi}$ -module

$$(129) \quad \mathbb{T}_{K_{r'}, \kappa', [\tau], \mathcal{O}_\pi} \otimes \widehat{I}_\pi \xrightarrow{\sim} \widehat{S}_{\kappa'}^{P\text{-a.ord}}(K_{r'}, [\tau]; \mathcal{O}_\pi)_\pi$$

such that for $r'' \geq r'$, the “change-of-level” diagram

$$\begin{array}{ccc} \mathbb{T}_{K_{r'', \kappa'}, [\tau], \mathcal{O}_\pi} \otimes \widehat{I}_\pi & \xrightarrow{\sim} & \widehat{S}_{\kappa'}^{P\text{-a.ord}}(K_{r'', [\tau]; \mathcal{O}_\pi})_\pi \\ \downarrow & & \downarrow \\ \mathbb{T}_{K_{r'}, \kappa', [\tau], \mathcal{O}_\pi} \otimes \widehat{I}_\pi & \xrightarrow{\sim} & \widehat{S}_{\kappa'}^{P\text{-a.ord}}(K_{r'}, [\tau]; \mathcal{O}_\pi)_\pi \end{array}$$

commutes.

Furthermore, tensoring (129) with $\mathbb{T}_{K_{r'}, \kappa', [\tau], \mathcal{O}_\pi} / \ker(\lambda_{\pi'})$ over $\mathbb{T}_{K_{r'}, \kappa', [\tau], \mathcal{O}_\pi}$, i.e. specializing this isomorphism at the \mathcal{O}_π -valued point $\lambda_{\pi'}$ of $\mathbb{T}_{K_{r'}, \kappa', [\tau], \mathcal{O}_\pi}$ corresponding to π' , yields the commutative diagram

$$\begin{array}{ccc} (\mathbb{T}_{K_{r'}, \kappa', [\tau], \mathcal{O}_\pi} / \ker(\lambda_{\pi'})) \otimes \widehat{I}_\pi & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{O}_\pi}(S_{\kappa'}^{P\text{-ord}}(K_{r'}, \tau; \mathcal{O}_\pi)[\lambda_{\pi'}], \mathcal{O}_\pi) \\ \downarrow = & & \downarrow \\ \mathcal{O}_\pi \otimes_{\mathcal{O}_\pi} \widehat{I}_\pi & \xrightarrow{\cong} & \mathrm{Hom}_{\mathcal{O}_\pi}(I_\pi, \mathcal{O}_\pi) \end{array},$$

where the bottom map is the tautological identification of \widehat{I}_π as the \mathcal{O}_π -dual of I_π .

Observe that all of this section can be rephrased for G_2 . Namely, we can rewrite all of the above for π^b as an anti-holomorphic P -anti-ordinary automorphic representation on G_2 . Then, considering the analogue of Hypothesis 8.23, we similarly obtain an isomorphism

$$(130) \quad \mathbb{T}_{\pi^b} \otimes_{\mathcal{O}_\pi} \widehat{I}_{\pi^b} \xrightarrow{\sim} \widehat{S}_{\kappa^b, -V}^{P\text{-a.ord}}(K^{b,p}, [\tau^b]; \mathcal{O}_\pi)_{\pi^b},$$

of finite free \mathbb{T}_{π^b} -modules. Considering (78) and the isomorphism $\mathbb{T}_{\pi^b} \cong \mathbb{T}_\pi$ induced from the second part of Lemma (8.3), we naturally identify \widehat{I}_{π^b} with I_π .

8.4.4. *I_π and test vectors.* Fix any $\varphi_S \in \pi_S^{K_S}$ and $\varphi_S^b \in \pi_S^{b, K_S}$ such that $j_\pi(\varphi_S) \in I_\pi$ and $j_\pi^b(\varphi_S^b) \in \widehat{I}_\pi$. Furthermore, let $\varphi_{l,0} \in \pi_l$ and $\varphi_{l,0}^b \in \pi_l^b$ be local test vectors at l for all finite places $l \notin S \cup \{p\}$ of \mathbb{Q} as well as $\varphi_\infty \in \pi_\infty$ and φ_∞^b be local test vector at ∞ , as in Section 7.

Fix a basis ι of $\mathrm{Hom}_{L_P}(\tau, \pi_p^{(P\text{-a.ord}, r)})$ and a basis ι^b of $\mathrm{Hom}_{L_P}(\tau^b, \pi_p^{b, (P\text{-a.ord}, r)})$. For any $v \in \tau$ and $v^b \in \tau^b$, let $\varphi_{p,v} = \iota(v)$ and $\varphi_{p,v^b}^b = \iota^b(v^b)$. By definition,

$$(131) \quad \varphi = \left(\bigotimes_{l \notin S \cup \{p\}} \varphi_{l,0} \right) \otimes \varphi_{p,v} \otimes \varphi_\infty \otimes \varphi_S$$

and

$$(132) \quad \varphi^b = \left(\bigotimes_{l \notin S \cup \{p\}} \varphi_{l,0}^b \right) \otimes \varphi_{p,v^b}^b \otimes \varphi_\infty^b \otimes \varphi_S^b$$

are test vectors of π and π^b respectively. By construction and (118), the inner product between φ and φ^b only depend on the choice of φ_S and φ_S^b , i.e.

$$(133) \quad \langle \varphi, \varphi^b \rangle = C \cdot \mathrm{Vol}(I_{P,r}^0) \cdot \langle \varphi_S, \varphi_S^b \rangle_S,$$

where C is the constant from (61) and $\langle \cdot, \cdot \rangle_S = \bigotimes_{l \in S} \langle \cdot, \cdot \rangle_{\pi_l}$.

By abuse of terminology, we still refer to $j_\pi(\varphi_S)$ and $j_\pi^b(\varphi_S^b)$ as “test vectors”, leaving the choice of basis of $\mathrm{Hom}_{\mathcal{L}}(\tau, \pi_p^{(P\text{-a.ord}, r)})$ and $\mathrm{Hom}_{\mathcal{L}}(\tau^b, \pi_p^{b, (P\text{-a.ord}, r)})$ implicit.

Let $\pi' \in \mathcal{S}(K_r, \kappa, [\tau], \pi)$ be a P -anti-ordinary automorphic representation of P -anti-WLT (κ, K_r, τ') . Using Remarks 1.5 and 8.24, one readily sees that φ_v and φ_v^b similarly determine test vectors of π' and π'^b , which we again denote φ_v and φ_v^b .

This yields embeddings

$$I_{\pi^b} = \widehat{I}_\pi \hookrightarrow (\pi'_S)^{K_S} \quad \text{and} \quad \widehat{I}_{\pi^b} = I_\pi \hookrightarrow (\pi'_S)^{b, K_S}$$

into the subspaces of test vectors. Therefore, using Proposition 8.25, we identify

$$\widehat{I}_\pi \otimes I_\pi = \text{End}_{\mathcal{O}_\pi}(\widehat{I}_\pi) = \text{End}_{\mathcal{O}_\pi}(I_{\pi^b})$$

as the space of test vectors in $\pi'_S \otimes \pi'^b_S$, for all $\pi' \in \mathcal{S}(K^p, \pi)$.

Part III. P -ordinary family of Siegel Eisenstein series

9. SIEGEL EISENSTEIN SERIES FOR THE DOUBLING METHOD.

Given any number field F/\mathbb{Q} , we write $|\cdot|_F$ for the standard absolute value on \mathbb{A}_F^\times (instead of $|\cdot|_{\mathbb{A}_F}$). For $F = \mathbb{Q}$, we keep writing \mathbb{A} for $\mathbb{A}_\mathbb{Q}$.

9.1. Siegel Eisenstein series.

9.1.1. *Siegel parabolic.* Let $W = V \oplus V$, equipped with $\langle \cdot, \cdot \rangle_W := \langle \cdot, \cdot \rangle_V \oplus (-\langle \cdot, \cdot \rangle_V)$, be the Hermitian vector space associated to G_4 . We work with G_4 for most of what follows, hence we set $G := G_4$ in all of Section 9.

Consider the subspaces $V^d = \{(x, x) \in W : x \in V\}$ and $V_d = \{(x, -x) \in W : x \in V\}$. We identify both with V via projection on their first factor. The direct sum $W = V_d \oplus V^d$ is a polarization of $\langle \cdot, \cdot \rangle_W$.

Let $P_{\text{Sgl}} \subset G$ denote the stabilizer of V^d under the right-action of G , a maximal \mathbb{Q} -parabolic subgroup. Let $M \subset P_{\text{Sgl}}$ denote the Levi subgroup that also stabilizes V_d . The unipotent radical of P_{Sgl} is the subgroup N that fixes both V^d and W/V^d and clearly, $P_{\text{Sgl}}/N \cong M$. Furthermore, there is a canonical identification $M \xrightarrow{\sim} \text{GL}_{\mathcal{K}}(V) \times \mathbb{G}_m$ via $m \mapsto (\Delta(m), \nu(m))$, where Δ is the projection

$$P_{\text{Sgl}} \rightarrow \text{GL}_{\mathcal{K}}(V^d) = \text{GL}_{\mathcal{K}}(V) ,$$

whose inverse is given by $(A, \lambda) \mapsto \text{diag}(\lambda(A^*)^{-1}, A)$, where $A^* = {}^t A^c$.

9.1.2. *Induced Representations.* Let $\chi : \mathcal{K}^\times \backslash \mathbb{A}_{\mathcal{K}}^\times \rightarrow \mathbb{C}^\times$ be a unitary Hecke character. It factors as $\chi = \bigotimes_w \chi_w$, where w runs over all places of \mathcal{K} . In later section, we assume that χ is of type A_0 , i.e. we impose certain conditions on $\chi_\infty = \bigotimes_{w|\infty} \chi_w$.

For convenience, define the character ∇ of $P_{\text{Sgl}}(\mathbb{A})$ as

$$\nabla(-) = |\text{Nm}_{\mathcal{K}/\mathbb{Q}} \circ \det \circ \Delta(-)|_{\mathcal{K}^+} \cdot |\nu(-)|_{\mathcal{K}^+}^{-n} = |\det \circ \Delta(-)|_{\mathcal{K}} \cdot |\nu(-)|_{\mathcal{K}}^{-n/2} ,$$

where $\text{Nm}_{\mathcal{K}/\mathbb{Q}}$ is the usual norm homomorphism $\mathbb{A}_{\mathcal{K}} \rightarrow \mathbb{A}_{\mathbb{Q}}$. One readily checks that $G_1(\mathbb{A})$, via its natural diagonal inclusion in $G_4(\mathbb{A})$, is in the kernel of ∇ . Moreover, the modulus character δ_{Sgl} of $P_{\text{Sgl}}(\mathbb{A}_{\mathbb{Q}})$ equals ∇^n .

Let $s \in \mathbb{C}$, and define the smooth and normalized induction

$$(134) \quad I(\chi, s) = \iota_{P_{\text{Sgl}}(\mathbb{A})}^{G(\mathbb{A})} \left(\chi(\det \circ \Delta(-)) \cdot \nabla(-)^{-s} \right).$$

This degenerate principal series is identical to the one in [Ehls20, Section 4.1.2]. It is also equal to the smooth, unnormalized parabolic induction

$$(135) \quad I(\chi, s) = \text{Ind}_{P_{\text{Sgl}}(\mathbb{A})}^{G(\mathbb{A})} \left(\chi(\det \circ \Delta(-)) \cdot \nabla(-)^{-s - \frac{n}{2}} \right),$$

and factors as a restricted tensor product of local induced representations

$$I(\chi, s) = \bigotimes_v I_v(\chi_v, s),$$

where v runs over all places of \mathbb{Q} and $\chi_v = \bigotimes_{w|v} \chi_w$. The definition of $I_v(\chi_v, s)$ is given by the obvious local analogue of (135) at v .

Remark 9.1. To compare with results in [Eis15] and [EL20], let us write s_E and s_{EL} for the variable s appearing in the unnormalized parabolic induction functor for these articles respectively. Then, the relations with our variable s are $s_E = s + \frac{n}{2}$ and $s_{EL} = -s$.

9.1.3. Siegel-Weil sections and Eisenstein series. Given a Siegel-Weil section $f = f_{\chi, s}$ of $I(\chi, s)$, one constructs the *standard (nonnormalized) Eisenstein series*

$$(136) \quad E_f(g) = \sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} f(\gamma g)$$

as a function on $G(\mathbb{A})$. It converges on the half-plane $\text{Re}(s) > n/2$ and if f is right- K -finite, for some maximal compact open subgroup $K \subset G$, it admits a meromorphic continuation on \mathbb{C} .

Remark 9.2. In the following sections, we choose explicit $f_v \in I_v(\chi_v, s)$ for each place v of \mathbb{Q} . Our choices are parallel to the ones in [Ehls20, Section 4] and are standard in the literature, especially at finite unramified places and at archimedean places.

However, our choice of section at p requires several adjustments to construct a Siegel Eisenstein series that interpolates properly p -adically along a P -ordinary family.

The main difference is that the locally constant function μ in [Ehls20, Section 4.3.1], which is essentially the nebentypus of an ordinary cuspidal automorphic representation, is replaced by a (matrix coefficient of a) type of a P -ordinary cuspidal automorphic representations.

9.1.4. *Zeta integrals.* Let $f = f_{\chi,s} \in I(\chi, s)$. Let π be any cuspidal automorphic representation for G_1 and let $\varphi \in \pi$ and $\varphi^\vee \in \pi^\vee$ be any vectors. The doubling method consists of relating the Rankin-Selberg integral

$$I(\varphi, \varphi^\vee, f; \chi, s) := \int_{Z_3(\mathbb{A})G_3(\mathbb{Q}) \backslash G_3(\mathbb{A})} E_f(g_1, g_2) \varphi(g_1) \varphi^\vee(g_2) \chi^{-1}(\det g_2) d(g_1, g_2)$$

and relate it to the L -function associated to π and χ . In Section 12, we reinterpret this integral algebraically as a pairing between a holomorphic modular form on G_3 and a anti-holomorphic cusp form on G_3 . To do so, as explained in [GPSR87], we use that for $\text{Re}(s)$ large enough,

$$I(\varphi, \tilde{\varphi}, f; \chi, s) = \int_{U_1(\mathbb{A})} f_{\chi,s}(u, 1) \langle \pi(u) \varphi, \varphi^\vee \rangle_\pi du.$$

In the following sections, we choose some f for which this can be done. More precisely, we construct f as a pure tensor $f = \bigotimes_l f_l$ over all places l of \mathbb{Q} . Assuming that π is P -anti-ordinary of P -anti-WLT (κ, K_r, τ) , we construct these local Siegel-Weil sections so that f_p depends on χ_p and τ , f_∞ depends on χ_∞ and κ , and for all finite prime l away from p , f_l depends on K_r^p .

Assume φ and φ^\vee are “pure tensors”, i.e. $\varphi = \bigotimes_l \varphi_l$ and $\varphi^\vee = \bigotimes_l \varphi_l^\vee$ according to the factorization (59), e.g. φ and φ^\vee are test vectors as in Section 7.

Then

$$I(\varphi, \varphi^\vee, f; \chi, s) = \prod_l I_l(\varphi_l, \varphi_l^\vee, f_l; \chi_l, s) \cdot \langle \varphi, \varphi^\vee \rangle,$$

where

$$(137) \quad I_l(\varphi_l, \varphi_l^\vee, f_l; \chi_l, s) = \frac{\int_{U_{1,l}} f_{\chi,s,l}(u, 1) \langle \pi_l(u) \varphi_l, \varphi_l^\vee \rangle_{\pi_l} du}{\langle \varphi_l, \varphi_l^\vee \rangle_{\pi_l}},$$

for any place l of \mathbb{Q} . Let Z_l denote the numerator of the fraction on the right-hand side of (137). We compute each zeta integral Z_l individually (by factoring it over places of \mathcal{K}^+ above l), in Section 10.

9.2. Local Siegel-Weil section at p . For each places $w \in \Sigma_p$ of \mathcal{K} , fix an isomorphism $\mathcal{K}_w = \mathcal{K}_{\bar{w}}$. Then, the identification (5) for G_4 induces an identification of $P_{\text{Sgl}}(\mathbb{Q}_p)$ with $\mathbb{Q}_p^\times \times \prod_{w \in \Sigma_p} P_n(\mathcal{K}_w)$, where $P_n \subset \text{GL}_{\mathcal{K}}(W)$ is the parabolic subgroup stabilizing V^d .

Let $\chi_p = \otimes_{w|p} \chi_w$ and, given $s \in \mathbb{C}$, view $\chi_p \cdot |\cdot|_p^{-s}$ as a character of $P_{\text{Sgl}}(\mathbb{Q}_p)$. One readily checks that its restriction to $\prod_{w \in \Sigma_p} P_n(\mathcal{K}_w)$ corresponds to the product over $w \in \Sigma_p$ of the characters $\psi_{w,s} : P_n(\mathcal{K}_w) \rightarrow \mathbb{C}^\times$ defined as

$$\psi_{w,s} \left(\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \right) = \chi_w(\det D) \chi_{\bar{w}}(\det A^{-1}) \cdot |\det A^{-1} D|_w^{-s},$$

by writing element of P_n according to the direct sum decomposition $W = V_d \oplus V^d$.

Let $W_w = W \otimes_{\mathcal{K}} \mathcal{K}_w$ and choose any $f_{w,s} \in \iota_{P_n(\mathcal{K}_w)}^{\mathrm{GL}_{\mathcal{K}_w}(W_w)} \psi_{w,s}$, for each $w \in \Sigma$. Then, it is clear that the section

$$(138) \quad f_p(g) = f_{p,\chi,s}(g) := |\nu|_p^{(s+\frac{n}{2})\frac{n}{2}} \prod_{w \in \Sigma_p} f_{w,s}(g_w), \quad g = (\nu, (g_w)_w) \in G(\mathbb{Q}_p),$$

is in $I_p(\chi_p, s)$.

Remark 9.3. The strategy below is to construct such $f_{w,s} = f_{w,s}^{\Phi_w}$, and hence $f_p = f_p$, from a specific Schwartz function $\Phi_w = \Phi_w^{\tau_w}$ (that depends on the type τ_w), see (147). This approach is already used in [Eis15, Section 2.2.8] and [EHLS20, Section 4.3.1]. In fact, our argument owes a great deal to their work and the details they carefully provide.

The novelty here is that we associate Schwartz functions to finite dimensional representations (namely the SZ types from Section 1.2.3), instead of characters.

9.2.1. *Locally Constant Matrix Coefficients.* In what follows, we use the notation of Section 7.2 freely. Let $\chi_{w,1} := \chi_w$ and $\chi_{w,2} := \chi_w^{-1}$. Increasing the level r of the SZ-types τ and τ^\vee at p if necessary, we assume that the following inequality holds :

$$(139) \quad r \geq \max(1, \mathrm{ord}_w(\mathrm{cond}(\chi_{w,1})), \mathrm{ord}_w(\mathrm{cond}(\chi_{w,2}))),$$

for each $w \in \Sigma_p$. In what follows, we consider $\chi_{w,1}$ and $\chi_{w,2}$ as characters of general linear groups of any rank via composition with the determinant without comment.

Let $\mu'_{w,j} : K_{w,j} \rightarrow \mathbb{C}$ be the matrix coefficient defined as

$$\mu'_{w,j}(X) = \begin{cases} \langle \phi_{w,j}, \tau_{w,j}^\vee(X) \phi_{w,j}^\vee \rangle_{\sigma_{w,j}}, & \text{if } j = 1, \dots, t_w, \\ \langle \tau_{w,j}(X) \phi_{w,j}, \phi_{w,j}^\vee \rangle_{\sigma_{w,j}}, & \text{if } j = t_w + 1, \dots, r_w. \end{cases}$$

Remark 9.4. We do not make the choice of $\phi_{w,j}$ and $\phi_{w,j}^\vee$ (see (106) and (110)) explicit in our notation for $\mu'_{w,j}$. See Remark 9.9 below for further details.

The products $\mu'_{a_w} = \bigotimes_{j=1}^{t_w} \mu'_{w,j}$ and $\mu'_{b_w} = \bigotimes_{j=t_w+1}^{r_w} \mu'_{w,j}$ on K_{a_w} and K_{b_w} , respectively, are the matrix coefficients

$$\mu'_{a_w}(X) = (\phi_{a_w}^0, \tau_{a_w}^\vee(X) \phi_{a_w}^{\vee,0})_{a_w} \quad ; \quad \mu'_{b_w}(X) = (\tau_{b_w}(X) \phi_{b_w}^0, \phi_{b_w}^{\vee,0})_{b_w}$$

of $\tilde{\tau}_{a_w}$ and τ_{b_w} respectively.

We now consider μ'_{a_w} as a locally constant function on $M_{a_w}(\mathcal{K}_w)$ supported on $\mathfrak{X}_w^{(1)} := {}^t I_{a_w, r}^0 I_{a_w, r}^0$. More precisely, one readily verifies that given $X \in \mathfrak{X}_w^{(1)}$ and any ${}^t \gamma_1, \gamma_2 \in I_{a_w, r}^0$ such that $X = \gamma_1 \gamma_2$, then

$$(140) \quad \mu'_{a_w}(X) := (\tau_{a_w}(\gamma_1^{-1}) \phi_{a_w}^0, \tau_{a_w}^\vee(\gamma_2) \phi_{a_w}^{\vee,0})_{a_w}$$

is well-defined. Indeed, given ${}^t\gamma'_1, \gamma'_2 \in I_{a_w, r}^0$ such that $X = \gamma_1\gamma_2 = \gamma'_1\gamma'_2$, we have

$$\begin{aligned} (\tau_{a_w}(\gamma_1^{-1})\phi_{a_w}^0, \tau_{a_w}^\vee(\gamma_2)\phi_{a_w}^{\vee, 0})_{a_w} &= (\tau_{a_w}(\gamma_2(\gamma'_2)^{-1})\tau_{a_w}((\gamma'_1)^{-1})\phi_{a_w}^0, \tau_{a_w}^\vee(\gamma_2)\phi_{a_w}^{\vee, 0})_{a_w} \\ &= (\tau_{a_w}((\gamma'_1)^{-1})\phi_{a_w}^0, \tau_{a_w}^\vee(\gamma'_2\gamma_2^{-1})\tau_{a_w}^\vee(\gamma_2)\phi_{a_w}^{\vee, 0})_{a_w} \\ &= (\tau_{a_w}((\gamma'_1)^{-1})\phi_{a_w}^0, \tau_{a_w}^\vee(\gamma'_2)\phi_{a_w}^{\vee, 0})_{a_w} \end{aligned}$$

where the first and second equality holds since $\gamma_1^{-1}\gamma'_1 = \gamma_2(\gamma'_2)^{-1}$ modulo \mathfrak{p}_w^r lies in $L_{a_w}(\mathcal{O}_w/\mathfrak{p}_w^r\mathcal{O}_w)$.

Similarly, we extend μ'_{b_w} to a locally constant function on $M_{b_w}(\mathcal{K}_w)$ supported on $\mathfrak{X}_w^{(4)} := {}^tI_{b_w, r}^0 I_{b_w, r}^0$ via

$$(141) \quad \mu'_{b_w}(X) := (\tau_{b_w}(\gamma_2)\phi_{b_w}^0, \tau_{b_w}^\vee(\gamma_1^{-1})\phi_{b_w}^{\vee, 0})_{b_w},$$

where $X \in \mathfrak{X}_w^{(4)}$ and ${}^t\gamma_1, \gamma_2 \in I_{b_w, r}^0$ are any elements such that $X = \gamma_1\gamma_2$.

Let $\mu_{a_w}(A) := \chi_{2, w}^{-1}\mu'_{a_w}$ and $\mu_{b_w} := \chi_{1, w}\mu'_{b_w}$. Let $\mathfrak{X}_w \subset M_n(\mathcal{O}_w)$ be the set of matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that $A \in \mathfrak{X}_w^{(1)}$, $B \in M_{a_w \times b_w}(\mathcal{O}_w)$, $C \in M_{b_w \times a_w}(\mathcal{O}_w)$ and $D \in \mathfrak{X}_w^{(4)}$.

We define a locally constant function μ_w on $M_n(\mathcal{K}_w)$ supported on \mathfrak{X} via

$$(142) \quad \mu_w \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) = \mu_{a_w}(A)\mu_{b_w}(D),$$

for all $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{X}_w$.

Observe that the set \mathfrak{X}_w contains the subgroup $\mathfrak{G}_w = \mathfrak{G}_w(r) \subset \mathrm{GL}_n(\mathcal{O}_w)$ consisting of matrices whose terms below the $(n_{w, j} \times n_{w, j})$ -blocks along the diagonal are in \mathfrak{p}_w^r and such that the upper right $(a_w \times b_w)$ -block is also in \mathfrak{p}_w^r . Similarly, let $\mathfrak{G}_{l, w} = \mathfrak{G}_{l, w}(r)$ (resp. $\mathfrak{G}_{u, w} = \mathfrak{G}_{u, w}(r)$) be the largest subgroup of $\mathrm{GL}_n(\mathcal{O}_w)$ such that $\mathfrak{G}_{l, w} \cap P_{\mathfrak{d}_w}^u = 1$ (resp. $\mathfrak{G}_{l, w} \cap P_{\mathfrak{d}_w}^{u, \mathrm{op}} = 1$).

