Categorifying Jacobi-Trudi

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Link

These slides can be found on my webpage:

math.columbia.edu/~fanzhou/files/beamer-AMS2025-JT.pdf

A Tale of Two Cities

symmetric functions $\longleftrightarrow$ symmetric group representations

The classical story – symmetric functions

We can consider two families of symmetric functions:

- Let Schur functions be s_λ
- and let complete homogeneous functions be $h_\alpha = h_{\alpha_1} \cdots h_{\alpha_k}$, where h_i is the sum of all monomials of degree i.

Jacobi-Trudi

The Jacobi-Trudi determinant identity:

$$s_\lambda = \det(h_{\lambda_i - i + j})_{i,j} = \det \begin{pmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \cdots & h_{\lambda_1+\ell-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & \cdots & \vdots \\ & & \ddots & \\ h_{\lambda_\ell-\ell+1} & \cdots & h_{\lambda_\ell-1} & h_{\lambda_\ell} \end{pmatrix},$$

This is an alternating sum.

The classical story – symmetric groups

- Two families of modules over S_n:
- (Over $\mathbb{C}$,) "Specht modules" Σ_λ exhaust irreducibles of S_n.
- Given a composition α and $S_\alpha = S_{\alpha_1} \times \cdots \times S_{\alpha_k}$, the "permutation module" E_α is

$$E_\alpha := \mathrm{Ind}_{S_\alpha}^{S_n} \mathrm{triv}\,.$$

  This has dimension $\dim E_\alpha = \binom{n}{\alpha_1,\cdots,\alpha_k}$.
- This decomposes as

$$E_\lambda = \Sigma_\lambda \oplus \bigoplus_{\mu \rhd \lambda} \Sigma_\mu^{\oplus m_\mu}.$$

Category $\mathcal{O}$

Compare to the JH filtration of Vermas $\Delta_{w\circ 0}$ in which $L_{w\circ 0}$ appears as the top layer quotient, and

$$[\Delta_w] = [L_w] + \sum_{u>w} m_u [L_u].$$

The classical story – functions versus groups

- Over $\mathbb{C}$, $\mathsf{Rep}\, S_n$ is equivalent to symmetric functions via the Frobenius character.
- Letting $z_\lambda = \prod_{i \in \mathbb{Z}_+} i^{m_i} m_i!$ where $m_i = \#\{j : \lambda_j = i\}$, $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}$,

$$\chi(M) = \sum_{\lambda \vdash n} \mathrm{tra}(\lambda|_M) \frac{p_\lambda}{z_\lambda} = \frac{1}{n!} \sum_{w \in S_n} \mathrm{tra}(w|_M) p_{\lambda(w)}.$$

- This sends

$$\chi\colon \bigoplus_n K_0(\mathsf{Rep}\, S_n) \xrightarrow{\sim} \Lambda$$

$$\Sigma_\lambda \longmapsto s_\lambda$$

$$E_\lambda \longmapsto h_\lambda$$

Question

Is there some highest-weight explanation/elucidation for this? More precisely:

Question

Find a quasi-hereditary A with a map $\mathbb{C}S_n \longrightarrow A$ such that

$$\text{standard module} \xmapsto{\text{Res}} \text{permutation module}$$

$$\text{simple module} \xmapsto{\text{Res}} \text{Specht module}$$

Moreover find a BGG resolution over A of simples by standards such that restriction gives a resolution of Spechts by permutations , categorifying

$$s_\lambda = \det(h_{\lambda_i+j-i})_{i,j}.$$

Spoilers

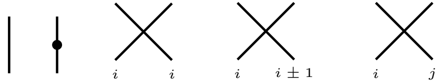
- This is done by considering a quotient of (cyclotomic) KLR.
- $\lambda \rightsquigarrow \alpha, \omega \rightsquigarrow \mathcal{R}_\alpha^\omega$
- Define a quotient $\mathring{\mathcal{R}}_\lambda$ by $\mathcal{R}_\alpha^\omega \twoheadrightarrow \mathring{\mathcal{R}}_\lambda$
- $\mathring{\mathcal{R}}_\lambda$ is quasi-hereditary with weight poset an ideal in $S_{\ell(\lambda)}$.
- The "dominant" simple L_1 will have a BGG resolution by Vermas Δ_w.
- Under $\mathbb{C}S_n \longrightarrow \widehat{\mathcal{H}}_n \longrightarrow \widehat{\mathcal{H}}_\alpha^\omega \xrightarrow{\sim} \mathcal{R}_\alpha^\omega \longrightarrow \mathring{\mathcal{R}}_\lambda$ (Brundan-Kleshchev [BK09]), this becomes a resolution of the Specht module Σ_λ by permutation modules.
- Homological computations are made diagrammatically via Soergel.