In particular, we have the natural decomposition $\mathfrak{G}_w = \mathfrak{G}_l(I_{a_w, r}^0 \times I_{b_w, r}^0)\mathfrak{G}_u$. By abuse of notation, given $B \in M_{a_w \times b_w}(\mathcal{K}_w)$ or $C \in M_{b_w \times a_w}(\mathcal{K}_w)$, we sometimes write $B \in \mathfrak{G}_{u, w}$ or $C \in \mathfrak{G}_{l, w}$ when we mean

$$\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \in \mathfrak{G}_{u, w} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix} \in \mathfrak{G}_{l, w}.$$

9.2.2. Choice of Schwartz functions. Let $\Phi_{1, w} : M_n(\mathcal{K}_w) \rightarrow \mathbb{C}$ be the locally constant function supported on \mathfrak{G}_w such that

$$(143) \quad \Phi_{1, w}(X) = \mu_w(X)$$

for all $X \in \mathfrak{G}_w$. Furthermore, define the locally constant functions

$$(144) \quad \nu_\bullet(z) = \chi_{w, 1}^{-1}\chi_{w, 2}\mu_\bullet(z) \quad ; \quad \phi_{\nu_\bullet}(z) = \nu_\bullet(-z),$$

where \bullet denotes a_w, b_w or w , and z is in the appropriate domain.

Let $\Phi_{2,w} : M_n(\mathbb{Q}_p) \rightarrow \mathbb{C}$ be

$$(145) \quad \Phi_{2,w}(x) = (\nu_w)^\wedge(x) = \int_{M_n(\mathcal{K}_w)} \phi_{\nu_w}(y) e_w(\text{tr}(yx)) dy$$

Remark 9.5. The definition of μ_w and its twist ν_w on \mathfrak{X}_w allows us to generalize the function denoted ϕ_{ν_v} in [EHL20, Section 4.3.1]. In *loc. cit.*, the SZ-types are all characters, in which case τ_\bullet is equal to μ'_\bullet (and there is no need to talk about types). The “telescoping product” in the definition of ϕ_{ν_v} is simply a formula that expresses the extension of these characters to I_\bullet^0 and ${}^t I_\bullet^0$ simultaneously. Our alternative is to use extensions of (matrix coefficients of) SZ-types such as in Equations (140), (141) and (142).

Remark 9.6. This Fourier transform in the definition of $\Phi_{2,w}$ is slightly different than the one in [Eis15, Section 2.2.8] and [EHL20, Section 4.3.1]. It is the same as the one involved in the Godement-Jacquet functional equation [Jac79].

Lemma 9.7. Given $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A \in M_{a_w \times a_w}(\mathcal{K}_w)$, $B, {}^t C \in M_{a_w \times b_w}(\mathcal{K}_w)$ and $D \in M_{b_w \times b_w}(\mathcal{K}_w)$, one can write

$$\Phi_{2,w}(X) = \Phi_w^{(1)}(A) \Phi_w^{(2)}(B) \Phi_w^{(3)}(C) \Phi_w^{(4)}(D)$$

with

$$\begin{aligned} \Phi_w^{(2)} &= \text{char}_{M_{a_w \times b_w}(\mathcal{O}_w)}, & \Phi_w^{(3)} &= \text{char}_{M_{b_w \times a_w}(\mathcal{O}_w)}, \\ \text{supp}(\Phi_w^{(1)}) &\subset \mathfrak{p}_w^{-r} M_{a_w \times a_w}(\mathcal{O}_w), & \text{supp}(\Phi_w^{(4)}) &\subset \mathfrak{p}_w^{-r} M_{b_w \times b_w}(\mathcal{O}_w), \end{aligned}$$

where r is as in Inequality (139).

Proof. The definitions of $\phi_{\nu_{a_w}}$, $\phi_{\nu_{b_w}}$ and ϕ_{ν_w} immediately imply

$$\begin{aligned} \Phi_{2,w}(X) &= \int_{\mathfrak{X}_w} \phi_{\nu_w} \left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right) e_w(\text{tr}(\alpha A + \beta B + \gamma C + \delta D)) d\alpha d\beta d\gamma d\delta \\ &= \int_{\mathfrak{X}_w^{(1)}} \phi_{\nu_{a_w}}(\alpha) e_w(\text{tr}(\alpha A)) d\alpha \int_{\mathfrak{X}_w^{(4)}} \phi_{\nu_{b_w}}(\delta) e_w(\text{tr}(\delta D)) d\delta \\ &\quad \times \text{char}_{M_{a_w \times b_w}(\mathcal{O}_w)}(B) \text{char}_{M_{b_w \times a_w}(\mathcal{O}_w)}(C) \end{aligned}$$

Then, we may conclude as in the proof of [EHL20, Lemma 4.3.2 (ii)] by observing

$$\begin{aligned} \Phi_w^{(1)}(A) &:= \int_{\mathfrak{X}_w^{(1)}} \phi_{\nu_{a_w}}(\alpha) e_w(\text{tr}(\alpha A)) d\alpha \\ &= \text{Vol}(\mathfrak{p}_w^r M_{a_w}(\mathcal{O}_w)) \sum_{\alpha \in \mathfrak{X}_w^{(1)} \bmod \mathfrak{p}_w^r} \phi_{\nu_{a_w}}(\alpha) e_w(\text{tr} \alpha A) \text{char}_{\mathfrak{p}_w^{-r} M_{a_w}(\mathcal{O}_w)}(A), \end{aligned}$$

and

$$\begin{aligned} \Phi_w^{(4)}(D) &:= \int_{\mathfrak{X}_w^{(4)}} \phi_{\nu_{b_w}}(\delta) e_w(\mathrm{tr}(\delta D)) d\delta \\ &= \mathrm{Vol}(\mathfrak{p}_w^r M_{b_w}(\mathcal{O}_w)) \sum_{\delta \in \mathfrak{X}_w^{(4)} \bmod \mathfrak{p}^r} \phi_{\nu_{b_w}}(\delta) e_w(\mathrm{tr} \delta D) \mathrm{char}_{\mathfrak{p}_w^{-r} M_{b_w}(\mathcal{O}_w)}(D). \end{aligned}$$

□

Define the Schwartz function $\Phi_w : M_{n \times 2n}(\mathbb{Q}_p) \rightarrow \mathbb{C}$ as

$$(146) \quad \Phi_w(X) = \Phi_w(X_1, X_2) = \frac{\dim \tau_w}{\mathrm{Vol}(\mathfrak{G}_w)} \Phi_{1,w}(-X_1) \Phi_{2,w}(X_2).$$

Remark 9.8. In this section, the local type τ_w is fixed, hence we do not include it in our notation. However, in Section 11, the type varies along a P -ordinary Hida family. Therefore, we write $\mu_w^{\tau_w}$, $\nu_w^{\tau_w}$, $\Phi_w^{\tau_w}$ and so on to emphasize the role of τ_w .

9.2.3. *Construction of $f_{w,s}$ for $w \in \Sigma_p$.* For each $w \in \Sigma_p$, write $V_w = V \otimes_{\mathcal{K}} \mathcal{K}_w$ and use similar notation for $V_{d,w}$ and V_w^d . Consider the decomposition

$$\mathrm{Hom}_{\mathcal{K}_w}(V_w, W_w) = \mathrm{Hom}_{\mathcal{K}_w}(V_w, V_{w,d}) \oplus \mathrm{Hom}_{\mathcal{K}_w}(V_w, V_w^d), \quad X = (X_1, X_2)$$

and its subspace

$$\mathbf{X} := \{X \in \mathrm{Hom}_{\mathcal{K}_w}(V_w, W_w) \mid X(V_w) = V_w^d\} = \{(0, X) \mid X : V_w \xrightarrow{\sim} V_w^d\}.$$

In fact, any $X \in \mathbf{X}$ can be viewed as an automorphism of V_w (by composing with the identification of V^d with V) and hence, we identify \mathbf{X} with $\mathrm{GL}_{\mathcal{K}_w}(V_w)$. Let $d^\times X$ be the Haar measure on the latter.

Furthermore, recall that we fixed an \mathcal{O}_w -basis of $L_{1,w}$ in Section 2.2.1. This provides a \mathcal{K}_w -basis of V_w and, via their identification to V , a \mathcal{K}_w -basis of $V_{d,w}$ and of V_w^d . Hence, it also induces a \mathcal{K}_w -basis of $W_w = V_{d,w} \oplus V_w^d$.

It identifies $\mathrm{Isom}(V_w^d, V_w)$ with $\mathrm{Isom}(V_{w,d}, V_w)$, $\mathrm{GL}_{\mathcal{K}_w}(V_w)$ with $\mathrm{GL}_n(\mathcal{K}_w)$, $\mathrm{GL}_{\mathcal{K}_w}(W_w)$ with $\mathrm{GL}_{2n}(\mathcal{K}_w)$, $P_n(\mathcal{K}_w)$ with the subgroup of $\mathrm{GL}_{2n}(\mathcal{K}_w)$ consisting of upper-triangular $n \times n$ -block matrices, and $\mathrm{Hom}_{\mathcal{K}_w}(V_w, W_w)$ with $M_{n \times 2n}(\mathbb{Q}_p)$

Therefore, we now view the Schwartz function $\Phi_w : M_{n \times 2n}(\mathbb{Q}_p) \rightarrow \mathbb{C}$ constructed above as a function on $\mathrm{Hom}_{\mathcal{K}_w}(V_w, W_w)$. We define $f_{w,s} = f_{w,s}^{\Phi_w} = f^{\Phi_w}$, an element of ${}^{\mathrm{GL}_{2n}(\mathcal{K}_w)}\psi_{P_n(\mathcal{K}_w)} \psi_{w,s}$, as

$$(147) \quad f^{\Phi_w}(g) = \chi_{2,w}(g) |\det g|_w^{\frac{n}{2}+s} \int_{\mathbf{X}} \Phi_w(Xg) \chi_{w,1}^{-1} \chi_{w,2}(X) |\det X|_w^{n+2s} d^\times X,$$

as in [Ehls20, Equation (55)]. To emphasize the role of χ_p and τ in the construction of the local Siegel-Weil section $f_p = f_{p,\chi,s}$ via (138) obtained from f^{Φ_w} in (147), for each $w \in \Sigma_p$, we sometimes denote it by

$$(148) \quad f_p(\bullet) = f_p(\bullet; \tau, \chi_p, s).$$

Remark 9.9. In Section 11, we consider the section $f_p(\bullet; \tau \otimes \psi, \chi_p, s)$ as τ remains fixed and $\psi = \otimes \psi_w$ varies over finite-order characters of $L_P(\mathbb{Z}_p)$. Note that this section depends on the choice of local vectors $\phi_{w,j}$ and $\phi_{w,j}^\vee$, for $w \in \Sigma_p$ and $1 \leq j \leq r_w$, with respect to $\tau \otimes \psi$ instead of τ , see (106) and (110).

However, we do not make this dependence explicit in our notation. This is because the choice of such local vectors $\phi_{w,j}^\vee$ are fixed given any τ , and our conventions ensure that the corresponding local vector for $\tau \otimes \psi$ are the local vectors $\phi_{w,j}^\vee \otimes 1$, i.e. essentially the “same” local vectors, see Remarks 1.5, 7.6, 7.10, and 8.24.

9.3. Local Siegel-Weil section at ∞ . In what follows, we continue with the notation of Section 7.3 (for $G = G_4$ instead of G_1). In particular, we have $G = G_4 \subset G^*$ and $G^*(\mathbb{R}) = \prod_\sigma G_\sigma$, where $G_\sigma \cong \mathrm{GU}^+(n, n)$, identifying $\sigma \in \Sigma$ with its restriction in $\Sigma_{\mathcal{K}^+} = \mathrm{Hom}(\mathcal{K}^+, \mathbb{R})$. Similarly, the homomorphism $h = h_4$ from the PEL datum \mathcal{P}_4 , valued in $G_{/\mathbb{R}}^*$, factors as $h = \prod_\sigma h_\sigma$.

By fixing a basis of L_1 , we naturally obtain bases for V_d and V^d (via their identification with $V = L_1 \otimes \mathbb{R}$) and $W = V_d \oplus V^d$. We use these to view G_σ as a subgroup of $\mathrm{GL}_{2n}(\mathbb{C})$ and write $g_\sigma = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$ where $a_\sigma, b_\sigma, c_\sigma$ and d_σ are all $n \times n$ -matrices. We always use this convention of symbols without comments. Furthermore, this choice of basis induces an identification between $\mathfrak{p}_\sigma^+ = \mathfrak{p}_{4,\sigma}^+$ and $M_n(\mathbb{C})$.

One readily checks that the $G^*(\mathbb{R})$ -conjugacy class of h is again equal to the $G(\mathbb{R})$ -conjugacy class X of h . Therefore, $X = \prod_{\sigma \in \Sigma} X_\sigma$, where X_σ is the G_σ -conjugacy class of h_σ . Let $X_\sigma^+ \subset X_\sigma$ be the connected component containing h_σ .

It is well-known that the space X_σ^+ is holomorphically isomorphic to a tube domain in $\mathfrak{p}_\sigma^+ \simeq M_n(\mathbb{C})$, see [Har86, (5.3.2)], [Eis15, Section 2.1] or [EHLS20, Section 4.4.2]. Namely, let $\mathfrak{J}_\sigma \in M_n(\mathbb{C})$ be the fixed point of $U_\sigma = U_\infty \cap G_\sigma$. Without loss of generality, we may assume that \mathfrak{J}_σ is a diagonal matrix whose entries are trace-zero elements of $\sigma(\mathcal{K})$. Then, X_σ^+ is naturally identified with

$$X_{n,n} := \{z \in M_n(\mathbb{C}) \mid \mathfrak{J}_\sigma({}^t \bar{z} - z) \text{ is positive-definite}\}.$$

The action of $g_\sigma \in G_\sigma$ on $z \in X_{n,n}$ is given by

$$g_\sigma(z) = (a_\sigma z + b_\sigma) \cdot (c_\sigma z + d_\sigma)^{-1}.$$

9.3.1. Unitary Hecke characters of type A_0 . Let $\chi = \otimes_w \chi_w$ be the unitary Hecke character introduced in Section 9.1.2. Let $\chi_\infty = \otimes_{\sigma \in \Sigma} \chi_\sigma$. We assume that for each $\sigma \in \Sigma$, there exists integer $k_\sigma \in \mathbb{Z}_{\geq 0}$ and $\nu_\sigma \in \mathbb{Z}$ such that

$$(149) \quad \chi_\infty(z) = \prod_{\sigma \in \Sigma} z_\sigma^{-(k_\sigma + 2\nu_\sigma)} (z_\sigma \bar{z}_\sigma)^{\frac{k_\sigma}{2} + \nu_\sigma} = \prod_{\sigma \in \Sigma} \left(\frac{|z_\sigma|_\sigma}{z_\sigma} \right)^{k_\sigma + 2\nu_\sigma},$$

for all $z = (z_\sigma)_\sigma \in \mathbb{A}_{\mathcal{K}, \infty}^\times = \prod_{\sigma \in \Sigma} \mathbb{C}$.

Remark 9.10. To compare with the notation with [Ehls20, Section 4.4.2], consider the Hecke character $\chi_\infty(\bullet) \mid \bullet |_\infty^{-s-\frac{n}{2}}$. Assume there exists some integer k such that $k = k_\sigma$ for all $\sigma \in \Sigma$. In that case, let $s = \frac{k-n}{2}$ so that

$$(150) \quad |z|_\infty^{\frac{k}{2}} \chi_\infty(z) = \prod_{\sigma \in \Sigma} z_\sigma^{-(k+\nu_\sigma)} \overline{z_\sigma}^{\nu_\sigma},$$

for all $z = (z_\sigma)_\sigma \in \mathbb{A}_{\mathcal{K}, \infty}^\times$.

For the expression (150) to be in the same form as the character denoted $|\bullet|^m \chi_0$ from [Ehls20, p.72], there is a lot of freedom on the integers m , $a(\chi_\sigma)$ and $b(\chi_\sigma)$ introduced in *loc.cit.* For instance, we can pick m arbitrarily and let $a(\chi_\sigma) = m + k + \nu_\sigma$ and $b(\chi_\sigma) = m - \nu_\sigma$.

Remark 9.11. The relations of the previous remark are the ones unstated in the explanations of [Ehls20, Section 5.3]. We prefer to work with the notation of k , ν_σ and a unitary character χ as it is easier to compare with the computations of [Eis15] and [EL20] and the work of Shimura more generally. However, for applications towards motivic conjectures, the notation with m , $a(\chi_\sigma)$ and $b(\chi_\sigma)$ is often more appropriate.

9.3.2. *Canonical automorphy factors for $\mathrm{GU}(n, n)$.* For $\sigma \in \Sigma$, $z \in X_{n, n}$ and $g_\sigma \in G_\sigma$, let

$$J_\sigma(g_\sigma, z) = c_\sigma z + d_\sigma \quad \text{and} \quad j_\sigma(g_\sigma, z) = \det(J_\sigma(g_\sigma, z)).$$

Similarly, for $z = (z_\sigma) \in X = \prod_\sigma X_\sigma$ and $g = (g_\sigma) \in G^*(\mathbb{R}) = \prod_\sigma G_\sigma$, let

$$J(g, z) = \prod_\sigma J_\sigma(g_\sigma, z_\sigma) \quad \text{and} \quad J'(g, z) = \prod_\sigma J'_\sigma(g_\sigma, z_\sigma).$$

The functions

$$J_\sigma(g_\sigma) = J_\sigma(g_\sigma, \mathfrak{I}_\sigma) \quad \text{and} \quad J'_\sigma(g_\sigma) = J'_\sigma(g_\sigma, \mathfrak{I}_\sigma)$$

are C^∞ -functions on G_σ valued in $\mathrm{GL}_n(\mathbb{C})$, and so the functions

$$j_\sigma(g_\sigma) = \det(J_\sigma(g_\sigma)) \quad \text{and} \quad j'_\sigma(g_\sigma) = \det(J'_\sigma(g_\sigma))$$

are C^∞ -functions on G_σ valued in \mathbb{C}^\times .

Using the integers k_σ and ν_σ from above, let

$$j_{\chi_\sigma}(g_\sigma, z) := \det(g_\sigma, z)^{-\nu_\sigma} j_\sigma(g_\sigma, z)^{-k_\sigma}$$

and given $s \in \mathbb{C}$, define $f_\sigma(g_\sigma; \mathfrak{I}_\sigma, \chi_\sigma, s)$ as

$$j_{\chi_\sigma}(g_\sigma, \mathfrak{I}_\sigma) \cdot |j_\sigma(g_\sigma, \mathfrak{I}_\sigma)|_\sigma^{s-\frac{k_\sigma-n}{2}} \cdot |\nu(g_\sigma)|_\sigma^{\frac{n}{2}(s+\frac{n}{2})},$$

a function on G_σ .

From this point on, assume χ satisfies the following hypothesis :

HYPOTHESIS 9.12. There exists some integer $k \geq 0$ such that $k_\sigma = k$ for all $\sigma \in \Sigma$.

Remark 9.13. This hypothesis is exactly the necessary condition to ensure that the function $\prod_{\sigma \in \Sigma} j_{\chi_\sigma}(g_\sigma, z) \cdot |j_\sigma(g_\sigma, z)|_\sigma^{s - \frac{k_\sigma - n}{2}}$ is holomorphic as a function of z at $s = \frac{k-n}{2}$.

Let $U(\mathfrak{g}_\sigma)$ denote the universal enveloping algebra of \mathfrak{g}_σ . Consider the $U(\mathfrak{g}_\sigma)$ -submodule $C_{\chi_\sigma}(G_\sigma)$ generated by $f_\sigma(g_\sigma; \mathfrak{J}_\sigma, \chi_\sigma, \frac{n-k}{2})$ of $C^\infty(G_\sigma)$. It naturally carries the structure of a $(U(\mathfrak{g}_\sigma), U_\sigma)$ -module and as explained in [EHL20, Section 4.4.2], it is isomorphic to the holomorphic $(U(\mathfrak{g}_\sigma), U_\sigma)$ -module $\mathbb{D}^2(\chi_\sigma)$ with highest U_σ -type

$$\Lambda(\chi_\sigma) := (\bullet; \nu_\sigma, \dots, \nu_\sigma; k + \nu_\sigma, \dots, k + \nu_\sigma),$$

where \bullet denotes some character of the \mathbb{R} -split center of U_σ (whose exact description is irrelevant for our purpose), the next n entries are identical, and the last n -entries are also identical.

One readily checks that

$$(151) \quad f_\infty(g) = f_\infty(g; \mathfrak{J}, \chi_\infty, s) := \prod_{\sigma \in \Sigma} f_\sigma(g_\sigma; \mathfrak{J}_\sigma, \chi_\sigma, s) \in I_\infty(\chi_\infty, s),$$

where $\mathfrak{J} = (\mathfrak{J}_\sigma)_\sigma \in X$ and $g = (g_\sigma)_\sigma \in G_4(\mathbb{R}) \subset G^*(\mathbb{R}) = \prod_\sigma G_\sigma$.

More generally, replacing \mathfrak{J} by any $z = (z_\sigma)_\sigma \in X$, we define $f_\sigma(g_\sigma; z_\sigma, \chi_\sigma, s)$ and $f_\infty(g; z, \chi_\infty, s)$ similarly.

9.3.3. C^∞ -differential operators.

Remark 9.14. The Eisenstein measure involved in the construction of our p -adic L -functions uses the Siegel section f_∞ constructed above at $s = \frac{k-n}{2}$. The idea is to view the corresponding Siegel Eisenstein series as a p -adic modular form, using the theory of Section 5, and apply p -adic differential operators.

However, to relate this measure to standard L -functions, we compare this p -adic Eisenstein series to a smooth (non-holomorphic) Eisenstein series obtained by replacing the p -adic differential operators with more familiar C^∞ differential operators. We obtain special values of L -functions by applying the doubling method to the C^∞ Eisenstein series.

As opposed to our choice of Siegel-Weil sections at p from Section 9.2 (and the corresponding computation of Zeta integrals in Section 10.1), the objects and calculus needed here are already explained thoroughly in the literature, see [Har97, Har08], [EHL20, Sections 4.4-4.5] or [EL20]. Therefore, we simply recall the material and results upon which we rely.

Let κ be a dominant character of T_{H_0} as in Section 2.3.1. We modify our notation slightly in this section by identifying κ as a tuple in $\mathbb{Z} \times \prod_{\sigma \in \Sigma} \mathbb{Z}^{a_\sigma} \times \mathbb{Z}^{b_\sigma}$. Therefore, we momentarily write

$$\kappa_\sigma = (\kappa_{\sigma,1}, \dots, \kappa_{\sigma,a_\sigma}; \kappa_{\sigma,1}^c, \dots, \kappa_{\sigma,b_\sigma}^c) \in \mathbb{Z}^{a_\sigma} \times \mathbb{Z}^{b_\sigma}$$

and $\kappa = (\kappa_0, (\kappa_\sigma)_{\sigma \in \Sigma})$.

Definition 9.15. We say a pair (κ, χ) is *critical* if $\kappa_\sigma \in C_3(\chi_\sigma)$ for all $\sigma \in \Sigma$, where $C_3(\chi_\sigma) \subset \mathbb{Z}^{a_\sigma} \times \mathbb{Z}^{b_\sigma}$ is defined as

$$\left\{ \begin{array}{l|l} (-\nu_\sigma - r_{a_\sigma}, \dots, -\nu_\sigma - r_1; & r_1 \geq \dots \geq r_{a_\sigma} \geq 0; \\ -k - \nu_\sigma + s_1, \dots, -k - \nu_\sigma + s_{b_\sigma}) & s_1 \geq \dots \geq s_{b_\sigma} \geq 0 \end{array} \right\}$$

Given a critical pair (κ, χ) , we set

$$(152) \quad \rho_\sigma := (-r_{a_\sigma}, \dots, -r_1; s_1, \dots, s_{b_\sigma}) \quad \text{and} \quad \rho_\sigma^v := (r_1, \dots, r_{a_\sigma}; s_1, \dots, s_{b_\sigma})$$

and write $\rho = (\rho_\sigma)_{\sigma \in \Sigma}$, $\rho^v = (\rho_\sigma^v)_{\sigma \in \Sigma}$. Obviously, given a fixed unitary Hecke character χ of type A_0 , the tuples κ_σ and ρ_σ determine one another, however we do not make this relation explicit in our notation. Informally, we think of ρ_σ as the “shift” from $\Lambda(\chi_\sigma)$ to κ_σ (but note the change of signs).

Remark 9.16. See [EHLS20, Remark 4.4.6] and the discussion that precedes it for more information about the relevance of the set $C_3(\chi_\sigma)$.

Fix any $\kappa_\sigma \in C_3(\chi_\sigma)$. Following the discussion in Section 7.3, note that $\mathbb{D}(\kappa_\sigma) \otimes \mathbb{D}(\kappa_\sigma^b \otimes \chi_\sigma)$ is a holomorphic discrete series for $U_{3,\sigma} := U(a_\sigma, b_\sigma) \times U(b_\sigma, a_\sigma)$, see Remark 7.12, for all $\kappa_\sigma \in C_3(\chi_\sigma)$. Furthermore, for k sufficiently large, the restriction of $\mathbb{D}^2(\chi_\sigma)$ to $U_{3,\sigma}$ is isomorphic to

$$\bigoplus_{\kappa_\sigma \in C_3(\chi_\sigma)} \mathbb{D}(\kappa_\sigma) \otimes \mathbb{D}(\kappa_\sigma^b \otimes \chi_\sigma).$$

Observe that $\mathbb{D}(\kappa_\sigma) \otimes \mathbb{D}(\kappa_\sigma^b \otimes \chi_\sigma)$ is defined over the field of definition $E(\kappa_\sigma, \chi_\sigma)$ of $\kappa_\sigma \boxtimes (\kappa_\sigma^b \otimes \chi_\sigma)$. Let $v_{\kappa_\sigma} \otimes v_{\kappa_\sigma^b \otimes \chi_\sigma}$ be a highest weight vector in the minimal $U_{3,\sigma}$ -type, rational over $E(\kappa_\sigma, \chi_\sigma)$. Note that v_{κ_σ} and $v_{\kappa_\sigma^b \otimes \chi_\sigma}$ are dual to the choice of anti-holomorphic test vectors from Section 7.3.3.

Similarly, let v_{χ_σ} be the tautological generator of the $\Lambda(\chi_\sigma)$ -isotypic subspace of $\mathbb{D}^2(\chi_\sigma)$. We fix a map $\iota_{\chi_\sigma} : \mathbb{D}^2(\chi_\sigma) \rightarrow C^\infty(G_\sigma)$ mapping v_σ to $f_\sigma(\bullet; \mathfrak{J}_\sigma, \chi_\sigma, \frac{k-n}{2})$.

For each κ_σ , there is a natural projection

$$\text{pr}_{\kappa_\sigma} : \mathbb{D}^2(\chi_\sigma) \rightarrow \mathbb{D}(\kappa_\sigma) \otimes \mathbb{D}(\kappa_\sigma^b \otimes \chi_\sigma)$$

and its composition with the orthogonal projection onto the highest weight component also yields

$$(153) \quad \text{pr}_{\kappa_\sigma}^{\text{hol}} : \mathbb{D}^2(\chi_\sigma) \rightarrow \text{span}(v_{\kappa_\sigma} \otimes v_{\kappa_\sigma^b \otimes \chi_\sigma}).$$

Then, as explained in [EHLS20, Section 4.4.7], there exists a differential operator $D(\rho_\sigma^v) \in U(\mathfrak{p}_{4,\sigma}^+)$ such that :

$$\text{pr}_{\kappa_\sigma}^{\text{hol}}(D(\rho_\sigma^v)v_{\chi_\sigma}) = P_{\kappa_\sigma, \chi_\sigma} \cdot v_{\kappa_\sigma} \otimes v_{\kappa_\sigma^b \otimes \chi_\sigma},$$

for some $P_{\kappa_\sigma, \chi_\sigma} \in E(\kappa_\sigma, \chi_\sigma)^\times$. Furthermore, let

$$D(\rho^v) = \prod_{\sigma} D(\rho_\sigma^v) \quad ; \quad D^{\text{hol}}(\rho^v) = \prod_{\sigma} \text{pr}_{\kappa_\sigma}^{\text{hol}} D(\rho_\sigma^v) \quad ; \quad D^{\text{hol}}(\rho^v) = \prod_{\sigma} D^{\text{hol}}(\rho_\sigma^v).$$

Then, for any other dominant weight $\kappa^\dagger \leq \kappa$ of T_{H_0} (in particular, (κ^\dagger, χ) is again critical), there exists a differential operator $\delta(\kappa, \kappa^\dagger) \in U(\mathfrak{p}_3^+)$ such that

$$(154) \quad D(\kappa, \chi) = \sum_{\kappa^\dagger \leq \kappa} \delta(\kappa, \kappa^\dagger) \circ D^{\text{hol}}(\kappa^\dagger, \chi),$$

and $\delta(\kappa, \kappa) = \prod_{\sigma} P_{\kappa_{\sigma}, \chi_{\sigma}}$. See [EHLS20, Corollary 4.4.9] for further details.

Remark 9.17. We sometimes denote $D(\rho_{\sigma}^v)$ and $D(\rho)$ by $D(\chi_{\sigma}, \kappa_{\sigma})$ and $D(\chi, \kappa)$ respectively. We have similar conventions for the holomorphic differential operators.

Lastly, if $\kappa_{\sigma} \in C_3(\chi_{\sigma})$, let

$$f_{\sigma, \kappa_{\sigma}}(g_{\sigma}; \mathfrak{J}_{\sigma}, \chi_{\sigma}, s) := D(\chi_{\sigma}, \kappa_{\sigma}) f_{\sigma}(g_{\sigma}; \mathfrak{J}_{\sigma}, \chi_{\sigma}, s)$$

where $s = \frac{k-n}{2}$ as previously set. For each critical pair (κ, χ) , our choice of Siegel-Weil section at ∞ is

$$(155) \quad f_{\infty, \kappa}(g) = f_{\infty, \kappa}(g; \mathfrak{J}, \chi_{\infty}, s) := \prod_{\sigma \in \Sigma} f_{\sigma, \kappa_{\sigma}}(g; \mathfrak{J}_{\sigma}, \chi_{\sigma}, s), \quad g = (g_{\sigma})_{\sigma},$$

which lies in $I_{\infty}(\chi_{\infty}, \frac{k-n}{2})$ by [EHLS20, Lemma 4.5.2].

9.4. Local Siegel-Weil section away from p and ∞ . Our choice of Siegel-Weil section away from p and ∞ is

$$f^{p, \infty}(\bullet) := \bigotimes_{l \neq p, \infty} f_l \in \bigotimes_{l \neq p, \infty} I_l(\chi_l, s),$$

where $f_l \in I_l(\chi_l, s)$ is defined as in the following two section, for each finite prime $l \neq p$ of \mathbb{Q} .

9.4.1. Local Siegel-Weil section at finite unramified places. Fix a prime $l \notin S \cup \{p\}$. Then, $G_4(\mathbb{Z}_l)$ is a hyperspecial maximal compact subgroup of $G_4(\mathbb{Q}_l)$. Furthermore, $G_4(\mathbb{Q}_l)$ factors as $\prod_{v|l} G_{4,v}$ and so does $I_l(\chi_l, s) = \bigotimes_{v|l} I_v(\chi_v, s)$. Our choice of local Siegel-Weil section at l is the unique $G_4(\mathbb{Z}_l)$ -invariant function

$$(156) \quad f_l = \bigotimes_{v|l} f_v$$

such that $f_l(G_4(\mathbb{Z}_l)) = 1$.

9.4.2. Local Siegel-Weil section at finite ramified places. For a prime $l \in S$, we choose the same section f_l as in [EHLS20, Section 4.2.2] and [Eis15, Section 2.2.9] (with some minor adjustments).

Consider $P_{U, \text{Sgl}} = P_{\text{Sgl}} \cap U$, where $U = U_4$ is the unitary subgroup of $G = G_4$. Then, $P_{\text{Sgl}} = \mathbb{G}_m \times P_{U, \text{Sgl}}$, where \mathbb{G}_m is the similitude factor.

We have $U_4(\mathbb{Q}_l) = \prod_{v|l} U_{4,v}$, where the products run over all primes v of \mathcal{K}^+ above l . There are similar decompositions $U_i(\mathbb{Q}_l) = \prod_{v|l} U_{i,v}$ for $i = 1, 2$ and 3 , with the obvious inclusions (see Section 4 for more details). Similarly, we have

$P_{U,\text{Sgl}}(\mathbb{Q}_l) = \prod_{v|l} P_{\text{Sgl},v}$. Fix any place $v \mid l$ of \mathcal{K}^+ and write $P_v := P_{\text{Sgl},v}$. Let N_v be the unipotent radical of P_v .

As explained in [Ehls20, Section 4.2.2], we know $A = P_v \cdot (U_{1,v} \times 1_n) \cap P_v w P_v$ is open in $U_{4,v}$. Furthermore,

$$P_v w = P_v \cdot (-1_n, 1_n) \subset P_v \cdot U_{3,v} \quad \text{and} \quad P_v \cap (U_{1,v} \times 1_n) = (1_n, 1_n) \in U_{3,v},$$

hence A is an open neighborhood of w in $P_v w P_v$ and can be written as $A = P_v w \mathfrak{U}$ for some open subgroup \mathfrak{U} of N_v .

Let $\varphi_v \in \underline{\pi}_v$ be any nonzero vector (i.e. a local test vector at v as in Section 7.1). Let $K_v \subset U_{1,v}$ be an open compact subgroup that fixes φ_v and fix any lattice L_v sufficiently small so that

$$N(L_v) := \left\{ \begin{pmatrix} 1_n & L_v \\ 0 & 1_n \end{pmatrix} \right\} \subset \mathfrak{U},$$

and

$$P_v w N(L_v) \subset P_v \cdot ((-1 \cdot K_v) \times 1_n) \subset P_v \cdot U_{3,v}.$$

For such L_v , we have $P_v w N(L_v) = P_v \cdot (\mathcal{U}_v \times 1_n) \subset -1 \cdot K_v$, for some open neighborhood \mathcal{U}_v of -1_n . Since $-1 \cdot K_v$ is open in $U_{1,v}$, so is \mathcal{U}_v .

Let $I_{U_{4,v}}(\chi_v, s)$ be the principal series defined as in (134) (and its local version) for P_v and $U_{4,v}$. Then, there exists some $f_{L_v} \in I_v(\chi_v, s)$ supported on $P_v w P_v$ such that

$$f_{L_v}(wx) = \delta_{L_v}(x),$$

for all $x \in N_v$, where δ_{L_v} is the characteristic function of $N(L_v)$, see [HLS06, p. 449-450].

Let

$$(157) \quad f_v = f_{L_v}^-,$$

where $f_{L_v}^-(g) = f_{L_v}(g \cdot (-1_n, 1_n))$ for all $g \in U_{4,v}$, and define $f_U = \otimes f_v$ on $U(\mathbb{Q}_l) = \prod_{v|l} U_v$.

Our choice of local Siegel-Weil section at l is any $f_l \in I_l(\chi_l, s)$ whose restriction from $G_4(\mathbb{Q}_l)$ to $U_4(\mathbb{Q}_l)$ equals f_U . This section depends on many choices which we do not make explicit in our notation.

Remark 9.18. One easily checks that any element of $\bigotimes_{v|l} I_v(\chi_v, s)$ can be extended to a function in $I_l(\chi_l, s)$. Furthermore, the choice of extension f_l of f_U is irrelevant for our purpose as all of our later constructions (Fourier coefficients and local zeta integrals) only depends on the restriction of f_l to $U_4(\mathbb{Q}_l)$. In fact, the Siegel-Weil section at l in [Ehls20, Section 4.2.2.] is only described on $U_4(\mathbb{Q}_l)$ as its full description on $G_4(\mathbb{Q}_l)$ is unnecessary.

9.4.3. *Comparison to other choices in the literature.* For each finite place v of \mathcal{K}^+ away from p , the local section f_v at v is constructed as in [EHLS20, Section 4.2]. However, they differ slightly from [Shi97, Section 18], [HLS06, Section (3.3.1)-(3.3.2)], [Eis15, Section 2.2.9], see [EHLS20, Section 4.2.2].

Namely, given an ideal \mathfrak{b} of $\mathcal{O}_{\mathcal{K}^+}$, let $f_v^{\mathfrak{b}} = f_v^{\mathfrak{b}}(\bullet; \chi_v, s)$ be the local Siegel-Weil section in [EHLS20, Section 2.2.9]. Then, one can choose \mathfrak{b} prime to p (depending on the lattice L_v for each ramified v) such that

$$f_v^{\mathfrak{b}} = \begin{cases} f_v, & v \text{ is unramified,} \\ f_{L_v}, & v \text{ is ramified.} \end{cases}$$

Since $f_v^{\mathfrak{b}}$ is at most a translation of f_v , the Fourier coefficients associated to each of them are equal, see Section 11.2.3. For more details, see [Eis15, Remark 12].

9.5. **Siegel Eisenstein series as C^∞ -modular forms.** Let π be a P -anti-ordinary automorphic representation of P -anti-WLT (κ, K_r, τ) . Furthermore, fix a unitary Hecke character χ as in Section 9.3.1 that satisfies Hypothesis 9.12 for some integer $k \geq 0$.

To K_r , τ and χ , we associate the Siegel-Weil section

$$(158) \quad f_\chi^\tau = f_\chi^\tau(\bullet; s) := f_p(\bullet; \tau; \chi_p, s) \otimes f_\infty(\bullet; i1_n, \chi_\infty, s) \otimes f^{p, \infty}(\bullet),$$

using notation from (148), (151), (156) and (157).

Similarly, if (κ, χ) is critical, we also define

$$(159) \quad f_\chi^{\tau, \kappa} = f_\chi^{\tau, \kappa}(\bullet; s) := f^{p, \infty}(\bullet) \otimes f_p(\bullet; \tau; \chi_p, s) \otimes f_{\infty, \kappa}(\bullet; i1_n, \chi_\infty, s),$$

using (155).

Let ψ be any finite order character of $L_P(\mathbb{Z}_p)$. When considering $\tau \otimes \psi$ instead of τ (thinking of τ as fixed and ψ as varying), we set

$$(160) \quad f_{\chi, \psi}^\tau := f_\chi^{\tau \otimes \psi} \quad \text{and} \quad f_{\chi, \psi}^{\tau, \kappa} := f_\chi^{\tau \otimes \psi, \kappa}.$$

Set $E_{f_{\chi, \psi}^\tau} = E_{f_{\chi, \psi}^\tau}(\bullet; \frac{k-n}{2})$ for the Eisenstein series associated to $f_{\chi, \psi}^\tau(\bullet; s)$ at $s = \frac{k-n}{2}$, for k as in Hypothesis 9.12.

Let $L(\chi)$ be the 1-dimensional vector space on which U_∞ acts via $\Lambda(\chi)$, and $\mathcal{L}(\chi)$ denote the automorphic line bundle on $\text{Sh}(G_4)$ determined by the dual of $\Lambda(\chi)$. Its fiber at the fixed point h_4 of U_∞ is isomorphic to $L(\chi)$. Let $\mathcal{L}(\chi)^{\text{can}}$ denote its canonical extension to the toroidal compactification $\text{Sh}(G_4)^{\text{tor}}$.

Remark 9.19. Although the notation is different, the automorphic line bundle $\mathcal{L}(\chi)$ is an automorphic bundle associated to a highest weight representation as in Section 2.4.1 (replacing G_1 with G_4).

Then, $E_{f_{\chi, \psi}^\tau}$ corresponds to an Eisenstein modular form

$$(161) \quad E_{\chi, \psi}^\tau \in H^0(\text{Sh}(G_4)^{\text{tor}}, \mathcal{L}(\chi)^{\text{can}})$$

via complex uniformization and pullback to functions on $G_4(\mathbb{A})$. It descends to a modular form of level $K_4 = K_{4,r} = I_{4,r}K_4^p \subset G_4(\mathbb{A}_f)$, for r as in (139) and where K_4^p contains the maximal subgroup $G_4(\mathbb{Z}_l)$ for each $l \notin S \cup \{p\}$ and depends on $K_v = K_{1,v}$ for each $v \mid l \in S$ (see Section 9.4.1).

Most importantly, if $K_r = K_{1,r}$ is the level of the anti-holomorphic P -anti-ordinary representation π on G_1 fixed in Section 9.1.4, and $K_{2,r} = K_{1,r}^b$, then $K_{4,r} \cap G_3(\mathbb{A}_f) \supset (K_{1,r} \times K_{2,r}) \cap G_3(\mathbb{A}_f)$.

The differential operators $D(\kappa, \chi)$ used to define $f_{\infty, \kappa}$ can be interpreted at the level of modular forms as well. Set

$$d = \sum_{\sigma \in \Sigma} r_{1,\sigma} + s_{1,\sigma} = \sum_{\sigma \in \Sigma} k - \kappa_{\sigma, a_\sigma} + \kappa_{1,\sigma}^c,$$

where $r_1 = r_{1,\sigma}$ and $s_1 = s_{1,\sigma}$ are the integers appearing in the definition of $\rho^v = (\rho_\sigma^v)_\sigma$ in (152). Consider the C^∞ -differential operator

$$\delta^d(\kappa, \chi) = \delta_\chi^d(\rho^v) : H^0(\mathrm{Sh}(V_4)^{\mathrm{tor}}, \mathcal{L}(\chi)^{\mathrm{can}}) \rightarrow \mathcal{A}(G, \mathcal{L}(\chi)_h)$$

defined [Ehls20, Equation (105)].

Furthermore, let $K_1 = K_{1,r}$, $K_2 = K_{2,r}$ and $K_4 = K_{4,r}$ be the level subgroups above, and set $K_3 = K_{3,r} := (K_{1,r} \times K_{2,r}) \cap G_3(\mathbb{A}_f)$. If we compose the above with restriction of functions from $G_4(\mathbb{A})$ to $G_3(\mathbb{A})$, we obtain

$$\mathrm{Res}_3 \circ \delta^d(\kappa, \chi) : H^0_{(K_4)}(\mathrm{Sh}(V_4)^{\mathrm{tor}}, \mathcal{L}(\chi)^{\mathrm{can}}) \rightarrow H^0_{(K_3)}(\mathrm{Sh}(3_4)^{\mathrm{tor}}, i_3^* \mathcal{L}(\chi)^{\mathrm{can}}),$$

where i_3 is as in (69). Then, [Ehls20, Proposition 4.4.11] show that the above is the pullback to functions on $G_4(\mathbb{A})$ of a differential operator

$$D(\kappa, \chi) : H^0_{(K_4)}(\mathrm{Sh}(V_4)^{\mathrm{tor}}, \mathcal{L}(\chi)^{\mathrm{can}}) \rightarrow M_{\kappa, V}^\infty(K_1, \mathbb{C}) \otimes M_{\kappa^b, -V}^\infty(K_2, \mathbb{C}) \otimes (\chi \circ \det)$$

where the superscript ∞ stands for smooth modular forms (as opposed to holomorphic ones). The above respect cuspidal forms, namely we also have

$$D(\kappa, \chi) : H^0_{(K_4)}(\mathrm{Sh}(V_4)^{\mathrm{tor}}, \mathcal{L}(\chi)^{\mathrm{can}}) \rightarrow S_{\kappa, V}^\infty(K_1, \mathbb{C}) \otimes S_{\kappa^b, -V}^\infty(K_2, \mathbb{C}) \otimes (\chi \circ \det).$$

Lastly, one can compose with the holomorphic projections from (153) to obtain

$$D^{\mathrm{hol}}(\kappa, \chi) : H^0_{(K_4)}(\mathrm{Sh}(V_4)^{\mathrm{tor}}, \mathcal{L}(\chi)^{\mathrm{can}}) \rightarrow M_{\kappa, V}(K_1, \mathbb{C}) \otimes M_{\kappa^b, -V}(K_2, \mathbb{C}) \otimes (\chi \circ \det)$$

and

$$D^{\mathrm{hol}}(\kappa, \chi) : H^0_{(K_4)}(\mathrm{Sh}(V_4)^{\mathrm{tor}}, \mathcal{L}(\chi)^{\mathrm{can}}) \rightarrow S_{\kappa, V}(K_1, \mathbb{C}) \otimes S_{\kappa^b, -V}(K_2, \mathbb{C}) \otimes (\chi \circ \det)$$

Therefore,

$$E_{\chi, \psi}^{\tau, \kappa} := D(\kappa, \chi) E_{\chi, \psi}^\tau$$

is (the restriction to G_3) of the Eisenstein C^∞ -modular form associated to $f_{\chi, \psi}^{\tau, \kappa}$.

9.5.1. *Family of Siegel Eisenstein series.* Fix two dominant weights κ and κ' such that $[\kappa] = [\kappa']$, say $\kappa' = \kappa + \theta$ for some P -parallel weight θ . Assume that both (κ, χ) and (κ', χ) are critical. In later section, it is convenient to think of κ as fixed and vary θ , see Remark 8.22, hence we sometimes write $D(\kappa, \theta, \chi)$ (resp. $D^{\text{hol}}(\kappa, \theta, \chi)$) instead of $D(\kappa', \chi)$ (resp. $D^{\text{hol}}(\kappa', \chi)$). Similarly, we set

$$E_{\chi, \psi, \theta}^{\tau, \kappa} := E_{\chi, \psi}^{\tau, \kappa'} = D(\kappa', \chi) E_{\chi, \psi}^{\tau} = D(\kappa, \theta, \chi) E_{\chi, \psi}^{\tau},$$

and

$$E_{\chi, \psi, \theta}^{\tau, \kappa, \text{hol}} = E_{\chi, \psi}^{\tau, \kappa', \text{hol}} := D^{\text{hol}}(\kappa', \chi) E_{\chi, \psi}^{\tau} = D^{\text{hol}}(\kappa, \theta, \chi) E_{\chi, \psi}^{\tau}.$$

10. ZETA INTEGRALS AND THE DOUBLING METHOD.

In this section, we compute the zeta integral Z_l introduced in Section 9.1.4, for each place l of \mathbb{Q} . The most technical case is objectively for $l = p$. Our method adapts the calculations of [EHLS20, Section 4.3.6], where we resolve various issues arising when working with a P -nebentypus (i.e. finite-dimensional representations) instead of an ordinary nebentypus (i.e. a character). The calculations of the other integrals are more common in the literature, and we recall the necessary results for our purposes.

10.1. Local zeta integrals at p .

10.1.1. *Construction of $f_{w,s}^+$.* Let f_p be the corresponding local Siegel-Weil section at p , as in Equation (138). Ahead of our computations in the next section, we write down an explicit expression for $f_p(u, 1)$ for any $u \in U_1(\mathbb{Q}_p)$.

Firstly, the isomorphism (5), restricted to U_1 , yields an identification $U_1(\mathbb{Q}_p) = \prod_{w \in \Sigma_p} U_{1,w}$, where $U_{1,w} = \text{GL}_{\mathcal{K}_w}(V_w) = \text{GL}_n(\mathcal{K}_w)$. We write $u = (u_w)_{w \in \Sigma_p}$ accordingly and wish to evaluate (147) at $g = (u, 1)$, where this notation is with respect to the embedding $G_1 \times G_2 \hookrightarrow G_3$. To simplify the expression, it is therefore more convenient to replace the decomposition $W_w = V_{w,d} \oplus V_w^d$ with $W_w = V_w \oplus V_w$. In that case, an element $X \in \text{GL}_n(\mathcal{K}_w) = \text{GL}_{\mathcal{K}_w}(V_w)$ corresponds to an element (X, X) in \mathbf{X} instead of $(0, X)$.

Secondly, using the decomposition $W_w = V_w \oplus V_w$ again and the corresponding identification $W_w = V_w \oplus V_w$, consider the element

$$S_w = \begin{pmatrix} 1_{a_w} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{b_w} \\ 0 & 0 & 1_{a_w} & 0 \\ 0 & 1_{b_w} & 0 & 0 \end{pmatrix}.$$

Remark 10.1. As explained in [HLS06, Section 2.1.11] and [EHLS20, Remark 3.1.4], the natural inclusion of Shimura varieties associated to G_3 and G_4 does not induce the natural inclusion on Igusa tower. In fact, one needs to twist the former by the matrix S_w to obtain the latter, see Section 5.1.2 (where we wrote γ_w for S_w).

Lastly, replace each $f_{w,s}$ by its translation $f_{w,s}^+$ via $g \mapsto gS_w$, and let f_p^+ be the corresponding local Siegel-Weil section at p defined by Equation (138). In that case, for $g = (u, 1)$, we obtain that $f_p^+(u, 1)$ is equal to a product over $w \in \Sigma_p$ of

$$\chi_{2,w}(\det u_w) |\det u_w|_w^{\frac{n}{2}+s} \int_{\mathrm{GL}_n(\mathcal{K}_w)} \Phi_w((Xu_w, X)S_w) \chi_{w,1}^{-1} \chi_{w,2}(\det X) |\det X|_w^{n+2s} d^\times X$$

and we denote the above expression by $f_{w,s}^+(u_w, 1) = f_w^+(u_w, 1)$, as a function of $u_w \in \mathrm{GL}_n(\mathcal{K}_w)$.