Previously

- This topic has been explored before in e.g. works of Zelevinsky, Arakawa-Suzuki, Orellana-Ram.
- Their works port the BGG resolution from category $\mathcal{O}$ to some variant of S_n by using an exact (“Arakawa-Suzuki”) functor.
- Those works inspired this project.
- We would like to elucidate the highest-weight structure ‘natively’.

The KLR algebra – generators

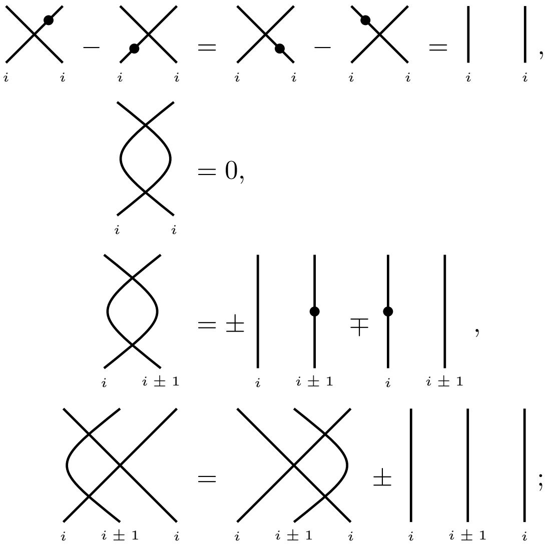
$\mathcal{R} = \bigoplus_\alpha \mathcal{R}_\alpha$. The monoidal generators are



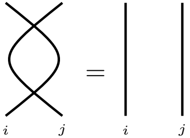
degree $\quad 0 \quad 2 \quad -2 \quad 1 \quad 0$

where $|j - i| > 1$.

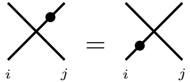
The KLR algebra – relations



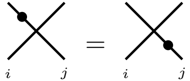
The KLR algebra – relations



for $|i-j|>1$,



for $i \neq j$,



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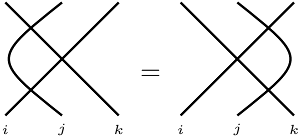


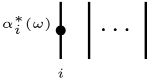
for $(j,k) \neq (i \pm 1, i)$.

The KLR algebra – cyclotomic relation

Given $\omega \in \Lambda_+$,

$$\mathcal{R}_\alpha^\omega := \mathcal{R}_\alpha \big/ \langle y_1^{\alpha^*_{c_1}(\omega)} e_c = 0 \rangle .$$

Diagrammatically:



$= 0.$

$\mathcal{R}_\lambda$

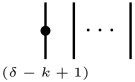
Let ϖ_i be fundamental weights, $\operatorname{cont}\lambda$ be the (multi)set of contents of λ where the top-left box has content δ.
We will let $\mathcal{R}_\lambda = \mathcal{R}^\omega_\alpha$, where

$$\alpha = \sum_{i \in \operatorname{cont}\lambda} \alpha_i$$

and

$$\omega = \varpi_\delta + \varpi_{\delta-1} + \cdots + \varpi_{\delta-\ell(\lambda)+1}.$$

This is requiring



$= 0$

for $1 \leq k \leq \ell(\lambda)$.

Dominance order

For partitions: $\lambda \trianglerighteq \mu$ if

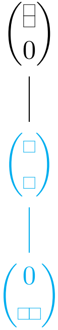
$$\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i \quad \forall\, k.$$

For multi-partitions: $\lambda \trianglerighteq \mu$ if

$$\sum_{j=1}^{m-1} |\lambda^{(j)}| + \sum_{i=1}^{k} \lambda_i^{(m)} \geq \sum_{j=1}^{m-1} |\mu^{(j)}| + \sum_{i=1}^{k} \mu_i^{(m)} \quad \forall\, m, k.$$

Cellular structure

$\mathcal{R}_\alpha^\omega$ is cellular under the dominance order due to Hu-Mathas [HM10].
Details are too involved.
As an example: let $n = \delta = 2$, $\lambda = \begin{smallmatrix}\square\\\square\end{smallmatrix}$, so $\alpha = \alpha_1 + \alpha_2$, $\omega = \varpi_2 + \varpi_1$; let 1 be black and 2 be red.