In the following two subsections, we prove the following formula :
(162)

$$I_p(\varphi_p, \varphi_p^\vee, f_p^+, \chi_p, s) = E_p \left(s + \frac{1}{2}, P\text{-ord}, \pi_p, \chi_p \right) \cdot (\dim \tau_p) \cdot \frac{\mathrm{Vol}(I_{P,r}^0) \mathrm{Vol}({}^t I_{P,r}^0)}{\mathrm{Vol}(I_{P,r}^0 \cap {}^t I_{P,r}^0)}$$

10.1.2. *Local integrals at places above p .* We proceed with the same notation as in Sections 7.2 and 9.2. In particular, for each $w \in \Sigma_p$, consider the local test vectors $\phi_w \in \pi_w$ and $\phi_w^\vee \in \tilde{\pi}_w$ at w defined in Section 7.2.5. We now compute the p -adic local zeta integral Z_p defined in (137) for the local section f_p^+ constructed above and the test vectors

$$\varphi_p = 1 \otimes \left(\bigotimes_{w \in \Sigma_p} \phi_w \right) \quad ; \quad \varphi_p^\vee = 1 \otimes \left(\bigotimes_{w \in \Sigma_p} \phi_w^\vee \right)$$

as in (114).

Then by definition, for

$$\begin{aligned} Z_w &:= \int_{\mathrm{GL}_n(\mathcal{K}_w)} f_{w,s}^+(g, 1) \langle \pi_w(g) \phi_w, \tilde{\phi}_w \rangle_{\pi_w} d^\times g \\ &= \int_{\mathrm{GL}_n(\mathcal{K}_w)} \chi_{2,w}(\det g) |\det g|_w^{\frac{n}{2}+s} \int_{\mathrm{GL}_n(\mathcal{K}_w)} \Phi_w((Xg, X)S_w) \\ &\quad \times \chi_{w,1}^{-1} \chi_{w,2}(\det X) |\det X|_w^{n+2s} \langle \pi_w(g) \phi_w, \tilde{\phi}_w \rangle_{\pi_w} d^\times X d^\times g, \end{aligned}$$

we have $Z_p = \prod_{w \in \Sigma_p} Z_w$.

According to the decomposition $M_{n \times n}(\mathcal{K}_w) = M_{n \times a_w}(\mathcal{K}_w) \times M_{n \times b_w}$, write $Z_1 := Xg = [Z'_1, Z''_1]$ and $Z_2 := X = [Z'_2, Z''_2]$, where Z'_1 and Z'_2 (resp. Z''_1 and Z''_2) are $n \times a$ -matrices (resp. $n \times b$ -matrices). Then,

$$(Xg, X)S_w = ([Z'_1, Z''_1], [Z'_2, Z''_2])$$

and

$$\langle \pi_w(g) \phi_w, \tilde{\phi}_w \rangle_{\pi_w} = \langle \pi_w(Xg) \phi_w, \tilde{\pi}_w(X) \tilde{\phi}_w \rangle_{\pi_w} = \langle \pi_w(Z_1) \phi_w, \tilde{\pi}_w(Z_2) \tilde{\phi}_w \rangle_{\pi_w}.$$

Therefore, using (146), we obtain

$$Z_w = \frac{\dim \tau_w}{\text{Vol}(\mathfrak{G}_w)} \int_{\text{GL}_n(\mathcal{K}_w)} \chi_{w,2}(Z_1) \chi_{w,1}(Z_2)^{-1} |\det Z_1 Z_2|_w^{s+\frac{n}{2}} \\ \times \Phi_{1,w}(Z'_1, Z''_1) \Phi_{2,w}(Z'_2, Z''_2) \langle \pi_w(Z_1) \phi_w, \tilde{\pi}_w(Z_2) \tilde{\phi}_w \rangle_{\pi_w} d^\times Z_1 d^\times Z_2 .$$

We take the integrals over the following open subsets of full measure. We take the integral in Z_1 over

$$\left\{ \begin{pmatrix} 1 & 0 \\ C_1 & 1 \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & D_1 \end{pmatrix} \begin{pmatrix} 1 & B_1 \\ 0 & 1 \end{pmatrix} \mid B_1, {}^t C_1 \in M_{a_w \times b_w}(\mathcal{K}_w), A_1 \in \text{GL}_{a_w}(\mathcal{K}_w), D_1 \in \text{GL}_{b_w}(\mathcal{K}_w) \right\},$$

with the measure

$$\left| \det A_1^{b_w} \det D_1^{-a_w} \right|_w dC_1 d^\times A_1 d^\times D_1 dB_1 .$$

Similarly, we take the integral in Z_2 over

$$\left\{ \begin{pmatrix} 1 & B_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_2 & 0 \\ 0 & D_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C_2 & 1 \end{pmatrix} \mid B_2, {}^t C_2 \in M_{a_w \times b_w}(\mathcal{K}_w), A_2 \in \text{GL}_{a_w}(\mathcal{K}_w), D_2 \in \text{GL}_{b_w}(\mathcal{K}_w) \right\},$$

with the measure

$$\left| \det A_2^{b_w} \det D_2^{-a_w} \right|_w dC_2 d^\times A_2 d^\times D_2 dB_2 .$$

Therefore, one has

$$\Phi_{1,w}(Z'_1, Z''_1) = \Phi_{1,w} \left(\begin{pmatrix} A_1 & B_2 D_2 \\ C_1 A_1 & D_2 \end{pmatrix} \right) \\ \Phi_{2,w}(Z'_2, Z''_2) = \Phi_{2,w} \left(\begin{pmatrix} A_2 + B_2 D_2 C_2 & A_1 B_1 \\ D_2 C_2 & C_1 A_1 B_1 + D_1 \end{pmatrix} \right)$$

and both can be simplified by considering their support.

Lemma 10.2. *The product*

$$\Phi_{1,w} \left(\begin{pmatrix} A_1 & B_2 D_2 \\ C_1 A_1 & D_2 \end{pmatrix} \right) \Phi_{2,w} \left(\begin{pmatrix} A_2 + B_2 D_2 C_2 & A_1 B_1 \\ D_2 C_2 & C_1 A_1 B_1 + D_1 \end{pmatrix} \right)$$

is zero unless all of the following conditions are met:

$$A_1 \in I_{a_w, r}^0 \quad ; \quad D_2 \in I_{b_w, r}^0 \quad ; \quad C_1 \in \mathfrak{G}_l \quad ; \quad B_2 \in \mathfrak{G}_u \\ B_1 \in M_{a_w \times b_w}(\mathcal{O}_w) \quad ; \quad C_2 \in M_{b_w \times a_w}(\mathcal{O}_w) \\ A_2 \in \mathfrak{p}_w^{-r} M_{a_w \times a_w}(\mathcal{O}_w) \quad ; \quad D_1 \in \mathfrak{p}_w^{-r} M_{b_w \times b_w}(\mathcal{O}_w)$$

Moreover, in this case, the product is equal to $\mu_{a_w}(A_1) \mu_{b_w}(D_2) \Phi_w^{(1)}(A_2) \Phi_w^{(4)}(D_1)$.

Proof. Using Lemma 9.7 and the definition of $\Phi_{w,1}$, it is clear that the product above is nonzero if and only if the conditions above are satisfied. Moreover, if they are satisfied, one has

$$\Phi_{1,w} \left(\begin{pmatrix} A_1 & B_2 D_2 \\ C_1 A_1 & D_2 \end{pmatrix} \right) = \mu_{a_w}(A_1) \mu_{b_w}(D_2)$$

by definition of μ_w . One also obtains

$$\Phi_{2,w} \left(\begin{pmatrix} A_2 + B_2 D_2 C_2 & A_1 B_1 \\ D_2 C_2 & C_1 A_1 B_1 + D_1 \end{pmatrix} \right) = \Phi_w^{(1)}(A_2) \Phi_w^{(4)}(D_1)$$

as in the proof of Lemma 9.7. \square

Lemma 10.3. *Under the conditions of Lemma 10.2, one has*

$$\langle \pi_w(Z_1) \phi_w, \pi_w^\vee(Z_2) \phi_w^\vee \rangle_{\pi_w} = \langle \pi_w \left(\begin{pmatrix} A_1 & 0 \\ 0 & 1 \end{pmatrix} \right) \phi_w, \pi_w^\vee \left(\begin{pmatrix} A_2 & 0 \\ 0 & D_1^{-1} D_2 \end{pmatrix} \right) \phi_w^\vee \rangle_{\pi_w}$$

Proof. We write

$$Z_1 = \begin{pmatrix} 1 & 0 \\ C_1 & 1 \end{pmatrix} \begin{pmatrix} A_1 & 0 \\ 0 & D_1 \end{pmatrix} \begin{pmatrix} 1 & B_1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad Z_2 = \begin{pmatrix} 1 & B_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_2 & 0 \\ 0 & D_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C_2 & 1 \end{pmatrix}$$

under the conditions of Lemma 10.2. As $\begin{pmatrix} 1 & B_1 \\ 0 & 1 \end{pmatrix} \in I_{w,r}$ and $\begin{pmatrix} 1 & 0 \\ C_2 & 1 \end{pmatrix} \in {}^t I_{w,r}$ fix ϕ_w and ϕ_w^\vee respectively, the pairing

$$\langle \pi_w(Z_1) \phi_w, \pi_w^\vee(Z_2) \phi_w^\vee \rangle_{\pi_w}$$

is equal to

$$\langle \pi_w \left(\begin{pmatrix} 1 & -B_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C_1 & D_1 \end{pmatrix} \right) \pi_w \left(\begin{pmatrix} A_1 & 0 \\ 0 & 1 \end{pmatrix} \right) \phi_w, \pi_w^\vee \left(\begin{pmatrix} A_2 & 0 \\ 0 & D_2 \end{pmatrix} \right) \phi_w^\vee \rangle_{\pi_w}$$

Furthermore, write

$$\begin{pmatrix} 1 & -B_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C_1 & D_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}$$

where

$$\begin{aligned} A &= 1 - B_2 C_1 \in 1 + \mathfrak{p}_w^{2r} M_{a_w}(\mathcal{O}_w), \\ CA &= C_1 \in \mathfrak{p}_w^r M_{b_w \times a_w}(\mathcal{O}_w), \\ AB &= -B_2 D_1 \in M_{a_w \times b_w}(\mathcal{O}_w) \\ D_1 &= D + CAB \in \mathfrak{p}_w^{-r} M_{b_w}(\mathcal{O}_w). \end{aligned}$$

Note that $1 = A^{-1} + B_2 C_1 A^{-1}$, so

$$\begin{aligned} A^{-1} &= 1 - B_2 C \in 1 + \mathfrak{p}_w^{2r} M_{a_w}(\mathcal{O}_w), \\ C &\in \mathfrak{p}_w^r M_{b_w \times a_w}(\mathcal{O}_w), \quad B \in M_{a_w \times b_w}(\mathcal{O}_w), \\ D &= (1 + CB_2) D_1 \in (1 + \mathfrak{p}_w^{2r}) M_{b_w}(\mathcal{O}_w) D_1. \end{aligned}$$

Therefore,

$$\begin{pmatrix} 1 & -B_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C_1 & D_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ C & 1 + CB_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & D_1 \end{pmatrix} \begin{pmatrix} A & AB \\ 0 & 1 \end{pmatrix}$$

Setting

$$\gamma_0 = \begin{pmatrix} A & AB \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \tilde{\gamma}_0 = \begin{pmatrix} 1 & 0 \\ C & 1 + CB_2 \end{pmatrix},$$

one obtains

$$\begin{aligned} & \langle \pi_w(Z_1)\phi_w, \pi_w^\vee(Z_2)\phi_w^\vee \rangle_{\pi_w} \\ &= \langle \pi_w \left(\gamma_0 \begin{pmatrix} A_1 & 0 \\ 0 & 1 \end{pmatrix} \right) \phi_w, \pi_w^\vee \left(\begin{pmatrix} 1 & 0 \\ 0 & D_1^{-1} \end{pmatrix} \tilde{\gamma}_0 \begin{pmatrix} A_2 & 0 \\ 0 & D_2 \end{pmatrix} \right) \phi_w^\vee \rangle_{\pi_w} \end{aligned}$$

The desired result follows since $\gamma_0, {}^t\tilde{\gamma}_0 \in I_{w,r}$. \square

Proposition 10.4. *Under the conditions of Lemma 10.2, we have*

$$\Phi_{w,1}(Z'_1, Z''_1) \Phi_{w,2}(Z'_2, Z''_2) \langle \pi_w(Z_1)\phi_w, \pi_w^\vee(Z_2)\phi_w^\vee \rangle_{\pi_w} = \text{Vol}(I_{a_w, b_w, r}^0) \cdot J_{a_w} \cdot J_{b_w}$$

where

$$\begin{aligned} J_{a_w} &= \mu_{a_w}(A_1) \Phi_w^{(1)}(A_2) |\det A_2|_w^{b_w/2} \langle \pi_{a_w}(A_1)\phi_{a_w}, \pi_{a_w}^\vee(A_2)\phi_{a_w}^\vee \rangle_{\pi_{a_w}}, \\ J_{b_w} &= \mu_{b_w}(D_2) \Phi_w^{(4)}(D_1) |\det D_1|_w^{a_w/2} \langle \pi_{b_w}(D_1)\phi_{b_w}, \pi_{b_w}^\vee(D_2)\phi_{b_w}^\vee \rangle_{\pi_{b_w}} \end{aligned}$$

Proof. Using the conditions on Z_1 and Z_2 , we have

$$\begin{aligned} & \langle \pi_w(Z_1)\phi_w, \pi_w^\vee(Z_2)\phi_w^\vee \rangle_{\pi_w} \\ &= \langle \pi_w \left(\begin{pmatrix} A_1 & 0 \\ 0 & 1 \end{pmatrix} \right) \phi_w, \pi_w^\vee \left(\begin{pmatrix} A_2 & 0 \\ 0 & D_1^{-1} D_2 \end{pmatrix} \right) \phi_w^\vee \rangle_{\pi_w} \\ &= \int_{\text{GL}_n(\mathcal{O}_w)} (\varphi_w \left(k \begin{pmatrix} A_1 & 0 \\ 0 & 1 \end{pmatrix} \right), \varphi_w^\vee \left(k \begin{pmatrix} A_2 & 0 \\ 0 & D_1^{-1} D_2 \end{pmatrix} \right))_w d^\times k, \end{aligned}$$

using Equation (105).

As the support of φ_w is $P_{a_w, b_w} I_{w,r}$ and its intersection with $\text{GL}_n(\mathcal{O}_w)$ is equal to $I_{a_w, b_w, r}^0$, the integrand above is nonzero if and only if $k \in I_{a_w, b_w, r}^0$. Write such a $k \in I_{a_w, b_w, r}^0$ as

$$k = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}$$

with $A \in \text{GL}_{a_w}(\mathcal{O}_w)$, $D \in \text{GL}_{b_w}(\mathcal{O}_w)$, $B \in M_{a_w \times b_w}(\mathcal{O}_w)$ and $C \in \mathfrak{p}_w^r M_{b_w \times a_w}(\mathcal{O}_w)$.

An short computation shows that

$$\begin{aligned} \varphi_w \left(k \begin{pmatrix} A_1 & 0 \\ 0 & 1 \end{pmatrix} \right) &= \varphi_w \left(\begin{pmatrix} AA_1 & 0 \\ 0 & D \end{pmatrix} \right), \quad \text{and} \\ \varphi_w^\vee \left(k \begin{pmatrix} A_2 & 0 \\ 0 & D_1^{-1} D_2 \end{pmatrix} \right) &= \varphi_w^\vee \left(\begin{pmatrix} AA_2 & 0 \\ 0 & DD_1^{-1} D_2 \end{pmatrix} \right). \end{aligned}$$

Observe that the determinant of the matrices A , D , A_1 and D_2 are all integral p -adic units. Therefore, using the definition of φ_w (resp. φ_w^\vee) and its relation to $\phi_{a_w} \otimes \phi_{b_w}$ (resp. $\tilde{\phi}_{a_w} \otimes \tilde{\phi}_{b_w}$), the integrand above is equal to

$$\begin{aligned} & |\det A_2|_w^{b_w/2} |\det D_1^{-1}|_w^{-a_w/2} \\ & \times \langle \pi_{a_w}(AA_1)\phi_{a_w} \otimes \pi_{b_w}(D)\phi_{b_w}, \pi_{a_w}^\vee(AA_2)\phi_{a_w}^\vee \otimes \pi_{b_w}^\vee(DD_1^{-1}D_2)\phi_{b_w}^\vee \rangle_{a_w, b_w} \\ = & |\det A_2|_w^{b_w/2} |\det D_1|_w^{a_w/2} \\ & \times \langle \pi_{a_w}(A_1)\phi_{a_w}, \pi_{a_w}^\vee(A_2)\phi_{a_w}^\vee \rangle_{\pi_{a_w}} \langle \pi_{b_w}(D_1)\phi_{b_w}, \pi_{b_w}^\vee(D_2)\phi_{b_w}^\vee \rangle_{\pi_{b_w}}, \end{aligned}$$

which does not depend on $k \in I_{a_w, b_w, r}^0$. The result follows by using the second part of Lemma 10.2. \square

Corollary 10.5. *The zeta integral Z_w is equal to*

$$\dim \tau_w \cdot \frac{\text{Vol}(I_{a_w, b_w, r}^0)}{\text{Vol}(I_{a_w, r}^0) \text{Vol}(I_{b_w, r}^0)} \cdot \mathcal{I}_{a_w} \cdot \mathcal{I}_{b_w}$$

where

$$\begin{aligned} \mathcal{I}_{a_w} &= \int_{I_{a_w, r}^0} \int_{\text{GL}_{a_w}(\mathcal{K}_w)} \mu_{a_w}(A_1) \chi_{w,2}(A_1) \chi_{w,1}^{-1}(A_2) \\ & \quad \times \Phi_w^{(1)}(A_2) |\det A_2|_w^{s + \frac{a_w}{2}} \langle \pi_{a_w}(A_1)\phi_{a_w}, \pi_{a_w}^\vee(A_2)\phi_{a_w}^\vee \rangle_{\pi_{a_w}} d^\times A_2 d^\times A_1 \\ \mathcal{I}_{b_w} &= \int_{I_{b_w, r}^0} \int_{\text{GL}_{b_w}(\mathcal{K}_w)} \mu_{b_w}(D_2) \chi_{w,2}(D_1) \chi_{w,1}^{-1}(D_2) \\ & \quad \times \Phi_w^{(4)}(D_1) |\det D_1|_w^{s + \frac{b_w}{2}} \langle \pi_{b_w}(D_1)\phi_{b_w}, \pi_{b_w}^\vee(D_2)\phi_{b_w}^\vee \rangle_{\pi_{b_w}} d^\times D_1 d^\times D_2 \end{aligned}$$

Proof. Using Lemma 10.2 and Proposition 10.4,

$$\begin{aligned} Z_w &= \frac{\dim \tau_w}{\text{Vol}(\mathfrak{G}_w)} \\ & \times \int_{A_1, B_1, C_1, A_2, B_2, D_1, C_2, D_2} \chi_{w,2}(A_1) \chi_{w,2}(D_1) \chi_{w,1}^{-1}(A_2) \chi_{w,1}^{-1}(D_2) \\ & \times |\det A_1 \det D_1 \det A_2 \det D_2|_w^{s + \frac{n}{2}} \\ & \times \text{Vol}(I_{a_w, b_w, r}^0) J_{a_w} J_{b_w} \\ & \times \left| \det A_1^{b_w} \det D_1^{-a_w} \right| d^\times A_1 dB_1 dC_1 d^\times D_1 \\ & \times \left| \det A_2^{-b_w} \det D_2^{a_w} \right| d^\times A_2 dB_2 dC_2 d^\times D_2 \end{aligned}$$

where the domain of integration for the matrices A_i , B_i , C_i and D_i ($i = 1, 2$) is given by the conditions of Lemma 10.2.

Note that the integrand is independent of $B_1 \in M_{a_w \times b_w}(\mathcal{O}_w)$, $B_2 \in \mathfrak{G}_{u,w}$, $C_1 \in \mathfrak{G}_{l,w}$ and $C_2 \in M_{b_w \times a_w}(\mathcal{O}_w)$. Moreover, the determinants of the matrices A_1 and

D_2 are both p -adic units. Therefore, the above simplifies to

$$\begin{aligned}
Z_w &= \frac{\dim \tau_w}{\text{Vol}(\mathfrak{G}_w)} \text{Vol}(I_{a_w, b_w, r}^0) \text{Vol}(M_{a_w \times b_w}(\mathcal{O}_w))^2 \text{Vol}(\mathfrak{G}_{l, w}) \text{Vol}(\mathfrak{G}_{u, w}) \\
&\times \int_{I_{a_w, r}^0} \int_{I_{b_w, r}^0} \int_{\text{GL}_{a_w}(\mathcal{K}_w)} \int_{\text{GL}_{b_w}(\mathcal{K}_w)} \chi_{w, 2}(A_1) \chi_{w, 2}(D_1) \chi_{w, 1}^{-1}(A_2) \chi_{w, 1}^{-1}(D_2) \\
&\times \mu_{a_w}(A_1) \Phi_w^{(1)}(A_2) |\det A_2|_w^{s + \frac{a_w}{2}} \langle \pi_{a_w}(A_1) \phi_{a_w}, \pi_{a_w}^\vee(A_2) \phi_{a_w}^\vee \rangle_{\pi_{a_w}} \\
&\times \mu_{b_w}(D_2) \Phi_w^{(4)}(D_1) |\det D_1|_w^{s + \frac{b_w}{2}} \langle \pi_{b_w}(D_1) \phi_{b_w}, \pi_{b_w}^\vee(D_2) \phi_{b_w}^\vee \rangle_{\pi_{b_w}} \\
&\times d^\times D_1 d^\times A_2 d^\times D_2 d^\times A_1,
\end{aligned}$$

and using the decomposition $\mathfrak{G}_w = \mathfrak{G}_{l, w}(I_{a_w, r}^0 \times I_{b_w, r}^0) \mathfrak{G}_{u, w}$, the result follows immediately. \square

Theorem 10.6. *The integrals \mathcal{I}_{a_w} and \mathcal{I}_{b_w} are equal to*

$$\begin{aligned}
\mathcal{I}_{a_w} &= \frac{\epsilon(-s + \frac{1}{2}, \pi_{a_w} \otimes \chi_{w, 1}) L(s + \frac{1}{2}, \pi_{a_w}^\vee \otimes \chi_{w, 1}^{-1})}{L(-s + \frac{1}{2}, \pi_{a_w} \otimes \chi_{w, 1})} \cdot \frac{\text{Vol}(\mathfrak{X}^{(1)}) \text{Vol}(I_{a_w, r}^0)}{(\dim \tau_{a_w})^2} \langle \phi_{a_w}, \phi_{a_w}^\vee \rangle_{\pi_{a_w}} \\
\mathcal{I}_{b_w} &= \frac{L(s + \frac{1}{2}, \pi_{b_w} \otimes \chi_{w, 2})}{\epsilon(s + \frac{1}{2}, \pi_{b_w} \otimes \chi_{w, 2}) L(-s + \frac{1}{2}, \pi_{b_w}^\vee \otimes \chi_{w, 2}^{-1})} \cdot \frac{\text{Vol}(\mathfrak{X}^{(4)}) \text{Vol}(I_{b_w, r}^0)}{(\dim \tau_{b_w})^2} \langle \phi_{b_w}, \phi_{b_w}^\vee \rangle_{\pi_{b_w}}
\end{aligned}$$

Therefore, by setting

$$\begin{aligned}
&L\left(s + \frac{1}{2}, \text{ord}, \pi_w, \chi_w\right) \\
&:= \frac{\epsilon(-s + \frac{1}{2}, \pi_{a_w} \otimes \chi_{w, 1}) L(s + \frac{1}{2}, \pi_{a_w}^\vee \otimes \chi_{w, 1}^{-1}) L(s + \frac{1}{2}, \pi_{b_w} \otimes \chi_{w, 2})}{L(-s + \frac{1}{2}, \pi_{a_w} \otimes \chi_{w, 1}) \epsilon(s + \frac{1}{2}, \pi_{b_w} \otimes \chi_{w, 2}) L(-s + \frac{1}{2}, \tilde{\pi}_{b_w} \otimes \chi_{w, 2}^{-1})}
\end{aligned}$$

one has

$$Z_w = \frac{1}{\dim \tau_w} L\left(s + \frac{1}{2}, \text{ord}, \pi_w, \chi_w\right) \cdot \frac{\text{Vol}(I_{w, r}^0) \text{Vol}({}^t I_{w, r}^0)}{\text{Vol}(I_{w, r}^0 \cap {}^t I_{w, r}^0)} \cdot \langle \varphi_w, \varphi_w^\vee \rangle_{\pi_w}$$

Proof. This proof is inspired by the argument of [EHL20, Theorem 4.3.10]. First, write

$$\mathcal{I}_{a_w} = \int_{I_{a_w, r}^0} \mu_{a_w}(A_1) \chi_{w, 2}(A_1) \mathcal{I}_{a_w, 2}(A_1) d^\times A_1,$$

where $\mathcal{I}_{a_w, 2} = \mathcal{I}_{a_w, 2}(A_1)$ is defined as

$$\int_{\text{GL}_{a_w}(\mathcal{K}_w)} \Phi_w^{(1)}(A_2) |\det A_2|_w^{s + \frac{a_w}{2}} \langle \pi_{a_w}(A_1) \phi_{a_w}, (\chi_{w, 1}^{-1} \otimes \pi_{a_w}^\vee)(A_2) \phi_{a_w}^\vee \rangle_{\pi_{a_w}} d^\times A_2.$$

The above is a ‘‘Godement-Jacquet’’ integral, as defined in [Jac79, Equation (1.1.3)]. Therefore, we use its functional equation to obtain

$$\begin{aligned} \mathcal{I}_{a_w,2} &= \frac{\epsilon(-s + \frac{1}{2}, \pi_{a_w} \otimes \chi_{w,1}) L(s + \frac{1}{2}, \pi_{a_w}^\vee \otimes \chi_{w,1}^{-1})}{L(-s + \frac{1}{2}, \pi_{a_w} \otimes \chi_{w,1})} \\ &\quad \times \int_{\mathrm{GL}_{a_w}(\mathcal{K}_w)} \left(\Phi_w^{(1)} \right)^\wedge (A_2) |\det A_2|_w^{-s + \frac{a_w}{2}} \\ &\quad \times \chi_{w,1}(A_2) \langle \pi_{a_w}(A_1) \phi_{a_w}, \pi_{a_w}^\vee(A_2^{-1}) \phi_{a_w}^\vee \rangle_{\pi_{a_w}} d^\times A_2 \end{aligned}$$

Let $L_{a_w, \mathrm{ord}}$ denote the quotient of L -factors and ϵ -factors leading the expression above. Recall that $\left(\Phi_w^{(1)} \right)^\wedge (A_2)$ is supported on $\mathfrak{X}^{(1)}$. Furthermore, for $A_2 \in \mathfrak{X}^{(1)}$, we have $\left(\Phi_w^{(1)} \right)^\wedge (A_2) = \nu_{a_w}(A_2)$ and $|\det A_2|_w = 1$. Then, $\mathcal{I}_{a_w,2}$ is equal to

$$L_{a_w, \mathrm{ord}} \times \int_{\mathfrak{X}^{(1)}} \chi_{w,1}(A_2) \nu_{a_w}(A_2) \langle \pi_{a_w}(A_1) \phi_{a_w}, \pi_{a_w}^\vee(A_2^{-1}) \phi_{a_w}^\vee \rangle_{\pi_{a_w}} d^\times A_2$$

By definition of $\mathfrak{X}^{(1)}$, we can write $A_2 = \gamma_1 k_2 \gamma_2$ uniquely for some $k_2 \in K_{a_w}$, $\gamma_1 \in \mathfrak{X}_l^{(1)} := {}^t I_{a_w, r}^0 \cap {}^t P_{a_w}^u$, and $\gamma_2 \in \mathfrak{X}_u^{(1)} := I_{a_w, r}^0 \cap P_{a_w}^u$. It follows that

$$\begin{aligned} \langle \pi_{a_w}(A_1) \phi_{a_w}, \pi_{a_w}^\vee(A_2^{-1}) \phi_{a_w}^\vee \rangle_{\pi_{a_w}} &= \langle \pi_{a_w}(k_2 \gamma_2 A_1) \phi_{a_w}, \pi_{a_w}^\vee(\gamma_1^{-1}) \phi_{a_w}^\vee \rangle_{\pi_{a_w}} \\ &= \langle \pi_{a_w}(k_2 A_1) \phi_{a_w}, \phi_{a_w}^\vee \rangle_{\pi_{a_w}} \\ &= \int_{\mathrm{GL}_{a_w}(\mathcal{O}_w)} (\varphi_{a_w}(k k_2 A_1), \varphi_{a_w}^\vee(k))_{a_w} d^\times k \end{aligned}$$

The support of φ_{a_w} is $P_{a_w} I_{a_w, r} = P_{a_w} I_{a_w, r}^0$. Since $k_2 A_1 \in I_{a_w, r}^0$, the integrand vanishes unless $k \in P_{a_w} I_{a_w, r} \cap \mathrm{GL}_{a_w}(\mathcal{O}_w) = I_{a_w, r}^0$. Using the fact that such k is in P_{a_w} as well as Equation (107), we obtain

$$(\varphi_{a_w}(k k_2 A_1), \varphi_{a_w}^\vee(k))_{a_w} = (\varphi_{a_w}(k_2 A_1), \varphi_{a_w}^\vee(1))_{a_w} = (\tau_{a_w}(k_2 A_1) \phi_{a_w}^0, \phi_{a_w}^{\vee, 0})_{a_w}.$$

Then, using the above, Equation (140), the definition of ν_{a_w} , and orthogonality relations of matrix coefficients, we obtain

$$\begin{aligned} \mathcal{I}_{a_w,2} &= L_{a_w, \mathrm{ord}} \times \int_{\mathfrak{X}^{(1)}} \mu'_{a_w}(A_2) \langle \pi_{a_w}(A_1) \phi_{a_w}, \pi_{a_w}^\vee(A_2^{-1}) \phi_{a_w}^\vee \rangle_{\pi_{a_w}} d^\times A_2 \\ &= L_{a_w, \mathrm{ord}} \mathrm{Vol}(I_{a_w, r}^0) \mathrm{Vol}(\mathfrak{X}_l^{(1)}) \mathrm{Vol}(\mathfrak{X}_u^{(1)}) \\ &\quad \times \int_{K_{a_w}} (\phi_{a_w}^0, \tau_{a_w}^\vee(k_2) \phi_{a_w}^{\vee, 0})_{a_w} (\tau_{a_w}(k_2) \tau_{a_w}(A_1) \phi_{a_w}^0, \phi_{a_w}^{\vee, 0})_{a_w} d^\times k_2 \\ &= L_{a_w, \mathrm{ord}} \mathrm{Vol}(I_{a_w, r}^0) \frac{\mathrm{Vol}(\mathfrak{X}^{(1)})}{\dim \tau_{a_w}} (\phi_{a_w}^0, \phi_{a_w}^{\vee, 0})_{a_w} (\tau_{a_w}(A_1) \phi_{a_w}^0, \phi_{a_w}^{\vee, 0})_{a_w} \end{aligned}$$

Using Equation (116), orthogonality relations of matrix coefficients once more, and the normalization $(\phi_{a_w}^0, \phi_{a_w}^{\vee,0})_{a_w} = 1$, we ultimately obtain that \mathcal{I}_{a_w} is equal to

$$\begin{aligned} & L_{a_w, \text{ord}} \langle \phi_{a_w}, \phi_{a_w}^{\vee} \rangle_{\pi_{a_w}} \frac{\text{Vol}(\mathfrak{X}^{(1)})}{\dim \tau_{a_w}} \int_{I_{a_w, r}^0} \mu'_{a_w}(A_1) (\tau_{a_w}(A_1) \phi_{a_w}^0, \phi_{a_w}^{\vee,0})_{a_w} d^\times A_1 \\ &= L_{a_w, \text{ord}} \frac{\text{Vol}(\mathfrak{X}^{(1)}) \text{Vol}(I_{a_w, r}^0)}{(\dim \tau_{a_w})^2} \langle \phi_{a_w}, \phi_{a_w}^{\vee} \rangle_{\pi_{a_w}} \end{aligned}$$

A similar argument yields

$$\mathcal{I}_{b_w} = L_{b_w, \text{ord}} \frac{\text{Vol}(\mathfrak{X}^{(4)}) \text{Vol}(I_{b_w, r}^0)}{(\dim \tau_{b_w})^2} \langle \phi_{b_w}, \phi_{b_w}^{\vee} \rangle_{\pi_{b_w}},$$

where

$$L_{b_w, \text{ord}} = \frac{L(s + \frac{1}{2}, \pi_{b_w} \otimes \chi_{w,2})}{\epsilon(s + \frac{1}{2}, \pi_{b_w} \otimes \chi_{w,2}) L(-s + \frac{1}{2}, \pi_{b_w}^{\vee} \otimes \chi_{w,2}^{-1})}.$$

Therefore, the result follows using Corollary 10.5, Equation (118), and the identity

$$\text{Vol}(\mathfrak{X}^{(1)}) \text{Vol}(\mathfrak{X}^{(4)}) = \frac{\text{Vol}(I_{a_w, r}^0) \text{Vol}({}^t I_{a_w, r}^0) \text{Vol}(I_{b_w, r}^0) \text{Vol}({}^t I_{b_w, r}^0)}{\text{Vol}(I_{a_w, r}^0 \cap {}^t I_{a_w, r}^0) \text{Vol}(I_{b_w, r}^0 \cap {}^t I_{b_w, r}^0)} = \frac{\text{Vol}(I_{w, r}^0) \text{Vol}({}^t I_{w, r}^0)}{\text{Vol}(I_{w, r}^0 \cap {}^t I_{w, r}^0)}$$

□

10.1.3. *Main Local Theorem.* Keeping with the notation of Theorem 10.6, define

$$I_p \left(s + \frac{1}{2}, P\text{-ord}, \pi, \chi \right) := \prod_{w \in \Sigma_p} L \left(s + \frac{1}{2}, P\text{-ord}, \pi_w, \chi_w \right).$$

Then, from Theorem 10.6 and (137), we immediately obtain our main result.

Theorem 10.7. *Let χ be a unitary Hecke character of \mathcal{K} , $\chi_p = \otimes_{w|p} \chi_w$, and let $s \in \mathbb{C}$. Let $f_p = f_p(\bullet; \tau, \chi_p, s) \in I_p(\chi_p, s)$ be the local Siegel-Weil section at p in (148). Let $\varphi_p \in \pi_p$ and $\varphi_p^{\vee} \in \pi_p^{\vee}$ be the test vectors defined in (114).*

Then, the p -adic local zeta integral $I_p(\varphi_p, \varphi_p^{\vee}, f_p; \chi_p, s)$ from (137) is equal to

$$(163) \quad \frac{1}{\dim \tau} I_p \left(s + \frac{1}{2}, P\text{-ord}, \pi, \chi \right) \cdot \frac{\text{Vol}(I_{P, r}^0) \text{Vol}({}^t I_{P, r}^0)}{\text{Vol}(I_{P, r}^0 \cap {}^t I_{P, r}^0)}$$

Remark 10.8. Using the same minor manipulation explained in [EHLS20, Remark 4.3.11], we see that the p -Euler factor $I_p(s + \frac{1}{2}, P\text{-ord}, \pi_p, \chi_p)$ takes the form of a modified Euler factor at p as predicted in [Coa89, Section 2, Equation (18b)] for the conjectures of Coates and Perrin-Riou on p -adic L -functions.

10.2. Local zeta integrals at ∞ . Assume the unitary character χ satisfies Hypothesis 9.12 for some $k \geq 0$. Let $f_{\infty, \kappa} = f_{\infty, \kappa}(\bullet; \mathfrak{J}, \chi_{\infty}, s)$ be the Siegel-Weil section in (155). Let $\varphi_{\infty} \in \pi_{\infty}$ and $\varphi_{\infty}^{\vee} \in \pi_{\infty}^{\vee}$ be test vectors as in (121).

If $k \geq n$ and (κ, χ) is critical, then [EL20, Theorem 1.3.1] yields

$$\begin{aligned} & Z_{\infty}(\varphi_{\infty}, \varphi_{\infty}^{\flat}, f_{\infty, \kappa}; \chi_{\infty}, s) \Big|_{s=\frac{k-n}{2}} \\ &= \frac{A(\pi_{\infty}, \chi_{\infty})}{\left(2^{(n-1)n} (-2\pi i)^{-nk} \pi^{\frac{n(n-1)}{2}} \prod_{j=0}^{n-1} \Gamma(k-j) \right)^{[\mathcal{K}^+:\mathbb{Q}]}} E_{\infty} \left(\frac{k-n+1}{2}, \pi, \chi \right) \end{aligned}$$

where $A(\pi_{\infty}, \chi_{\infty})$ is some algebraic number depending on π_{∞} and χ_{∞} , and E_{∞} is the modified archimedean Euler factor (both introduced in [EL20, Section 1.3]). Let $D_{\infty}(\pi_{\infty}, \chi_{\infty})$ denote the fraction on the right-hand side.

Remark 10.9. The denominator of the leading term on the right-hand side of the equation above also appears in the archimedean Fourier coefficients of the Siegel Eisenstein series associated to $f_{\infty} = f_{\infty}(\bullet; \mathfrak{J}, \chi_{\infty}, \frac{n-k}{2})$ (the holomorphic Siegel-Weil section introduced in Section 9.3.2), see (170).

We later normalize this Siegel-Weil section so that this terms does not appear in either the local archimedean zeta integral nor Fourier coefficient.

We set

$$(164) \quad I_{\infty} \left(\frac{k-n+1}{2}, \pi, \chi \right) := A(\pi_{\infty}, \chi_{\infty}) E_{\infty} \left(\frac{k-n+1}{2}, \pi, \chi \right).$$

10.3. Local zeta integrals away from p and ∞ .

10.3.1. Local zeta integrals at finite unramified places. For each $l \notin S \cup \{p\}$, let φ_l and φ_l^{\vee} be local test vectors at l as in Section 7.1.1. Similarly, let $f_l \in I_l(\chi_l, s)$ be as in Section 9.4.1.

It follows from [Jac79], [GPSR87, Section 6] and [Li92, Section 3] that

$$d^{S,p}(s + \frac{1}{2}, \chi) \prod_{l \notin S \cup \{p\}} I_l(\varphi_l, \varphi_l^{\vee}, f_l, s) = L^{S,p}(s + \frac{1}{2}, \pi, \chi),$$

where $d^{S,p}(s, \chi) = \prod_{l \notin S \cup \{p\}} \prod_{v|l} d_v(s, \chi)$ and

$$(165) \quad d_v(s, \chi) = \prod_{r=1}^n L_v(2s + n - r, \chi^+ \cdot \eta^r),$$

where χ^+ is the restriction of χ to $\mathbb{A}_{\mathcal{K}^+}$, and $\eta = \eta_{\mathcal{K}/\mathcal{K}^+}$ is the quadratic character of $\mathbb{A}_{\mathcal{K}^+}$ associated to the extension $\mathcal{K}/\mathcal{K}^+$. For more details, see [EHLS20, Section 4.2.1].

10.3.2. *Local zeta integrals at finite ramified places.* For each $l \in S$, let $\varphi_l = \bigotimes_{v|l} \varphi_v$ and $\varphi_l^\vee = \bigotimes_{v|l} \varphi_v^\vee$ be local test vectors at l as in Section 7.1.2. Similarly, let $f_l = \bigotimes_{v|l} f_v \in I_l(\chi_l, s)$ be as in Section 9.4.2.

For each place $v \mid l$ of \mathcal{K}^+ , let \mathcal{U}_v be the open neighborhood of -1 in K_v as in Section 9.4.2. It follows from [EHLS20, Lemma 4.2.3] that

$$I_l(\varphi_l, \varphi_l^\vee, f_l, \chi) = \prod_{v|l} \text{Vol}(\mathcal{U}_v),$$

where the volume is respect to the local Haar measure discussed in Section 2.7.1. We normalize the product over all primes in S as

$$(166) \quad I_S = \prod_{l \in S} \prod_{v|l} d_v(s + \frac{1}{2}, \chi) \text{Vol}(\mathcal{U}_v),$$

where $d_v(s, \chi)$ is defined as in (165). Note that I_S is independent of π , see Remark 7.2.

10.4. **Global formula.** Let $D^{p,\infty}(\chi) = \prod_{v \nmid p\infty} d_v(s + \frac{1}{2}, \chi)$, where the product runs over all finite places v of \mathcal{K}^+ away from p , and $d_v(s, \chi)$ is again as in (165).

Theorem 10.10. *Let π be a P -anti-ordinary, anti-holomorphic cuspidal automorphic representation for G_1 of P -anti-WLT (κ, K_r, τ) . Let $S = S(K^p)$ as in Section 3.1.1 and let χ be a unitary Hecke character of type A_0 satisfying Hypothesis 9.12 for some integer $k \geq n$. Assume that (κ, χ) is critical.*

Let $\varphi \in \pi$ and $\varphi^\vee \in \pi^\vee$ be test vectors as in Section 7. Let $f = f_\chi^{\tau, \kappa} \in I(\chi, s)$ be as in (159) for $s = \frac{k-n}{2}$. Then,

$$\begin{aligned} & D^{p,\infty}(\chi) I(\varphi, \varphi^\vee, f; \chi, s) \\ &= \frac{\langle \varphi, \varphi^\vee \rangle}{\dim \tau} \cdot \frac{\text{Vol}(I_{P,r}^0) \text{Vol}({}^t I_{P,r}^0)}{\text{Vol}(I_{P,r}^0 \cap {}^t I_{P,r}^0)} \cdot I_p \left(s + \frac{1}{2}, P\text{-ord}, \pi, \chi \right) \\ & \times D_\infty(\pi_\infty, \chi_\infty) E_\infty \left(s + \frac{1}{2}; \pi, \chi \right) I_S L^S \left(s + \frac{1}{2}; \pi, \chi_u \right) \end{aligned}$$

at $s = \frac{k-n}{2}$.

11. P -ORDINARY EISENSTEIN MEASURE.

We now construct an Eisenstein measure, in the sense of [EHLS20, Section 5], by p -adically interpolating the (holomorphic) Eisenstein series associated to the Siegel-Weil sections chosen in the previous section. To do so, we follow the approach of [Eis15] and in particular use several results of [Shi97]. Therefore, it is convenient to work with a specific choice of basis for the Hermitian vector space associated to G_4 .

Namely, let $(V, \langle \cdot, \cdot \rangle_V)$ and $(W, \langle \cdot, \cdot \rangle_W)$ be the Hermitian vector spaces associated to $G_{1/\mathbb{R}}$ and $G_{4/\mathbb{R}}$ respectively. Let $\mathcal{B}_1 := \{e_1, \dots, e_n\}$ be any orthogonal basis of V and let ϕ be the corresponding diagonal matrix for $\langle \cdot, \cdot \rangle_V$. Let $\mathcal{B}_4 :=$

$\{(e_1, 0), \dots, (e_n, 0), (0, e_1), \dots, (0, e_n)\}$ be the corresponding basis of W . Then, we momentarily identify $G_1(\mathbb{R})$ with the group of matrices (written with respect to \mathcal{B}_1) preserving some form ϕ up to scalar, and $G_4(\mathbb{R})$ with the group of matrices preserving

$$\begin{pmatrix} \phi & 0 \\ 0 & -\phi \end{pmatrix}.$$

Let $\alpha \in K$ be any totally imaginary element, define

$$S = \begin{pmatrix} 1_n & -\frac{\alpha}{2}\phi \\ -1_n & -\frac{\alpha}{2}\phi \end{pmatrix}$$

and consider the basis $\mathcal{B}'_4 := S\mathcal{B}_4$ of W . In this section, and only in this section, we identify $G_{4/\mathbb{R}}$ with the group of matrices preserving the matrix

$$\eta := \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}$$

This way, the unitary group $U_4(\mathbb{R})$ is equal to the group denoted $G(\eta)$ in [Eis15]. The results of Shimura to compute Fourier coefficients of Eisenstein series are stated with respect to this $G(\eta)$, motivating our change in notation. Our choice of local Siegel-Weil sections in Section 9.2, 9.3, and 9.4 make no mention of an explicit global basis for V or W , hence this does not introduce any unintentional technicalities.

Observe that U_4 is the restriction of scalar to \mathbb{Q} of an algebraic group $U := U_{\mathcal{K}^+}$ on \mathcal{K}^+ . In what follows, it is more convenient to work with $U_4(\mathbb{A}_{\mathbb{Q}}) = U(\mathbb{A}_{\mathcal{K}^+})$.

Fix a unitary Hecke character χ that satisfies Hypothesis 9.12 for some integer $k \geq 0$, a P -nebenotypus $\tau = \bigotimes_{w \in \Sigma_p} \tau_w$ of level $r \gg 0$ and a finite-order character $\psi = \bigotimes_{w \in \Sigma_p} \psi_w$ of $L_P(\mathbb{Z}_p)$. Let $f = f_{\chi, \psi}^\tau \in I(\chi, s)$ be the associated Siegel-Weil section in (160). By construction, its restriction f_U to U factors as

$$f_U = \bigotimes_v f_v$$

where the tensor product runs over all the places v of \mathcal{K}^+ .

Let $P_{U, \text{Sgl}} \subset U_4$ be the maximal \mathbb{Q} -parabolic subgroup that stabilizes V^d , i.e. $P_{U, \text{Sgl}} = P_{\text{Sgl}} \cap U$. Its Levi subgroup M_U is identified with $\text{GL}_{\mathcal{K}}(V)$ via Δ or equivalently, with $\text{GL}_n(\mathcal{K})$ using the basis \mathcal{B}_1 . Its unipotent radical N_U is identified with the group of matrices $\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$, where $m \in \text{Her}_n(\mathcal{K})$.

With this notation, we can adapt the content of Section 9.1 by replacing G_4 with U_4 . Let $E_{f, U}$ be the Eisenstein series associated to f_U on U_4 or equivalently, the restriction of E_f from $G_4(\mathbb{A})$ to $U_4(\mathbb{A}) = U(\mathbb{A}_{\mathcal{K}^+})$.

11.1. Fourier coefficients of Eisenstein series. The Siegel-Weil section f_U satisfies Conditions 4 and 5 of [Eis15, Section 2.2.3]. Therefore by [Shi97, Proposition

18.3], $E_{f,U}$ admits a Fourier expansion: For all $m \in \text{Her}_n(\mathbb{A}_{\mathcal{K}})$, $h \in \text{GL}_n(\mathbb{A}_{\mathcal{K}})$, we have

$$(167) \quad E_{f,U} \left(\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} {}^t\bar{h}^{-1} & 0 \\ 0 & h \end{pmatrix} \right) = \sum_{\beta \in \text{Her}_n(\mathcal{K})} c(\beta, h; f_U) e_{\mathbb{A}_{\mathcal{K}^+}}(\text{tr } \beta m),$$

where $c(\beta, h; f_U)$ is a complex number that depends on f_U , the Hermitian matrix β , and h .

Furthermore, by [Shi97, Sections 18.9, 18.10], for each non-degenerate matrix β , the Fourier coefficient $c(\beta, h; f)$ factors over the places v of \mathcal{K}^+ . More precisely, write $\beta = (\beta_v)_v$ and $h = (h_v)_v$ as v runs over the places v of \mathcal{K}^+ , and define $c(\beta_v, h_v; f_v)$ as

$$\int_{\text{Her}(\mathcal{K} \otimes_{\mathcal{K}^+} \mathcal{K}_v^+)} f_v \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & N_v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} {}^t\bar{h}_v^{-1} & 0 \\ 0 & h_v \end{pmatrix} \right) e_v(-\text{tr } \beta_v N_v) dN_v.$$

Then, we have

$$c(\beta, h; f) = C(n, \mathcal{K}) \prod_v c(\beta_v, h_v; f_v),$$

where

$$(168) \quad C(n, \mathcal{K}) = 2^{n(n-1)[\mathcal{K}^+:\mathbb{Q}]/2} |D_{\mathcal{K}^+}|^{-n/2} |D_{\mathcal{K}}|^{-n(n-1)/4},$$

and dN_v denotes the Haar measure on $\text{Her}(\mathcal{K} \otimes_{\mathcal{K}^+} \mathcal{K}_v^+)$ such that

$$\int_{\text{Her}_n(\mathcal{O}_{\mathcal{K}} \otimes_{\mathcal{O}_{\mathcal{K}^+}} \mathcal{O}_{\mathcal{K}_v^+})} dN_v = 1, \text{ for each finite place } v,$$

and

$$dN_v := \left| \bigwedge_{j=1}^n dN_{jj} \bigwedge_{j < k} 2^{-1} dN_{jk} \wedge d\bar{N}_{kj} \right|, \text{ for each archimedean place } v,$$

where N_{jk} is the (j, k) -th entry of the matrix N_v .

In the following sections, we generalize the approach of [Eis15] to compute the local Fourier coefficients at p corresponding to the local Siegel-Weil sections associated to types constructed in Section 9.2. Then, we rely on known formulas obtained by Shimura in [Shi97] and extended by Eischen in [Eis15] for the local coefficients at places away from p . We later combine these results with the discussion above to p -adically interpolate the Eisenstein series $E_{f,U}$.