Cellular structure

$\mathcal{R}^{\omega}_{\alpha}$ is cellular under the dominance order due to Hu-Mathas [HM10]. Details are too involved.
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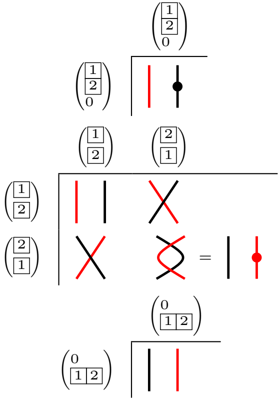
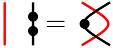


In fact,



$$= 0.$$

We want a quasi-hereditary A, so we want to kill this nilpotent cell.

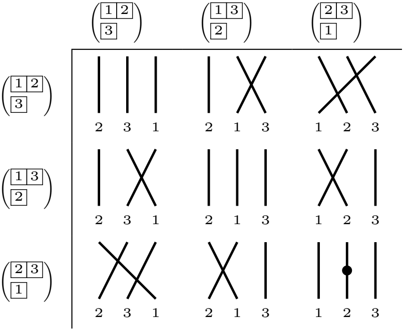
Quotienting the cyclotomic KLR

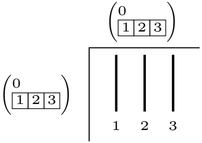
- There is an alternative ordering, the coarsened order of Uglov, on multi-partitions.
- A result of Bowman [Bow17] says the cellular structure of $\mathcal{R}_\lambda$ respects this.
- This justifies the construction

$$\mathring{\mathcal{R}}_\lambda := \mathring{\mathcal{R}} := \mathcal{R}_\lambda \big/ \langle \nu \rangle_{\nu \text{ multi-row}}.$$

- This algebra is very easy to write down explicitly, simply keep the cells labeled by one-row multi-partitions.
- This algebra is quasi-hereditary with poset $W_\lambda = \{w : w \leq w_\lambda\}$.
- It is also equivalent to a quotient of (a principal central block of) category $\mathcal{O}$ for $\mathfrak{sl}_{\ell(\lambda)}$.

Example: $\mathring{\mathcal{R}}_\lambda$ for $\lambda = \ydiagram{2,1}$, $n = 3$, $\delta = 2$





The BGG resolution

Theorem (Z.)

There is a (finite) BGG resolution in $\mathsf{Mod}\,\mathring{\mathcal{R}}_\lambda$, namely a resolution of the simple module L_1 by Vermas,

$$0 \longrightarrow \Delta_{w_\lambda} \longrightarrow \cdots \longrightarrow \bigoplus_{\ell(w)=k} \Delta_w \longrightarrow \cdots \longrightarrow \Delta_1 \longrightarrow L_1 \longrightarrow 0.$$

When restricted to the action of S_n via the map $\mathbb{C}S_n \hookrightarrow \widehat{\mathcal{H}}_n \twoheadrightarrow \widehat{\mathcal{H}}_\alpha^\omega \xrightarrow{\sim} \mathcal{R}_\alpha^\omega \twoheadrightarrow \mathring{\mathcal{R}}_\lambda$, this resolution becomes

$$\cdots \longrightarrow \bigoplus_{\ell(w)=k} E_{w\circ\lambda} \longrightarrow \cdots \longrightarrow E_\lambda \longrightarrow \Sigma_\lambda \longrightarrow 0.$$

Decategorifying this resolution via alternating sum of Frobenius character/cycle index series recovers Jacobi-Trudi:

$$s_\lambda = \det(h_{\lambda_i - i + j})_{i,j}.$$

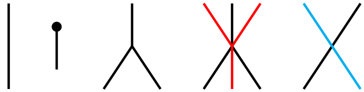
The homological information we need to prove this is

$$\mathrm{Ext}^{\bullet}_{\mathring{\mathcal{R}}_\lambda}(\Delta_w, L_1).$$

We will compute this by using Morita equivalence to a Soergel calculus.