11.2. Calculations of local Fourier coefficients. In this section, for each place v of \mathcal{K}^+ , we compute $c(\beta_v, h_v; f_v)$. It is more convenient to compute these coefficients for $h_v = 1$. One can use [Eis15, Lemma 9] to relate $c(\beta, h; f_U)$ to $c(\beta, 1; f_U)$ for arbitrary $h \in \text{GL}_n(\mathbb{A}_{\mathcal{K}})$.

11.2.1. *Local coefficients at p .* Assume $v \mid p$ and identify v with the unique place $w \mid v$ in Σ_p . Let $f_v = f_w = f^{\Phi_w}$ be as in (147), for $\Phi_w = \Phi_w^{\tau_w \otimes \psi_w}$, see Remark 9.8. Then, the local coefficient for $\beta_v = \beta_w$ is

$$\begin{aligned} c(\beta_w, 1; f_w) &= \int_{M_n(\mathcal{K}_w)} f_w \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \right) e_p(-\operatorname{tr} \beta_w N) dN \\ &= \int_{\operatorname{GL}_n(\mathcal{K}_w)} \chi_{w,1}^{-1} \chi_{w,2}(X) |\det X|_w^{n+2s} \\ &\quad \times \int_{M_n(\mathcal{K}_w)} \Phi_w \left((0, X) \begin{pmatrix} 0 & -1 \\ 1 & N \end{pmatrix} \right) e_p(-\operatorname{tr} \beta_w N) dN d^\times X \end{aligned}$$

From (146), we have

$$\Phi_w \left((0, X) \begin{pmatrix} 0 & -1 \\ 1 & N \end{pmatrix} \right) = \frac{\dim \tau_w}{\operatorname{Vol}(\mathfrak{G}_w)} \Phi_{1,w}(X) \Phi_{2,w}(XN)$$

which is nonzero if and only if $X \in \mathfrak{G}_w$. It follows that $c(\beta_w, 1; f_w)$ is equal to

$$\begin{aligned} &\frac{\dim \tau_w}{\operatorname{Vol}(\mathfrak{G}_w)} \int_{\mathfrak{G}_w} \chi_{w,1}^{-1} \chi_{w,2}(X) \Phi_{1,w}(X) \int_{M_n(\mathcal{K}_w)} \Phi_{2,w}(XN) e_p(-\operatorname{tr} \beta_w N) dN d^\times X \\ &= \frac{\dim \tau_w}{\operatorname{Vol}(\mathfrak{G}_w)} \int_{\mathfrak{G}_w} \chi_{w,1}^{-1} \chi_{w,2}(X) \Phi_{1,w}(X) \int_{M_n(\mathcal{K}_w)} \Phi_{2,w}(N) e_p(\operatorname{tr}(-\beta_w X^{-1}N)) dN d^\times X \\ &= \frac{\dim \tau_w}{\operatorname{Vol}(\mathfrak{G}_w)} \int_{\mathfrak{G}_w} \chi_{w,1}^{-1} \chi_{w,2}(X) \Phi_{1,w}(X) (\Phi_{2,w})^\wedge(-\beta_w X^{-1}) d^\times X \\ &= \frac{\dim \tau_w}{\operatorname{Vol}(\mathfrak{G}_w)} \int_{\mathfrak{G}_w} \chi_{w,1}^{-1} \chi_{w,2}(X) \Phi_{1,w}(X) \nu_w^{\tau_w \otimes \psi_w}(-\beta_w X^{-1}) d^\times X, \end{aligned}$$

using (145) in the last line and notation as in Remark 9.8 for $\nu_w = \nu_w^{\tau_w \otimes \psi_w}$.

Now, write

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for some $B, {}^t C \in M_{a_w \times b_w}(\mathbb{Z}_p)$, $A \in I_{a_w, r}^0$ and $D \in I_{b_w, r}^0$, so that the above equals

$$\frac{\dim \tau_w}{\operatorname{Vol}(\mathfrak{G}_w)} \int_{\mathfrak{G}_w} \chi_{w,1}^{-1} \chi_{w,2}(X) \mu_{a_w}(A) \mu_{b_w}(D) \nu_w(-\beta_w X^{-1}) d^\times X,$$

using (142) and (143).

In particular, the above is zero unless $-\beta_w X^{-1} \in \mathfrak{X}_w$. Thus, $\beta_w \in M_n(\mathcal{O}_w)$ and we can write

$$\beta_w = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}$$

with $\beta_1 \in M_{a_w}(\mathcal{O}_w)$, $\beta_2, {}^t \beta_3 \in M_{a_w \times b_w}(\mathcal{O}_w)$, and $\beta_4 \in M_{b_w}(\mathcal{O}_w)$.

In particular,

$$-\beta_w X^{-1} \equiv \begin{pmatrix} -\beta_1 A^{-1} & * \\ * & -\beta_4 D^{-1} \end{pmatrix}$$

modulo \mathfrak{p}_w^r , where the precise description of the bottom-left and upper-right corners is irrelevant in what follows.

Using (144) and the definitions of both μ_{a_w} and μ_{b_w} , we obtain

$$\begin{aligned} c(\beta_w, 1; f_w) &= \frac{\dim \tau_w}{\text{Vol}(\mathfrak{G}_w)} \chi_{w,1}^{-1} \chi_{w,2}(-\beta_w) \chi_2^{-1}(-\beta_1) \chi_1(-\beta_4) \\ &\quad \times \int_{\mathfrak{G}_w} \chi_{w,1}(A) \chi_{w,2}^{-1}(D) \mu'_{a_w}(A) \mu'_{b_w}(D) \mu'_{a_w}(-\beta_1 A^{-1}) \mu'_{b_w}(-\beta_4 D^{-1}) d^\times X. \end{aligned}$$

Using orthogonality relations between matrix coefficients, as in the end of the proof of Theorem 10.6, it follows that

$$c(\beta, 1; f) = \chi_{w,1}^{-1} \chi_{w,2}(-\beta_w) \chi_2^{-1}(-\beta_1) \chi_1(-\beta_4) \mu'_{a_w}(-\beta_1) \mu'_{b_w}(-\beta_4),$$

and using (144) once more, we ultimately obtain

$$c(\beta_w, 1; f_w) = \nu_w(-\beta_w).$$

From now on, we write $\nu_w(\bullet; \tau_w, \psi_w)$ for $\nu_w(\bullet) = \nu_w^{\tau_w \otimes \psi_w}(\bullet)$.

11.2.2. *Local coefficients at ∞ .* Assume Hypothesis 9.12 and consider the Siegel-Weil section $f_{\infty, \mathfrak{J}} = f_{\infty}(\bullet; \mathfrak{J}, \chi_{\infty}, s)$ defined at the end of Section 9.3.2.

Let $g_0 \in U_4(\mathbb{R})$ be any element such that $g_0 \mathfrak{J} = i1_n$. Then, we have

$$(169) \quad f_{\infty}(g; \mathfrak{J}, \chi_{\infty}, s) = f_{\infty}(gg_0^{-1}; i1_n, \chi_{\infty}, s) f_{\infty}(g_0^{-1}; i1_n, \chi_{\infty}, s)^{-1},$$

where $f_{\infty, i1_n} = f_{\infty}(\bullet; i1_n, \chi_{\infty}, s)$ is defined by replacing \mathfrak{J} with $i1_n$ in (151). In particular, $f_{\infty, \mathfrak{J}}$ and $f_{\infty, i1_n}$ only differ by nonzero constant.

Remark 11.1. In what follows, we use $f_{\infty}(\bullet; i1_n, \chi_{\infty}, s)$ instead of $f_{\infty}(\bullet; \mathfrak{J}, \chi_{\infty}, s)$ to state the results of Shimura directly. However, the Eisenstein series appearing in the previous section is still the one associated to $f_{\infty}(\bullet; \mathfrak{J}, \chi_{\infty}, s)$.

As we are currently trying to p -adically interpolate its Fourier coefficients, this change is not an issue as the two sections are related the Fourier coefficients of each are related by a nonzero constant.

Let $f_{\infty, U} = \prod_{\sigma \in \Sigma} f_{\sigma}$ be the restriction of $f_{\infty, i1_n}$ to $U_4(\mathbb{R}) = \prod_{\sigma \in \Sigma} U(\mathbb{R})$. It follows from [Shi83, Equation (7.12)] (see [Eis15, Section 2.2.6] as well) that at $s = \frac{k-n}{2}$, the archimedean Fourier coefficients at $\beta = \beta_{\sigma}$ is

$$c(\beta_{\sigma}, 1; f_{\sigma}) = \left(2^{(n-1)n} (2\pi i)^{nk} \pi^{\frac{n(n-1)}{2}} \prod_{j=0}^{n-1} \Gamma(k-j) \right)^{-1} \sigma(\det \beta)^{k-n} e^{i \text{tr}(\sigma(\beta))}.$$

Let $\beta_{\infty} = (\beta_{\sigma})_{\sigma \in \Sigma}$. The product $c(\beta_{\infty}, 1; f_{\infty, U}) = \prod_{\sigma \in \Sigma} c(\beta_{\sigma}, 1; f_{\sigma})$ is equal to

$$(170) \quad \left(2^{(n-1)n} (-2\pi i)^{-nk} \pi^{\frac{n(n-1)}{2}} \prod_{j=0}^{n-1} \Gamma(k-j) \right)^{-[\mathcal{K}^+ : \mathbb{Q}]} \prod_{\sigma} \det(\beta_{\sigma})^{k-n} e^{i \text{tr}(\beta_{\sigma})}.$$

Using [Eis15, Lemma 9], we see that given any $h_\infty = (h_\sigma)_{\sigma \in \Sigma} \in \mathrm{GL}_n(\mathbb{A}_{\mathcal{K}, \infty})$, if $k > n$, then $c(\beta_\infty, h_\infty, f_\infty) \neq 0$ if and only if $\det \beta_\infty \neq 0$. In particular, the Fourier coefficients are nonzero only if β is non-degenerate.

11.2.3. *Local coefficients at places away from p and ∞ .* Assume v is a finite place of \mathcal{K}^+ away from p . Let f_v be the local Siegel-Weil section at v constructed in Section 9.4.1 and 9.4.2, for v unramified and ramified respectively. As explained in Section 9.4.3, see [Ehls20, Section 4.2.2] as well, we have

$$c(\beta_v, 1; f_v) = c(\beta_v, 1; f_v^{\mathfrak{b}}),$$

for some ideal \mathfrak{b} of $\mathcal{O}_{\mathcal{K}^+}$ prime to p .

As explained in [Eis15], it follows from [Shi97, Proposition 19.2] that

$$\begin{aligned} \prod_{v \nmid p\infty} c(\beta_v, 1; f_v^{\mathfrak{b}}) &= \mathrm{Nm}_{\mathbb{Q}}^{\mathcal{K}^+}(\mathfrak{b})^{-n^2} \prod_{i=0}^{n-1} L^p(2s + n - i, (\chi^+)^{-1} \eta^i)^{-1} \\ &\times \prod_{v \nmid p\infty} P_{\beta_v, \mathfrak{b}}(\chi^+(\varpi_v)^{-1} |\varpi_v|^{2s+n}), \end{aligned}$$

where

- (i) χ^+ is the restriction of the unitary Hecke character χ from $\mathbb{A}_{\mathcal{K}}$ to $\mathbb{A}_{\mathcal{K}^+}$;
- (ii) η is the quadratic character of $\mathbb{A}_{\mathcal{K}^+}$ associated to the extension $\mathcal{K}/\mathcal{K}^+$;
- (iii) ϖ_v is a uniformizer of $\mathcal{O}_{\mathcal{K}^+, v}$, viewed as an element of \mathcal{K}^\times prime to p . In what follows, we identify ϖ_v with its image in $(\mathcal{O}_{\mathcal{K}} \otimes \mathbb{Z}_p)^\times$;
- (iv) $P_{\beta_v, \mathfrak{b}}$ is a polynomial, depending on β_v and \mathfrak{b} , with \mathbb{Z} -coefficients and constant term 1, which is identically 1 for all but finitely many v ; and
- (v) $L^p(r, \chi^+ \eta^r) = \prod_{v \nmid p\infty \text{ cond } \eta} d_v(s + \frac{1}{2}, \chi)$, where $d_v(s, \chi)$ is as in (165).

Furthermore, note that only

$$\alpha(\beta; \chi, s) = \alpha_{\mathfrak{b}}(\beta; \chi, s) := \prod_{v \nmid p\infty} P_{\beta_v, \mathfrak{b}}(\chi^+(\varpi_v)^{-1} |\varpi_v|^{2s+n})$$

depends on β_v in the expression on the right-hand side above. For future reference, we set $\alpha(\beta; \chi) = \alpha_{\mathfrak{b}}(\beta; \chi) := \alpha(\beta; \chi, \frac{k-n}{2})$ for k as in Hypothesis 9.12.

As explained in [Eis15, Section 2.2.10], $\alpha(\beta; \chi)$ is a (finite) \mathbb{Z} -linear combination of terms of the form

$$\prod_{v \nmid p\infty} \chi_v(\varpi)^{-1} |\varpi|_v^k,$$

where ϖ is a p -integral element of the integer ring of \mathcal{K}^+ . Furthermore, using (149), we have

$$(171) \quad \prod_{v \nmid p\infty} \chi_v(\varpi)^{-1} |\varpi|_v^k = \chi_1 \chi_2^{-1}(\varpi) \prod_{\sigma \in \Sigma} \sigma(\varpi)^{-k},$$

where $\chi_i = \bigotimes_{w \in \Sigma_p} \chi_{i,w}$ for $i = 1$ and 2 , see [Eis15, Equation (28)]. In particular, from the definition of $\chi_{w,1}$ and $\chi_{w,2}$ in Section 9.2.1, we have $\chi_p = \bigotimes_{w|p} \chi_w = \chi_1 \otimes \chi_2^{-1}$. We can thus rewrite $\alpha(\beta; \chi)$ as

$$(172) \quad \alpha(\beta; \chi) = \prod_{v \nmid p \infty} P_{\beta_v, \mathbf{b}}(\chi_p(\varpi_v) \mathrm{Nm}_{\mathbb{Q}}^{\mathcal{K}^+}(\varpi_v)^{-k})$$

11.2.4. *Global Fourier coefficients.* Assume Hypothesis 9.12. Using the same notation as in the previous sections, let

$$\begin{aligned} D(n, \mathcal{K}, \mathbf{b}, p, k) &= C(n, \mathcal{K}) \mathrm{Nm}_{\mathbb{Q}}^{\mathcal{K}^+}(\mathbf{b}) \\ &\times \prod_{\sigma \in \Sigma} \left(2^{(n-1)n} (-2\pi i)^{-nk} \pi^{\frac{n(n-1)}{2}} \prod_{j=0}^{n-1} \Gamma(k-j) \right)^{[\mathcal{K}^+ : \mathbb{Q}]} \\ &\times \prod_{i=0}^{n-1} L^p(k-i, (\chi^+)^{-1} \eta^i)^{-1}. \end{aligned}$$

Proposition 11.2. *Assume $k > n$ and let $f_{\chi, \psi}^\tau$ be the Siegel-Weil section $f_{\chi, \tau}^\tau(\bullet; \frac{k-n}{2})$ as in (160). For $\beta \in \mathrm{Her}_n(\mathcal{K})$, all the nonzero Fourier coefficients $c(\beta, 1; f_\chi^\tau)$ are given by*

$$(173) \quad D(n, \mathcal{K}, \mathbf{b}, p, k) \alpha(\beta) \nu_p(-\beta_p; \tau, \psi) \prod_{\sigma \in \Sigma} (\det \beta_\sigma)^{k-n} e^{i \mathrm{tr}_{\mathbb{Q}}^{\mathcal{K}^+}(\beta)},$$

where

$$\nu_p(-\beta_p; \tau, \psi) := \prod_{w \in \Sigma_p} \nu_w(-\beta_w; \tau_w, \psi_w).$$

Furthermore,

$$\nu_w(\bullet; \tau_w, \psi_w) = \nu_w^{\tau_w \otimes \psi_w} = \chi_{w,1}^{-1} \chi_{w,2} \mu_w(\bullet; \tau_w, \psi_w)$$

is as in (144), where $\mu_w(\bullet; \tau_w, \psi_w)$ is the product of matrix coefficients constructed in (142) with respect to $\tau_w \otimes \psi_w$.

Let $E_{\chi, \psi}^\tau$ be the Eisenstein modular form in (161). It follows from Proposition 11.2 that the algebraic q -expansion of

$$(174) \quad G_{\chi, \psi}^\tau := D(n, \mathcal{K}, \mathbf{b}, p, k)^{-1} E_{\chi, \psi}^\tau$$

at a cusp L is

$$(175) \quad G_{\chi, \psi}^\tau(q) = \sum_{\beta \in L} \left(\alpha(\beta, \chi) \nu_p(-\beta_p; \tau, \psi) \prod_{\sigma \in \Sigma} \det(\beta_\sigma)^{k-n} \right) q^\beta,$$

for $k > n$, see [Eis15, Section 2.2.11]. In particular, the coefficients are algebraic.

Remark 11.3. Recall that ψ is a finite-order character of $L(\mathbb{Z}_p)$. From now on, we identify ψ as a character of the center Z_P of L_P , see Remark 2.7 and the comments that follow.

Note that all of the above, especially the content of Section 11.2.1, remains valid if ψ is only a locally constant function on Z_P , and not necessarily a character. See Remark 9.9.

11.2.5. *p -adic shifts of Hecke characters.* In this section, we recall the notion of the “ p -adic shift” of χ , see [Eis15, Section 2.2.13] and [Ehls20, Section 8.2].

Assume the conductor of χ divides $p^m N_0$ for some $m \geq 0$ and integer N_0 prime to p . Let

$$U_{m,N_0} = (1 + N_0 \mathcal{O} \otimes \widehat{\mathbb{Z}}^p)^\times \times (1 + p^m \mathcal{O} \otimes \mathbb{Z}_p) \subset (\mathcal{K} \otimes \widehat{\mathbb{Z}})^\times,$$

where $\mathcal{O} = \mathcal{O}_{\mathcal{K}}$, and consider

$$X_p = X_{p,N_0} := \varprojlim_m \mathcal{K}^\times \backslash (\mathcal{K} \otimes \widehat{\mathbb{Z}})^\times / U_{m,N_0},$$

the ray class group of \mathcal{K} of conductor $p^\infty N_0$. The p -adic shift of χ will be a character of X_p . We often decompose a character α of X_p as $\alpha = \bigotimes_w \alpha_w$, where the tensor product runs over all the finite places of \mathcal{K} .

Now, assume as usual that χ satisfies Hypothesis 9.12 for some integer $k \geq 0$. Let $\chi_0 = \chi| \cdot |_{\mathbb{A}_{\mathcal{K}}}^{-k/2}$ and write $\chi_0 = \prod_w \chi_{0,w}$, as the product runs over all the places of \mathcal{K} . Similarly, for any place v of \mathbb{Q} , let $\chi_{0,v} = \prod_{w|v} \chi_{0,w}$, so that

$$\chi_{0,\infty}(a) = \prod_{\sigma \in \Sigma} \sigma(a)^{-k - \nu_\sigma} \bar{\sigma}(a)^{\nu_\sigma},$$

for all $a \in \mathcal{K}$, where $\bar{\sigma} = \sigma c$ and $\nu = (\nu_\sigma)_\sigma$ is as in (149) (this sequence of integers ν should not be confused with the locally constant function $\nu_p(\bullet; \tau, \psi)$). We say that the ∞ -type χ_0 is

$$\Psi_{k,\nu} = \prod_{\sigma \in \Sigma} \sigma^{-k} \left(\frac{\bar{\sigma}}{\sigma} \right)^{\nu_\sigma},$$

viewed as a function of $(\mathcal{O}_{\mathcal{K}} \otimes \mathbb{Z}_p)^\times$. Note that for all $\varpi \in \mathcal{O}_{\mathcal{K}^+}^\times$, we have

$$(176) \quad \Psi_{k,\nu}(\varpi) = \text{Nm}_{\mathbb{Q}}^{\mathcal{K}^+}(\varpi)^{-k}.$$

Let $\tilde{\chi}_{0,\infty} : (\mathcal{K} \otimes \mathbb{Z}_p)^\times \rightarrow \overline{\mathbb{Q}}_p^\times$ be the unique p -adically continuous character such that

$$\tilde{\chi}_{0,\infty}(a) = \text{incl}_p \circ \chi_{0,\infty}(a),$$

for all $a \in \mathcal{K}^\times$. In particular, $\tilde{\chi}_{0,\infty}(a) \in \mathcal{O}_{\mathbb{C}_p}^\times$ for all $a \in (\mathcal{O}_{\mathcal{K}} \otimes \mathbb{Z}_p)^\times$.

The p -adic shift of χ is defined as the p -adic character $\tilde{\chi}_0 : X_p \rightarrow \mathcal{O}_{\mathbb{C}_p}^\times$ for which

$$\tilde{\chi}_0(a) = \tilde{\chi}_{0,\infty}((a_w)_{w|p}) \prod_{w \nmid \infty} \chi_{0,w}(a_w),$$

for all $a = (a_w)_{w \nmid \infty} \in X_p$.

We now express the Fourier coefficients of $G_{\chi, \psi}^\tau$ in terms of $\tilde{\chi}_0$. In the process, we also use the definition of $\nu_w(\bullet; \tau, \psi)$ in (144) to express these coefficients in terms of

$$\mu_p(\bullet; \tau, \psi) := \prod_{w \in \Sigma_p} \mu_w(\bullet; \tau_w, \psi_w),$$

where $\mu_w(\bullet; \tau_w, \psi_w) = \mu_w^{\tau_w \otimes \psi_w}$ is as in (142) and Remark 9.8.

Firstly, observe that using (172) and (176), we have

$$\alpha(\beta, \chi) = \prod_{v \nmid p \infty} P_{\beta_v, \mathfrak{b}}(\tilde{\chi}_{0,p}(\varpi_v)),$$

where

$$\tilde{\chi}_{0,p} := \prod_{w \mid p} \tilde{\chi}_{0,w}.$$

Similarly, we have

$$\begin{aligned} \nu_p(-\beta; \tau, \psi) \operatorname{Nm}(\det \beta)^k &= \chi_p(-\beta^{-1}) \mu_p(-\beta; \tau, \psi) \operatorname{Nm}(\det \beta)^k \\ &= \chi_p(-1) \tilde{\chi}_{0,p}(\beta^{-1}) \mu_p(-\beta; \tau, \psi). \end{aligned}$$

Therefore, the β -th coefficient of the q -expansion of $G_{\chi, \psi}^\tau$ at a cusp L can be rewritten as a finite \mathbb{Z} -linear combinations of terms of the form

$$(177) \quad \tilde{\chi}_{0,p}(\varpi) \tilde{\chi}_{0,p}(\beta^{-1}) \mu_p(-\beta; \tau, \psi) \operatorname{Nm}(\det \beta)^{-n},$$

where the linear combination is over a finite set (which depends on β and L) of p -adic units $\varpi \in \mathcal{K}^\times$.

Note that if ψ and ψ' are two finite-order characters of Z_P such that $\psi \equiv \psi'$ modulo p^r , then the $\mu_p(\bullet; \tau, \psi) \equiv \mu_p(\bullet; \tau, \psi')$ modulo p^r .

Remark 11.4. The Fourier coefficients in (177) can be compared to the ones of the Eisenstein series constructed in [Eis14, Theorem 2]. The main difference is the level at p of the Eisenstein series considered.

From (177) and the q -expansion principle, it is clear that $G_{\chi, \psi}^\tau$ is a modular form on G_4 over the p -adic ring \mathcal{O}_π introduced in Section 8.2. We identify it with its image in the space of p -adic modular forms.

11.3. p -adic differential operators. In this section, we discuss the p -adic differential operators constructed in [EFMV18, Section 5] and their relevant properties for our purpose. The goal is to obtain a family of p -adic modular forms related to $E_{\chi, \psi, \theta}^{\tau, \kappa}$, see (160), using these p -adic differential operators (depending on p -adic weights κ and τ) and $G_{\chi, \psi}^\tau$.

Let $R = \mathcal{O}_\pi$ and let $K = K_4 \subset G_4(\mathbb{A}_f)$ be any neat open compact subgroup. Consider the space

$$\mathcal{V} := \mathcal{V}(G_4, K^p; R)$$

of (scalar-valued) p -adic modular forms on G_4 (with respect to the parabolic $P_4 \subset H_4$, see Section 4.1.3 and (68)), as in (87).

Now, let κ be an R -valued dominant weight for G_1 , as in Section 2.3.1. Then, $\tilde{\kappa} = (\kappa, \kappa^b)$ is a dominant weight of G_3 . Let $\tilde{\kappa}_p$ be the corresponding p -adic weight of $T_{H_3}(\mathbb{Z}_p) = T_{H_4}(\mathbb{Z}_p)$.