Soergel calculus – generators

Monoidally generated by:

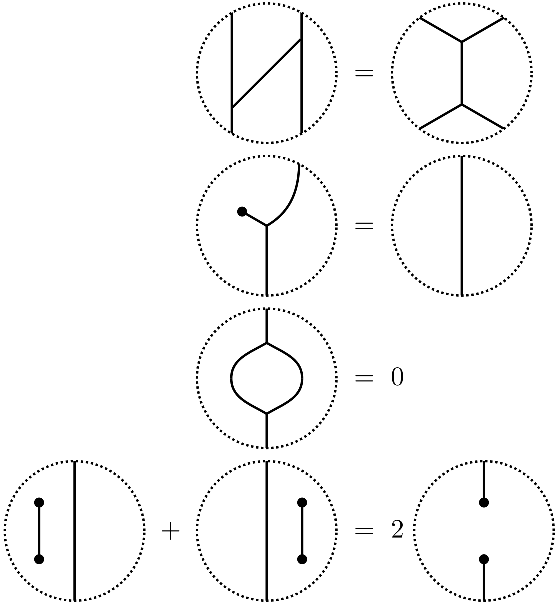


degree 0 $\quad 1 \quad -1 \quad 0 \quad 0$

as well as their upside-down flips.

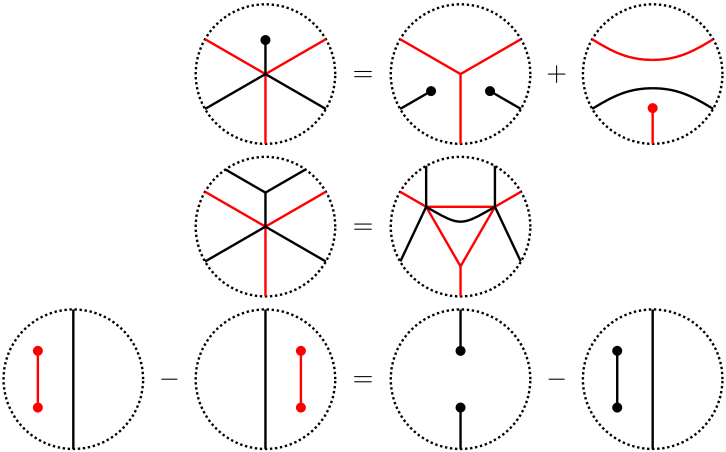
Soergel calculus – relations

1-color:



Soergel calculus – relations

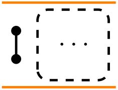
2-color (adjacent):



Distant colors pull apart.

Soergel calculus – cyclotomic relation

The “cyclotomic condition” for Soergel calculus is setting barbells at the far left to zero:

 $= 0$

Our choice for Soergel

Let

$S_\lambda = \{\underline{w} : \ell(\underline{w}) \leq \ell(w_\lambda),\ \underline{w}$ is a subword of some reduced word for $w_\lambda\}$.

We will consider

$$\mathcal{S}_\lambda := \mathbb{C} \otimes_R \operatorname{End}\left(\bigoplus_{\underline{w} \in S_\lambda} \mathrm{BS}_{\underline{w}} \right).$$

In other words we take cyclotomic Soergel calculus with endpoints $\underline{w} \in S_\lambda$.
This is a cellular algebra via the light leaves basis of Elias-Williamson [EW16].
It is Morita-equivalent to $\mathring{\mathcal{R}}_\lambda$:

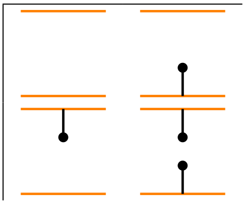
$$\mathsf{Mod}\, \mathring{\mathcal{R}}_\lambda \cong \mathsf{Mod}\, \mathcal{S}_\lambda,$$

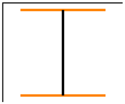
so that

$$\operatorname{Ext}^\bullet_{\mathcal{S}_\lambda}(\Delta_w, L_1) = \operatorname{Ext}^\bullet_{\mathring{\mathcal{R}}_\lambda}(\Delta_w, L_1).$$

Example: $\mathfrak{sl}_2$

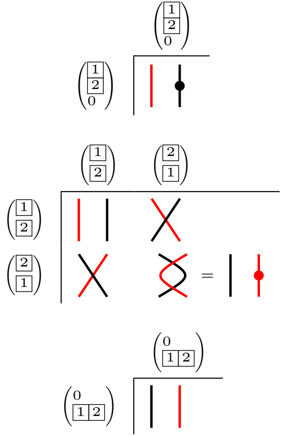
When we set $\lambda = \begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix}$, we get the classical description of category $\mathcal{O}$ for $\mathfrak{sl}_2$:





Cellular structure

Compare this to: let $n = \delta = 2$, $\lambda = \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$, so $\alpha = \alpha_1 + \alpha_2$, $\omega = \varpi_2 + \varpi_1$; let 1 be black and 2 be red.