Assume that (κ, χ) is critical, as in Definition 9.15, and let $\rho = (\rho_\sigma)_{\sigma \in \Sigma}$ and $\rho^v = (\rho_\sigma^v)_{\sigma \in \Sigma}$ be as in (152). As above, let $\tilde{\rho} = (\rho, \rho^b)$, $\tilde{\rho}^v = (\rho^v, \rho^{v,b})$, and denote the corresponding p -adic weights as $\tilde{\rho}_p$ and $\tilde{\rho}_p^v$.

Proposition 11.5. *Assume Conjecture 5.3. Keeping the same notation as above, there exists a p -adic differential operator*

$$\theta_\chi^d(\rho^v) : \mathcal{V}_\chi(G_4, K^p; R) \rightarrow \mathcal{V}(G_4, K^p; R),$$

compatible with change of level subgroups, such that

$$(178) \quad \Omega_{\tilde{\kappa}, r, J'_0, h_0} \circ \text{Res}_{J'_0, h_0} \circ \delta_\chi^d(\rho^v)(f) = \text{Res}_{p, J'_0, h_0} \circ \theta_\chi^d(\rho^v) \circ \Omega_{\tilde{\kappa}, r, G_4, X_4}(f)$$

for any $f \in M_\chi(G_4, K_{4,r}; R)$ and any ordinary CM pair (J'_0, h_0) , using the notation from Section 5.2.5. Here, $\delta_\chi^d(\rho^v)$ is as in Section 9.5.

Proof. The differential operators $\theta_\chi^d(\rho^v)$ are exactly the operators denoted $\Theta^{\tilde{\kappa}}$ in [EFMV18, Theorem 5.1.3].

However, one need to show that these extend to the space of p -adic modular form \mathcal{V} considered here, which is larger in general than the space “ V^N ” in *ibid.* This follows immediately if we assume Conjecture 5.3 (which replaces the use of [EFMV18, Theorem 2.6.1]).

Lastly, (178) is exactly [EHL20, Theorem 8.1.1 (a)]. □

Proposition 11.6. *In the setting of Proposition 11.5, we have*

- (i) *Fix any neat open compact subgroups $K_1 \subset G_1(\mathbb{A}_f)$ and $K_2 \subset G_2(\mathbb{A}_f)$ such that $K_1 \times K_2 \subset G_3(\mathbb{A}_f) \cap K_4$. Then,*

$$\theta(\kappa, \chi) := \text{Res}_3 \circ \theta_\chi^d(\rho^v)$$

defines a differential operator

$$\mathcal{V}_\chi(G_4, K_4^p; R) \rightarrow \mathcal{V}_\kappa(G_1, K_1^p; R) \otimes \mathcal{V}_{\kappa^b}(G_2, K_2^p; R) \otimes (\chi \circ \det),$$

where Res_3 is the pullback of the first embedding $\gamma_p \circ \iota_3$ in (84).

- (ii) *There is a differential operator*

$$\theta^{\text{hol}}(\kappa, \chi) : \mathcal{V}_\chi(G_4, K^p; R) \rightarrow \mathcal{V}(G_4, K^p; R)$$

whose composition with Res_3 coincides with $D^{\text{hol}}(\kappa, \chi)$ from Section 9.5, via pullback to functions on $G_4(\mathbb{A})$, restrictions to functions on $G_3(\mathbb{A})$ and (90) for G_3 .

(iii) For all $\kappa^\dagger \leq \kappa$, there exists an operator

$$\theta(\kappa, \kappa^\dagger) : \mathcal{V}(G_4, K^p; R) \rightarrow \mathcal{V}(G_4, K^p; R)$$

such that

$$\theta(\kappa, \chi) = \sum_{\kappa^\dagger \leq \kappa} \text{Res}_3 \circ \theta(\kappa, \kappa^\dagger) \circ \theta^{\text{hol}}(\kappa, \chi).$$

Proof. This is simply [EHLS20, Proposition 8.1.1 (b), (d)] and [EHLS20, Corollary 8.1.2] in our settings. The proof remains the same using the existence of the differential operators in Proposition 11.5. \square

For any dominant weight κ as above, using these differential operators, we define

$$G_{\chi, \psi}^{\tau, \kappa} := \theta(\kappa, \chi) G_{\chi, \psi}^\tau.$$

and let $K_3 = K_{3,r} \subset G_3(\mathbb{A}_f)$ be its level, see (174) and the comments below (161).

Furthermore, let θ be any P -parallel weight. Then, following the same logic as in Section 9.5, in what follows we set $\theta(\kappa, \theta, \chi) := \theta(\kappa + \theta, \chi)$ and

$$G_{\chi, \psi, \theta}^{\tau, \kappa} := G_{\chi, \psi}^{\tau, \kappa + \theta} = \theta(\kappa, \theta, \chi) G_{\chi, \psi}^\tau.$$

The action of $\theta(\kappa, \chi)$ on p -adic q -expansion is described in [EFMV18, Corollary 5.2.10]. Their work considers p -adic modular forms in

$$\mathcal{V}_{\infty, \infty} = \varprojlim_m \varinjlim_r \mathcal{V}_{r, m}(\mathcal{O}_{\mathbb{C}_p})^{B_H^u(\mathbb{Z}_p)},$$

for the Borel B_H associated to the trivial partition, see Remark 2.8, but their computations hold for all $f \in \mathcal{V}(G_4, K^p; R)$ if one assumes Conjecture 5.3 (which we do in this paper).

Namely, there exists a polynomial ϕ^κ (on $n \times n$ -matrices) such that for each $\beta \in L$, the β -th coefficient of $G_{\chi, \psi}^{\tau, \kappa}$ is equal to $\phi^\kappa(\beta)$ times the β -th coefficient of $G_{\chi, \psi}^\tau$, see [EFMV18, Theorem 5.1.3 (1)] and [EFMV18, Section 5.2.2].

Remark 11.7. In our notation, the polynomial in [EFMV18, Corollary 5.2.10] should be written $\phi_{\tilde{\kappa}}$ for $\tilde{\kappa} = (\kappa, \kappa^b)$. However, we only consider polynomials associated to such characters, hence we only emphasize their dependence on κ .

As above, considering κ as fixed and considering any $\kappa' = \kappa + \theta$ in the P -parallel lattice $[\kappa]$, we set $\phi_\theta^\kappa := \phi^{\kappa + \theta}$. Then, it follows from [EFMV18, Remark 5.2.11] that if θ and θ' are two P -parallel weights such that $\theta \equiv \theta'$ modulo $p^r(p-1)$, then $\phi_\theta^\kappa \equiv \phi_{\theta'}^\kappa$ modulo p^{r+1} .

Using the above and (177), one therefore readily checks that the β -th coefficient of $G_{\chi, \psi, \theta}^{\tau, \kappa}$ satisfy the “usual” Kummer congruences ([Kat78, (4.0.8)]) as $(\tilde{\chi}_0, \psi \cdot \theta)$ vary p -adically as characters of $X_p \times Z_P$. See [EHLS20, Section 5] for further details.

We obtain the following as a consequence of [Kat78, Proposition (4.1.2)] and using the same logic as in the construction of the analogous Eisenstein measures of [Kat78, Eis15, EFMV18].

Proposition 11.8. *Assume conjecture 5.5. Fix a p -adic weight κ of $T_{H_1}(\mathbb{Z}_p)$ and a P -nebentypus τ with central character ω_τ . There is a $\mathcal{V}_3(K_3^P; R)$ -valued measure $d\text{Eis}^{[\kappa, \tau]}$ on $X_p \times Z_P$ such that*

$$(179) \quad \int_{X_p \times Z_P} (\tilde{\chi}_0, (\omega_\tau \psi) \cdot \rho_{\kappa, \theta}^v) d\text{Eis}^{[\kappa, \tau]} = G_{\chi, \psi, \theta}^{\tau, \kappa},$$

for any p -adic shift $\tilde{\chi}_0$ of a Hecke character χ , as in Section 11.2.5, and for any arithmetic characters on Z_P whose finite-order part is $\omega_\tau \psi$ and algebraic part is $(\kappa_p + \theta_p)|_{Z_P}$ for some critical pair $(\kappa + \theta, \chi)$. Here, $\rho_{\kappa, \theta}^v$ is the “shift” associated to $\kappa + \theta$ and χ as in (152).

Both sides of (179) are independent of the choice of decompositions $\kappa' = \kappa + \theta \in [\kappa]$ and $\omega_{\tau'} = \omega_\tau \psi$ for the central character of some $\tau' \in [\tau]$.

Remark 11.9. When $P = B$ as in Remark 2.8, the above agrees with the measure in [Ehls20, Theorem 8.2.2].

11.3.1. *Comparison to classical Eisenstein series.* We first compare $\theta(\kappa, \chi)$ to the C^∞ -differential operators from Section 9.3.3.

Proposition 11.10. *Assume Conjecture 5.5. With the same setting as in Proposition 11.6, let $\theta(\kappa, \chi)^{\text{cusp}}$ denote the restriction of $\theta(\kappa, \chi)$ to $\mathcal{V}_\chi^{\text{cusp}}(G_4, K_4; R)$. Then,*

$$e_\kappa^{P\text{-ord}} \circ \theta(\kappa, \chi)^{\text{cusp}} = e_\kappa^{P\text{-ord}} \circ \delta_\chi^d(\rho^v)$$

as operators

$$\mathcal{V}_\chi^{\text{cusp}}(G_4, K_4; R) \rightarrow S_\kappa(G_1, K_1; R) \otimes S_{\kappa^b}(G_2, K_2; R) \otimes (\chi \circ \det).$$

Furthermore, for any cuspidal $F \in H_1^0(\text{Sh}(V_4), \mathcal{L}(\chi))$, we have

$$e_\kappa^{P\text{-ord}} \circ \theta(\kappa, \chi)(F) = e_\kappa^{P\text{-ord}} \circ D^{\text{hol}}(\kappa, \chi)(F)$$

Proof. The first part is exactly [Ehls20, Theorem 8.1.1 (c)] with the obvious modifications to our setting. For the second part, we follow the same logic as in the proof of [Ehls20, Proposition 8.1.3].

All one needs is the decompositions from (154) and Proposition 11.6 (iii), the first part of the above and the fact that for any $\kappa^\dagger < \kappa$,

$$e_\kappa^{P\text{-ord}} := \lim_N \left(\prod_{w \in \Sigma_p} \prod_{j=1}^{r_w} u_{w, D_w(j), \kappa} \right)^{N!}$$

converges absolutely to 0 on $S_{\kappa^\dagger}(K_r; R)$. This last claim follows from the fact that for $\kappa^\dagger < \kappa$, we have

$$u_{w, D_w(j), \kappa} = \kappa'(t_{w, D_w(j)}) U_{w, D_w(j)} = (\kappa' \cdot (\kappa^{\dagger'})^{-1})(t_{w, D_w(j)}) u_{w, D_w(j), \kappa^\dagger}$$

and, using the definition of κ' as in (55), that

$$\prod_{w \in \Sigma_p} \prod_{j=1}^{r_w} (\kappa' \cdot (\kappa^{\dagger, \prime})^{-1})(t_{w, D_w(j)}) = \prod_{w \in \Sigma_p} \prod_{j=1}^{r_w} (\kappa_p \cdot \kappa_p^{\dagger, -1})(t_{w, D_w(j)}) = p^m$$

for some strictly negative integer m . This is clear from the definition of each $t_{w, D_w(j)}$ and the relation (26). \square

We now wish to apply the previous proposition to an ordinary cusp form closely related to $G_{\chi, \psi, \theta}^{\tau, \kappa}$, using the notation of Proposition 11.8.

Let π be a P -anti-ordinary anti-holomorphic cuspidal automorphic form of P -anti-WLT (κ, K_r, τ) . Let $\mathfrak{m} = \mathfrak{m}_\pi$ be the non-Eisenstein maximal ideal of $\mathbf{T}_{K^p, [\kappa, \tau], \mathcal{O}_\pi}^{P\text{-ord}}$ associated to π as in Remark 8.10.

Assuming Conjecture 5.10, it follows from Proposition 11.10 and (174) that for κ very regular, after localization at \mathfrak{m} , both

$$(180) \quad e_\kappa^{P\text{-ord}} G_{\chi, \psi}^{\tau, \kappa} \quad \text{and} \quad D(n, \mathcal{K}, \mathfrak{b}, p, k)^{-1} e_\kappa^{P\text{-ord}} D^{\text{hol}}(\kappa, \chi) E_{\chi, \psi}^\tau$$

lie in

$$S_{\kappa, V}(K_{1, r}, \mathcal{O}_{\mathbb{C}_p})_{\mathfrak{m}} \otimes S_{\kappa^{\flat}, -V}(K_{2, r}, \mathcal{O}_{\mathbb{C}_p})_{\mathfrak{m}} \otimes (\chi \circ \det),$$

and are equal.

In particular, using (49) on G_3 , we can view

$$\langle e_\kappa^{P\text{-ord}} G_{\chi, \psi}^{\tau, \kappa}, \bullet \rangle_{\tilde{\kappa}, K_{3, r}}$$

as an element in the \mathcal{O}_π -dual of

$$\widehat{S}_{\kappa, V}(K_{1, r}, \mathcal{O}_\pi)_{\mathfrak{m}} \otimes \widehat{S}_{\kappa^{\flat}, -V}(K_{2, r}, \mathcal{O}_\pi)_{\mathfrak{m}} \otimes (\chi^{-1} \circ \det)$$

and the above is closely related to the integral involved in the doubling method.

Observe that together with the tautological pairing $\mathcal{M}_\tau \otimes \mathcal{M}_{\tau^{\flat}} = \mathcal{M}_\tau \otimes \mathcal{M}_\tau^{\vee} \rightarrow \mathcal{O}_\pi$, the above can in fact be identified as an element in the dual of

$$(181) \quad \widehat{S}_{\kappa, V}(K_{1, r}, [\tau]; \mathcal{O}_\pi)_{\mathfrak{m}} \otimes \widehat{S}_{\kappa^{\flat}, -V}(K_{2, r}, [\tau^{\flat}]; \mathcal{O}_\pi)_{\mathfrak{m}} \otimes (\chi^{-1} \circ \det).$$

Now, fix any $F \in I_\pi$ and $F^{\flat} \in I_{\pi^{\flat}}$, using the notation from Section 8.4.4. Fix non-zero elements $\iota \in \text{Hom}_{L_P}(\tau, \pi_p^{(P\text{-a.ord}, r)})$ and $\iota^{\flat} \in \text{Hom}_{L_P}(\tau^{\flat}, \pi_p^{\flat, (P\text{-a.ord}, r)})$, i.e. a basis for each of these two 1-dimensional spaces.

Fix a basis $\mathcal{B}_\tau = \{v_1, \dots, v_r\}$ of τ (where $r = \dim \tau$) and a dual basis $\mathcal{B}_\tau^{\flat} = \{v_1^{\flat}, \dots, v_r^{\flat}\}$ of τ^{\flat} . For each $1 \leq i \leq r$, let φ_i (resp. φ_i^{\flat}) be the (anti-holomorphic P -anti-ordinary) test vector of π (resp. π^{\flat}) determined by F , ι (resp. ι^{\flat}) and v_i (resp. v_i^{\flat}), as in (131) (resp. (132)). As explained at the end of Section 8.4.4, we have

$$\langle \varphi, \varphi^{\flat} \rangle_\pi = \langle \varphi_i, \varphi_i^{\flat} \rangle_\pi$$

for all $1 \leq i \leq \dim \tau$, where $\varphi := \varphi_1$ and $\varphi^{\flat} := \varphi_1^{\flat}$.

It follows from (48), using the identifications (127) and (128), that pairing the element in the dual of (181) corresponding to $e_\kappa^{P\text{-ord}}G_{\chi,\psi}^{\tau,\kappa}$ with $F \otimes F^b$ is equal to

$$\frac{1}{\text{Vol}(I_{r,V}^0)\text{Vol}(I_{r,-V}^0)} \sum_{i=1}^{\dim \tau} \int_{[G_3]} D(n, \mathcal{K}, \mathfrak{b}, p, k)^{-1} E_{\chi,\psi}^{\tau,\kappa,\text{hol}} \left(g_1, g_2; s + \frac{1}{2} \right) \times \varphi_i(g_1) \varphi_i^b(g_2) \chi^{-1}(\det(g_2)) \|\nu(g_2)\|^{a(\kappa)} dg_1 dg_2,$$

where $s = \frac{k-n}{2}$ and $[G_3] = G_3(\mathbb{Q})Z_{G_3}(\mathbb{R}) \backslash G_3(\mathbb{A})$. By definition, this is equal to

$$\frac{1}{\text{Vol}(I_{r,V}^0)\text{Vol}(I_{r,-V}^0)D(n, \mathcal{K}, \mathfrak{b}, p, k)} \sum_{i=1}^{\dim \tau} I \left(\varphi_i, \varphi_i^b \|\nu(g_2)\|^{a(\kappa)}, f_{\chi,\psi}^{\tau,\kappa,\text{hol}}, s + \frac{1}{2} \right),$$

for $s = \frac{k-n}{2}$. Lastly, by Theorem 10.10, it is equal to (182)

$$\frac{\langle \varphi, \varphi^b \rangle_\pi}{\text{Vol}(I_{r,V}^0 \cap I_{r,-V}^0)} I_p \left(s + \frac{1}{2}, P\text{-ord}, \pi, \chi \right) I_\infty \left(s + \frac{1}{2}; \pi, \chi \right) I_S L^S \left(s + \frac{1}{2}; \pi, \chi \right),$$

at $s = \frac{k-n}{2}$, where I_p, I_∞ and I_S are as in (163), (164) and (166) respectively.

Part IV. p -adic L -functions for P -ordinary families.

12. PAIRING, PERIODS AND MAIN RESULT.

12.1. Eisenstein measures and p -adic L -functions. In this section, we adapt the material of [EHLS20, Section 7.4] to the P -ordinary setting. The goal is to obtain an analogue of [EHLS20, Proposition 7.4.10] in the P -ordinary setting and interpret the Eisenstein measure $d\text{Eis}^{[\kappa,\tau]}$ of Proposition 11.8 as an element of the Hecke algebra \mathbb{T} from Section 8.

The idea is to view $d\text{Eis}^{[\kappa,\tau]}$ as a collection of linear transformations on locally constant functions compatible with the projective limit structure of \mathbb{T} over Hecke algebras of finite level.

12.1.1. Equivariance and the Garrett map. Throughout this section, we fix a neat open compact subgroup $K_1^p \subset G(\mathbb{A}_f^p)$ and set $K_{1,r} := K_1^p I_{P,r}$ for all $r \gg 0$. We let $K_{2,r} := K_{1,r}^b$ and set $K_{3,r} := (K_{1,r} \times K_{2,r}) \cap G_3(\mathbb{A}_f)$. We often write K_r for $K_{3,r}$. Furthermore, we set

$$\mathcal{V}_3 := \mathcal{V}^{P\text{-ord},\text{cusp}}(G_3, K^p; \mathcal{O}).$$

Consider the center $Z = Z_P$ of $L_P(\mathbb{Z}_P)$ and for each $r \geq 1$, let $Z^r = 1 + p^r Z$. In particular, $Z_{P,r} = Z/Z^r$ as in Section 8.1.1.

Let $\Lambda = \Lambda_\pi = \mathcal{O}_\pi[[Z_P]]$ be the Iwasawa algebra in Section 8.4.1 for $R = \mathcal{O}_\pi$. Since the ring $\mathcal{O} = \mathcal{O}_\mathcal{K}$ does not appear in this section, we set $\mathcal{O} = \mathcal{O}_\pi$ in what follows.

As usual, one identifies Λ as the algebra of distributions on Z with \mathcal{O} -coefficients, equipped with a canonical perfect pairing $\Lambda \otimes C(Z, \mathcal{O}) \rightarrow \mathcal{O}$, where $C(Z, \mathcal{O})$ denotes the module of continuous \mathcal{O} -valued functions on \mathbb{T} .

Let $\mathcal{I}_r \subset \Lambda$ be the augmentation ideal associated to Z^r , and set $\Lambda_r = \Lambda/\mathcal{I}_r$. Furthermore, define

$$C_r(Z, \mathcal{O}) = C(Z/Z^r, \mathcal{O}) := \{\text{continuous } Z^r\text{-invariant functions on } Z\},$$

a free \mathcal{O} -module of locally constant functions on Z . Let $\eta_r : C_r(Z, \mathcal{O}) \hookrightarrow C_{r+1}(Z, \mathcal{O})$ be the natural inclusion.

The restriction of the perfect pairing above to $\Lambda \otimes C_r(Z, \mathcal{O}) \rightarrow \mathcal{O}$ factors through a perfect pairing

$$\Lambda_r \otimes C_r(Z, \mathcal{O}) \rightarrow \mathcal{O},$$

identifying Λ_r with the algebra of distributions $\text{Hom}_{\mathcal{O}}(C_r(Z, \mathcal{O}), \mathcal{O})$.

Now, fix some critical pair (κ, χ) . Set

$$(183) \quad \phi = \phi_\chi := e_P \circ \int_{Z^P} (\tilde{\chi}_0, \bullet) d\text{Eis}^{[\kappa, \tau]}$$

as linear functional on $C(Z, \mathcal{O})$ valued in $\mathcal{V}_3^{P\text{-ord}}$.

Let $\rho = (\rho_\sigma)_\sigma$ and $\rho^v = (\rho_\sigma^v)_\sigma$ be as in (152). We identify ρ and ρ^v with p -adic weights of $T_{H_1}(\mathbb{Z}_p)$, as in Section 2.3.3. In fact, in this section, we are mostly concerned with the restriction of ρ and ρ^v to Z , which we still denote ρ and ρ^v respectively by abuse of notation.

For any $r \geq 0$, consider the subset $C_r(Z, \mathcal{O}) \cdot \rho^v \subset C(Z, \mathcal{O})$. By [EHLS20, Lemma 7.4.2], the measure $\phi = \phi_\chi$ on Z is equivalent to a collection $\phi_{\chi, \rho} = \phi_\rho = (\phi_{\rho, r})_{r \geq 0}$, such that

$$\phi_{\rho, r} \in \text{Hom}_{\Lambda}(C_r(Z, \mathcal{O}) \cdot \rho^v, \mathcal{V}_3^{P\text{-ord}}) \quad \text{and} \quad \eta_r^*(\phi_{\rho, r+1}) = \phi_{\rho, r},$$

where the equivalence is given by $\phi(\psi) = \phi_{r, \rho}(\psi \cdot \rho^v)$ for all $\psi \in C_r(Z, \mathcal{O})$. For χ fixed, ρ and κ determine one another, hence we sometimes write $\phi_{\chi, \rho}$ by $\phi_{\chi, \kappa}$.

Let $\mathcal{I}_{\rho, r} \subset \Lambda$ be the annihilator of $C_r(Z, \mathcal{O}) \cdot \rho^v$ with respect to the pairing $\Lambda \otimes C(Z, \mathcal{O}) \rightarrow \mathcal{O}$, and let $\Lambda_{\rho, r} = \Lambda/\mathcal{I}_{\rho, r}$. By definition, $\Lambda_{\rho, r}$ is identified with $\text{Hom}_{\mathcal{O}}(C_r(Z, \mathcal{O}) \cdot \rho^v, \mathcal{O})$.

As explained at the end of Section 11.3.1, we see that for all κ very regular and $\psi \in C_r(Z, \mathcal{O})$, we have

$$\phi_{\rho, r}(\psi) \in \text{Hom}_{\mathcal{O}}(\widehat{S}_{\kappa, V}^{P\text{-ord}}(K_{1, r}, [\tau]; \mathcal{O})_{\mathfrak{m}}, S_{\kappa^b, -V}^{P\text{-ord}}(K_{2, r}, [\tau^b]; \mathcal{O})_{\mathfrak{m}} \otimes (\chi \circ \det)),$$

where $\mathfrak{m} = \mathfrak{m}_\pi$ is as in Remark 8.10.

In fact, it follows from the work of [Gar84, GPSR87] on the Garrett map, see [EHLS20, Theorem 9.1.3–Corollary 9.1.4], that the measure $\phi_{\rho, r}$ satisfies a stronger equivariance property with respect to the appropriate Hecke algebra, namely

$$\phi_{\rho, r}(\psi) \in \text{Hom}_{\mathbb{T}_{K_r, \kappa, [\tau], \mathcal{O}}}(\widehat{S}_{\kappa, V}^{P\text{-ord}}(K_{1, r}, [\tau]; \mathcal{O})_{\mathfrak{m}}, S_{\kappa^b, -V}^{P\text{-ord}}(K_{2, r}, [\tau^b]; \mathcal{O})_{\mathfrak{m}} \otimes (\chi \circ \det)),$$

where $\mathbb{T}_{K_r, \kappa, [\tau], \mathcal{O}}$ is as in Proposition 8.19, for all κ very regular and $\psi \in C_r(Z, \mathcal{O})$.

To lighten notation, we omit κ and $[\tau]$ from our notation momentarily (as they do not vary in this section), and write

$$\widehat{S}_{r, V, \pi}^{P\text{-ord}} := \widehat{S}_{\kappa, V}^{P\text{-ord}}(K_{1, r}, [\tau]; \mathcal{O})_{\mathfrak{m}} \quad \text{and} \quad S_{r, -V, \pi^b}^{P\text{-ord}} := S_{\kappa^b, -V}^{P\text{-ord}}(K_{2, r}, [\tau^b]; \mathcal{O})_{\mathfrak{m}},$$

both modules over $\mathbb{T}_r := \mathbb{T}_{K_r, \kappa, [\tau], \mathcal{O}}$ using Lemma 8.3.

By definition of the finite free \mathcal{O} -modules $\widehat{I}_\pi = I_{\pi^b}$, we have isomorphisms

$$\mathbb{T}_r \otimes \widehat{I}_\pi \xrightarrow{\sim} \widehat{S}_{r, V, \pi}^{P\text{-ord}} \quad \text{and} \quad \widehat{\mathbb{T}}_r \otimes I_{\pi^b} \xrightarrow{\sim} S_{r, -V, \pi^b}^{P\text{-ord}},$$

see (126) and (the \mathcal{O} -dual) of (130), where $\widehat{\mathbb{T}}_r$ denotes the \mathcal{O} -dual of \mathbb{T}_r . Therefore, tensoring with $(\chi^{-1} \circ \det)$, we obtain

$$(184) \quad \text{Hom}_{\mathbb{T}_r}(\widehat{S}_{\kappa, V, \pi}^{P\text{-ord}}, S_{\kappa^b, -V, \pi^b}^{P\text{-ord}} \otimes (\chi \circ \det)) \xrightarrow{\sim} \text{Hom}_{\mathbb{T}_r}(\mathbb{T}_r \otimes \widehat{I}_\pi, \widehat{\mathbb{T}}_r \otimes I_{\pi^b}),$$

and setting $C_r = C_r(Z, \mathcal{O})$, we can then identify $\phi_{\rho, r}$ as an element of

$$\begin{aligned} \text{Hom}_{\mathbb{T}_r}(C_r \otimes_{\Lambda} \widehat{S}_{\kappa, V, \pi}^{P\text{-ord}}, S_{\kappa^b, -V, \pi^b}^{P\text{-ord}}) &\xrightarrow{\sim} \text{Hom}_{\mathbb{T}_r}(C_r \otimes_{\Lambda} \mathbb{T}_r \otimes_{\mathcal{O}} \widehat{I}_\pi, \widehat{\mathbb{T}}_r \otimes_{\mathcal{O}} I_{\pi^b}) \\ &= \text{Hom}_{\mathbb{T}_r}(C_r \otimes_{\Lambda} \mathbb{T}_r, \widehat{\mathbb{T}}_r) \otimes_{\mathcal{O}} \text{End}_{\mathcal{O}}(I_{\pi^b}) \\ &\xrightarrow{\sim} \widehat{\mathbb{T}}_r \otimes \text{End}_{\mathcal{O}}(I_{\pi^b}), \end{aligned}$$

and the next step is to study the compatibility on both sides as $r \gg 0$ varies.

To understand the left-hand side of the above as r varies, consider the inclusions $\eta_r : C_r \hookrightarrow C_{r+1}$ and $\iota_r : S_{r, V}^{P\text{-ord}} \hookrightarrow S_{r+1, V}^{P\text{-ord}}$, as well as the dual maps ι_r^* and η_r^* respectively, for all $r \gg 0$. Then, as explained in [EHLS20, Fact 7.4.7], we have

$$(185) \quad (\eta_r^* \otimes \text{id}_{r+1})(\phi_{\rho, r+1}) = \iota_r \circ \phi_{\rho, r} \circ (\text{id}_{C_r} \otimes \iota_r^*).$$

Furthermore, it follows from our work in Sections 2.7.2–2.7.3 that the map $\iota_r^* : \widehat{S}_{\kappa, V}^{P\text{-ord}}(K_{r+1}, \mathcal{O}_\pi) \rightarrow \widehat{S}_{\kappa, V}^{P\text{-ord}}(K_{r+1}, \mathcal{O}_\pi)$ is given by the trace map

$$t_r(h) = \frac{\#(I_{P, r}^0 / I_{P, r})}{\#(I_{P, r+1}^0 / I_{P, r+1})} \sum_{\gamma \in K_{P, r} / K_{P, r+1}} \gamma \cdot h,$$

for all $h \in \widehat{S}_{\kappa, V}^{P\text{-ord}}(K_{r+1}, \mathcal{O}_\pi)$. Comparing this with first commutative diagram in Proposition 8.25, we obtain the following result.

Proposition 12.1. *Let (κ, χ) be a critical pair such that κ is a very regular weight and assume Conjectures 8.12 and 8.17. Let ρ be the weight determined by (κ, χ) in (152). With respect to the identification*

$$\text{Hom}_{\mathbb{T}_r}(C_r \otimes_{\Lambda} \widehat{S}_{\kappa, V, \pi}^{P\text{-ord}}, S_{\kappa^b, -V, \pi^b}^{P\text{-ord}}) \xrightarrow{\sim} \widehat{\mathbb{T}}_r \otimes \text{End}_{\mathcal{O}}(I_{\pi^b})$$

the identity (185), the isomorphism $G_r : \widehat{\mathbb{T}}_r \xrightarrow{\sim} \mathbb{T}_r$ provided by Hypothesis 8.23, the collection $\phi_{\chi, \kappa} = \phi_{\rho} = (\phi_{\rho, r})_r$ defines an element

$$L(\phi_{\chi, \kappa}) = L(\phi_{\rho}) \in \varprojlim_r \mathbb{T}_r \otimes \text{End}_{\mathcal{O}}(I_{\pi^b}) \xrightarrow{\sim} \mathbb{T} \otimes \text{End}_{\mathcal{O}}(I_{\pi^b}).$$

Moreover, if κ' is another very regular weight in the same P -parallel lattice as κ , i.e. $[\kappa] = [\kappa']$, then $L(\phi_{\chi, \kappa}) = L(\phi_{\chi, \kappa'})$ as elements of

$$\mathbb{T} = \mathbb{T}_{K^p, [\kappa, \tau], \mathcal{O}} \xrightarrow{\sim} \varprojlim_r \mathbb{T}_{K_r, \kappa, [\tau], \mathcal{O}} \xrightarrow{\sim} \varprojlim_r \mathbb{T}_{K_r, \kappa', [\tau], \mathcal{O}}.$$

Therefore, $(\phi_{\chi, \kappa, r})_r$ and $(\phi_{\chi, \kappa', r})_r$ define the same element $L(\phi_{\chi, [\kappa]}) \in \mathbb{T} \otimes \text{End}_{\mathcal{O}}(I_{\pi^b})$. Conversely, any $L \in \mathbb{T} \otimes \text{End}_{\mathcal{O}}(I_{\pi^b})$ is induced as above from a collection $(\phi_{\chi, \kappa, r})_r$ associated to some very regular κ .

Remark 12.2. This is the P -ordinary analogue of [Ehls20, Proposition 7.4.10] and is proved the exact same way, namely unfolding definitions using the identifications introduced in this section.

Lastly, to consider the variation of χ as a character of X_p , consider the Iwasawa algebra $\Lambda_{X_p} = \mathbb{Z}_p[[X_p]]$. Then, it follows from [Ehls20, Proposition 7.4.13] and Proposition 12.1 that the \mathcal{V}_3 -valued measure $d\text{Eis}^{[\kappa, \tau]} = \phi = \phi_{\bullet}$ on $X_p \times \mathbb{Z}_p$ corresponds to an element

$$(186) \quad L(\text{Eis}^{[\kappa, \tau]}) \in \Lambda_{X_p} \widehat{\otimes} \mathbb{T} \otimes \text{End}_{\mathcal{O}}(I_{\pi^b}).$$

12.1.2. *Evaluation at classical points.* Let π be a anti-holomorphic P -anti-ordinary automorphic representation π for G_1 of P -anti-WLT (κ, K_r, τ) . Let $\lambda_{\pi} : \mathbb{T} \rightarrow \mathcal{O}_{\pi}$ be its associated Hecke character, see Section 8.3.1, and let \mathfrak{m}_{π} be the kernel of λ_{π} . Consider the set of classical points $\mathcal{S}(K^p, \pi)$ defined in (125).

Note that for any \mathcal{O} -algebra R ,

$$\text{End}_R(I_{\pi^b}) = \text{Hom}(\widehat{I}_{\pi}, I_{\pi^b}) \cong \text{Hom}(\widehat{I}_{\pi} \otimes \widehat{I}_{\pi^b}, R)$$

so given any test vectors $\varphi \in \widehat{I}_{\pi}$ and $\varphi^b \in \widehat{I}_{\pi^b}$ as in Section 8.4.4, we can define

$$L(\text{Eis}^{[\kappa, \tau]}; \varphi, \varphi^b) := [L(\text{Eis}^{[\kappa, \tau]}), \varphi \otimes \varphi^b]_{\text{loc}} \in \Lambda_{X_p} \widehat{\otimes} \mathbb{T},$$

and

$$L(\text{Eis}^{[\kappa, \tau]}, \chi, \kappa; \varphi, \varphi^b) := [L(\phi_{\chi, [\kappa]}), \varphi \otimes \varphi^b]_{\text{loc}} \in \mathbb{T},$$

where $[\bullet, \bullet]_{\text{loc}}$ is induced from the tautological pairing in both cases (abusing notation), and we recall that the relation between $d\text{Eis}^{[\kappa, \tau]}$ and $\phi_{\chi, [\kappa]}$ is given by (183) and Proposition 12.1.

Given R -valued character $\widehat{\chi}_0 : X_p \rightarrow R$ and any classical $\pi' \in \mathcal{S}(K^p, \pi)$ of P -anti-WLT $(\kappa', K_{r'}, \tau')$ such that (κ', χ) is critical and $\lambda_{\pi'}$ is R -valued, the image of $L(\text{Eis}^{[\kappa, \tau]}; \varphi, \varphi^b)$ under the homomorphism $\widehat{\chi}_0 \otimes \lambda_{\pi'} : \Lambda_{X_p, R} \otimes \mathbb{T}_{\pi, R} \rightarrow R$ induced by $(\widehat{\chi}_0, \lambda_{\pi'})$ is equal to

$$\lambda_{\pi'}(L(\text{Eis}^{[\kappa, \tau]}, \chi, \kappa; \varphi, \varphi^b)) \in R$$

and our computations at the end of Section 11.3.1 show that the latter is equal to the expression in (182).

12.2. **Normalized periods and congruence ideals.** We are now ready to state our main theorem to summarize the construction of the p -adic L -function in (186). However, we first adjust the definitions of certain periods studied in [Ehls20, Section 6.7] to generalize the theory to P -anti-ordinary representation.

Fix an anti-holomorphic P -anti-ordinary automorphic representation π on $G = G_1$ with P -anti-WLT (κ, K_r, τ) . In what follows, we use the notation of Sections 8.2-8.3 freely.

Consider the orthogonal complement

$$\widehat{S}_{\kappa, V}^{P\text{-a.ord}}(K_r, \tau; R)[\pi]^\perp \subset \widehat{S}_{\kappa^b, -V}^{P\text{-a.ord}}(K_r^b, \tau^b; R)_{\pi^b}$$

of $\widehat{S}_{\kappa, V}^{P\text{-a.ord}}(K_r, \tau; R)[\pi]$ with respect to $\frac{1}{\text{Vol}(I_{V,r}^0 \cap I_{-V,r}^0)} \langle \cdot, \cdot \rangle_{\kappa, \tau}^{\text{Ser}}$, see Lemma 8.11 and Section 4.2.3

Definition 12.3. The congruence ideal $C(\pi) \subset R$ associated to π is the annihilator of

$$\widehat{S}_{\kappa^b, -V}^{P\text{-a.ord}}(K_r^b, \tau^b; R)_{\pi^b} / \left(\widehat{S}_{\kappa, V}^{P\text{-a.ord}}(K_r, \tau; R)[\pi]^\perp + \widehat{S}_{\kappa^b, -V}^{P\text{-a.ord}}(K_r^b, \tau^b; R)[\pi^b] \right)$$

Lemma 12.4. Let $R \subset \mathbb{C}$ be a ring as in Proposition 8.8, then

$$L[\pi] := \frac{1}{\text{Vol}(I_{V,r}^0 \cap I_{-V,r}^0)} \langle \widehat{S}_{\kappa, V}^{P\text{-a.ord}}(K_r, \tau; R)[\pi], \widehat{S}_{\kappa^b, -V}^{P\text{-a.ord}}(K_r^b, \tau^b; R)[\pi^b] \rangle_{\kappa, \tau}^{\text{Ser}}, \text{ and}$$

$$L_\pi := \frac{1}{\text{Vol}(I_{V,r}^0 \cap I_{-V,r}^0)} \langle \widehat{S}_{\kappa, V}^{P\text{-a.ord}}(K_r, \tau; R)[\pi], \widehat{S}_{\kappa^b, -V}^{P\text{-a.ord}}(K_r^b, \tau^b; R)_{\pi^b} \rangle_{\kappa, \tau}^{\text{Ser}}$$

are rank one R -submodules of \mathbb{C} , generated by positive real numbers $Q[\pi]$ and Q_π , respectively. Any $c(\pi) \in R$ such that $c(\pi)Q_\pi = Q[\pi]$ generates the congruence ideal $C(\pi)$.

Obviously, $Q[\pi]$, Q_π and $c(\pi)$ are only well-defined up to units in R . However, our p -adic L -function does not depend on those choices. Furthermore, given a Hecke character χ , one has analogues $Q[\pi, \chi]$, $Q_{\pi, \chi}$, $C(\pi, \chi)$ and $c(\pi, \chi)$ upon twisting by $\chi^{-1} \circ \det$ as explained in [Ehls20, Section 6.7.6].

Proposition 12.5. Given anti-holomorphic P -anti-ordinary test vectors $\varphi \in \widehat{I}_\pi$ and $\varphi^b \in \widehat{I}_{\pi^b}$ as in Section 8.4.4, the period

$$\Omega_{\pi, \chi}(\varphi, \varphi^b) = \frac{\dim \tau \cdot \langle \varphi, \varphi^b \rangle_\chi}{\text{Vol}(I_{r, V}^0 \cap I_{r, -V}^0) \cdot Q[\pi, \chi]}$$

is independent of r and is p -integral. It is a p -adic unit for an appropriate choice of φ and φ^b .

Proof. The independence on r follows from the properties of the Serre pairing and φ^b under the trace map as r increases.

Furthermore, the fact that it is p -integral (resp. a p -adic unit) follows from the fact that the factor $\dim \tau$ in the above expression cancels with the factor $\dim \tau$ in the definition of $Q[\pi, \chi]$. \square

12.3. Statement of the main theorem. Our main theorem is simply a summary of the properties of the p -adic L -function constructed in (186) and incorporate the periods introduced above.

In the following statement, we refer to the Conjectures 5.3, 5.5, 5.10, 8.12 and 8.17 as the “standard conjectures of P -ordinary Hida theory”.

Theorem 12.6. *Let π be an anti-holomorphic, P -anti-ordinary cuspidal automorphic form $G_1(\mathbb{A})$ whose P -anti-WLT is (κ, K_r, τ) , where τ is the SZ-type of π . Assume that the standard conjectures of P -ordinary Hida theory hold. Assume that π satisfy Hypothesis 6.4, Hypothesis 8.5, Hypothesis 8.23, and Proposition-Hypothesis 8.25.*

Let ω_τ denote the central character of τ . Let \mathfrak{m}_π denote the maximal ideal of the P -ordinary Hecke algebra $\mathbf{T}_{K_r, \kappa, \tau}^{P\text{-a.ord}}$ corresponding to π and let \mathbb{T}_π be the localization of $\mathbf{T}_{K_r, \kappa, \tau}^{P\text{-a.ord}}$ at \mathfrak{m}_π .

Given test vectors $\varphi \in \widehat{I}_\pi$, $\varphi^b \in \widehat{I}_{\pi^b}$ as in Section 8.4.4, there exists a unique element

$$L(\text{Eis}^{[\kappa, \tau]}, P\text{-ord}; \varphi \otimes \varphi^b) \in \Lambda_{X_p, R} \widehat{\otimes} \mathbb{T}_\pi$$

satisfying the following property :

Let $\chi = \|\cdot\|^{\frac{n-k}{2}} \chi_u : X_p \rightarrow R^\times$ be the p -adic shift of a Hecke character as in Section 11.2.5. Let $\pi' \in \mathcal{S}(K^p, \pi)$ be a classical point of the P -ordinary Hida family \mathbb{T}_π .

Then, $L(\text{Eis}^{[\kappa, \tau]}, P\text{-ord}; \varphi \otimes \varphi^b)$ is mapped under the character $\chi \otimes \lambda_{\pi'}$ to

$$\begin{aligned} & c(\pi', \chi) \Omega_{\pi', \chi}(\varphi, \varphi^b) L_p \left(\frac{k-n+1}{2}, P\text{-ord}, \pi', \chi_u \right) \\ & \times L_\infty \left(\frac{k-n+1}{2}; \chi_u, \kappa' \right) I_S \frac{L^S \left(\frac{k-n+1}{2}, \pi', \chi_u \right)}{P_{\pi', \chi}}, \end{aligned}$$

where $P_{\pi', \chi} = Q_{\pi', \chi}^{-1}$.

REFERENCES

- [BC09] J. Bellaïche and G. Chenevier, *Families of galois representations and selmer groups*, Astérisque, vol. 324, Soc. Math. France, Paris, 2009.
- [BHR94] D. Blasius, M. Harris, and D. Ramakrishnan, *Coherent cohomology, limits of discrete series, and Galois conjugation*, Duke Math. J. **73** (1994), no. 3, 647–685.
- [BK93] C. J. Bushnell and P. C. Kutzko, *The admissible dual of $GL(N)$ via compact open subgroups*, Annals of Mathematics Studies, vol. 129, Princeton University Press, 1993.
- [BK98] ———, *Smooth representations of reductive p -adic groups : Structure theory via types*, Proceedings of the London Mathematical Society **77** (1998), no. 3, 582–634.
- [BK99] ———, *Semisimple types in $GL(n)$* , Compositio Mathematica **119** (1999), 57–106.
- [Cas95] W. Casselman, *Introduction to the Theory of Admissible Representations of p -adic Reductive Groups*, Unpublished manuscript. <https://personal.math.ubc.ca/~cass/research/pdf/p-adic-book.pdf>, 1995, 80 pages.
- [CEF⁺16] A. Caraiani, E. Eischen, J. Fintzen, E. Mantovan, and I. Varma, *p -adic q -expansion Principles on Unitary Shimura Varieties*, Directions in Number Theory, Association for Women in Mathematics Series, vol. 3, Springer International Publishing, Cham, 2016, pp. 197–243.
- [Coa89] J. Coates, *On p -adic L -functions attached to motives over \mathbf{Q} . ii*, Bol. Soc. Brasil. Mat. (N.S.) **20** (1989), no. 1, 101–112.

- [EFMV18] E. Eischen, J. Fintzen, E. Mantovan, and I. Varma, *Differential operators and families of automorphic forms on unitary groups of arbitrary signature*, Doc. Math. **23** (2018), 445–495.
- [EHLS20] E. Eischen, M. Harris, J.-S. Li, and C. Skinner, *p -adic L -functions for unitary groups*, Forum of Mathematics, Pi **8** (2020), e9.
- [Eis12] E. Eischen, *p -adic differential operators on automorphic forms on unitary groups*, Ann. Inst. Fourier (Grenoble) **62** (2012), no. 1, 177–243.
- [Eis14] ———, *A p -adic Eisenstein measure for vector-weight automorphic forms*, Algebra Number Theory **8** (2014), no. 10, 2433–2469.
- [Eis15] ———, *A p -adic Eisenstein measure for unitary groups*, J. Reine Angew. Math. **699** (2015), 111–142.
- [EL20] E. Eischen and Z. Liu, *Archimedean Zeta Integrals for Unitary Groups.*, arXiv:2006.04302 [math.NT], June 2020, Preprint available at [arxiv:2006.04302](https://arxiv.org/abs/2006.04302).
- [Gar84] P. B. Garrett, *Pullbacks of Eisenstein series; applications*, Automorphic Forms of Several Variables (Katata, 1983), Progress in Mathematics, vol. 46, Birkhäuser, Boston, MA, 1984, pp. 114–137.
- [GPSR87] S. Gelbart, I. Piatetski-Shapiro, and S. Rallis, *Explicit Constructions of Automorphic L -functions*, Lecture Notes in Mathematics, vol. 1254, Springer, Berlin, 1987.
- [Har86] M. Harris, *Arithmetic vector bundles and automorphic forms on Shimura varieties. II*, Compositio Math. **60** (1986), no. 3, 323–378.
- [Har90] ———, *Automorphic forms of $\bar{\partial}$ -cohomology type as coherent cohomology classes*, J. Differential Geom. **32** (1990), no. 1, 1–63.
- [Har97] ———, *L -functions and periods of polarized regular motives*, J. Reine Angew. Math. **483** (1997), 75–161.
- [Har08] ———, *A simple proof of rationality of Siegel–Weil Eisenstein series*, Eisenstein Series and Applications, Progress in Mathematics, vol. 258, Birkhäuser, Boston, MA, 2008, p. 149–185.
- [Hid98] H. Hida, *Automorphic induction and Leopoldt type conjectures for $GL(n)$* , Asian J. Math. **2** (1998), no. 4, 667–710, Mikio Sato: a great Japanese mathematician of the twentieth century.
- [Hid04] ———, *p -adic automorphic forms on Shimura varieties*, Springer Monographs in Mathematics, Springer, New York, NY, 2004.
- [HLLM23] B. L. Hung, D. Le, B. Levin, and S. Morra, *Local models for Galois deformation rings and applications*, Inventiones mathematicae **231** (2023), 1277–1488.
- [HLS06] M. Harris, J.S. Li, and C. Skinner, *p -adic L -functions for unitary Shimura varieties, I: Construction of the Eisenstein Measure*, Doc. Math. **Extra Vol.** (2006), 393–464, (electronic).
- [Jac79] H. Jacquet, *Principal L -functions of the linear group*, in Automorphic Forms, Representations and L -functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proceedings of Symposia in Applied Mathematics, XXXIII, (American Mathematical Society, Providence, RI), 1979, pp. 63–86.
- [Jan03] J. C. Jantzen, *Representations of Algebraic Groups*, second ed., Mathematical surveys and monographs, vol. 107, American Mathematical Society, Providence, RI, 2003.
- [Kat78] N. M. Katz, *p -adic L -functions for CM fields*, Invent. Math. **49** (1978), no. 3, 199–297.
- [Kot92] R. E. Kottwitz, *Points on Some Shimura Varieties Over Finite Fields*, J. Amer. Math. Soc. **5** (1992), no. 2, 373–444.
- [Lan12] K.-W. Lan, *Comparison between analytic and algebraic constructions of toroidal compactifications of PEL-type Shimura varieties*, Journal für die reine und angewandte Mathematik (Crelles Journal) **664** (2012), 163–228.

- [Lan13] ———, *Arithmetic Compactifications of PEL-type Shimura Varieties*, London Mathematical Society Monographs, vol. 36, Princeton University Press, Princeton, 2013.
- [Lan16] ———, *Higher Koecher's principle*, Math. Res. Lett. **23** (2016), no. 1, 163–199.
- [Li92] J.-S. Li, *Nonvanishing theorems for the cohomology of certain arithmetic quotients*, J. Reine Angew. Math. **428** (1992), 177–217.
- [LR20] Z. Liu and G. Rosso, *Non-cuspidal Hida theory for Siegel modular forms and trivial zeros of p -adic L -functions*, Math. Ann. **378** (2020), 153–231.
- [Pil12] V. Pilloni, *Sur la théorie de Hida pour le groupe $GS_{p,2g}$* , Bull. Soc. Math. France **140** (2012), no. 3, 335–400.
- [Ren10] D. Renard, *Représentations des groupes réductifs p -adiques*, Collection SMF. Cours spécialisés, vol. 17, Société Mathématique de France, Paris, France, 2010.
- [Shi83] G. Shimura, *On Eisenstein Series*, Duke Math. J. **50** (1983), no. 2, 417–476.
- [Shi97] ———, *Euler Products and Eisenstein Series*, CBMS Regional Conference Series in Mathematics, vol. 93, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1997.
- [SU02] D. Skinner and E. Urban, *Sur les déformations p -adiques des formes de Saito-Kurokawa*, C. R. Math. Acad. Sci. Paris **335** (2002), no. 7, 581–586.
- [SZ99] P. Schneider and E. W. Zink, *K -types for the tempered components of a p -adic general linear group*, J. Reine Angew. Math. **517** (1999), 161–208.

DAVID MARCIL, DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, 2990 BROADWAY,
NEW YORK, NY 10027, USA

Email address: `dmarcil@math.columbia.edu`