Koszulity

A graded algebra is "quadratic" if it is generated in degree 1 and relations are generated in degree 2.

Definition

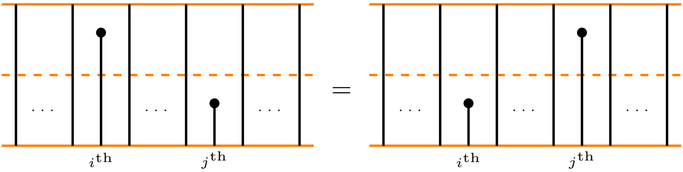
A quadratic graded algebra A ($A_0 = \Bbbk$) is "Koszul" if $\mathrm{Ext}_A(\Bbbk, \Bbbk)$ is nonzero only when the homological degree agrees with the Koszul degree.

This is the "homological concentration" we want.

Cyclotomic Soergel is nil-Koszul

Theorem (Z.)

There is a "lower-half subalgebra" $\mathcal{S}_\lambda^{\mathbf{\dagger}}$ of $\mathcal{S}_\lambda$ generated by lollipops which is essentially a polynomial ring.



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i^{th}
j^{th}

Cyclotomic Soergel is nil-Koszul

Theorem (Z.)

There is a "lower-half subalgebra" $\mathcal{S}_\lambda^{\dot{\uparrow}}$ of $\mathcal{S}_\lambda$ generated by lollipops which is essentially a polynomial ring.

$\mathcal{S}_\lambda^{\dot{\uparrow}}$ is Koszul under the Soergel grading.

Moreover

$$\mathcal{S}_\lambda \overset{\mathsf{L}}{\otimes}_{\mathcal{S}_\lambda^{\dot{\uparrow}}} \Bbbk e^w = \Delta_w,$$

so that

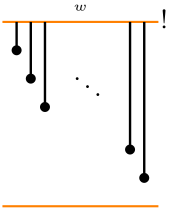
$$\mathsf{RHom}_{\mathcal{S}_\lambda}(\Delta_w, \square) = \mathsf{RHom}_{\mathcal{S}_\lambda^{\dot{\uparrow}}}(\Bbbk e^w, \square).$$

In particular

$$\mathrm{Ext}^\bullet_{\mathcal{S}_\lambda}(\Delta_w, L_1) = \mathrm{Ext}^\bullet_{\mathcal{S}_\lambda^{\dot{\uparrow}}}(\Bbbk e^w, \Bbbk e^1) = e^w \mathcal{S}_\lambda^{\dot{\uparrow},!} e^1 = \Bbbk[-\ell(w)].$$

This is the homological information needed to prove the BGG resolution.

The space $e^w \mathcal{S}_\lambda^{\bullet,!} e^1 = \Bbbk[-\ell(w)]$ is spanned by



Projective resolutions of Vermas

Incidentally the statement that $\mathcal{S}_\lambda \overset{\mathsf{L}}{\otimes}_{\mathcal{S}_\lambda^{\dagger}} \Bbbk e^w = \Delta_w$ gives us a projective resolution of Vermas , using that $\mathcal{S}^{\dagger}$ is essentially a polynomial ring.

- There is an explicit Koszul resolution

$$\mathcal{S}^{\dagger} \otimes_{\mathbb{K}} \mathcal{S}^{\dagger,!,\vee,\bullet} \simeq \Bbbk e^w.$$

- This is a free $\mathcal{S}^{\dagger}$-resolution, so use it to compute the derived tensor.

Projective resolutions of Vermas

Corollary

We have a resolution

$$\mathcal{S} \otimes_{\mathbb{K}} \mathcal{S}^{\lollipop,!,\vee,\bullet} e^w \simeq \Delta_w;$$

the maps are

$$\begin{aligned} \mathrm{d}\colon \mathcal{S} \otimes_{\mathbb{K}} \mathcal{S}_k^{\lollipop,!,\vee} e^w &\longrightarrow \mathcal{S} \otimes_{\mathbb{K}} \mathcal{S}_{k-1}^{\lollipop,!,\vee} e^w \\ x \otimes P(\lollipop_1 e^w, \cdots, \lollipop_n e^w) &\longmapsto \sum_{\text{ways to write } P = \lollipop_i \cdot P'} x \cdot \lollipop_i \otimes P', \end{aligned}$$

where $P \in \mathcal{S}_k^{\lollipop,!,\vee}$ is a degree k anti-commutative polynomial in the lollipops.

Proof strategy

1. First key idea: "weight theory" $\longrightarrow$ stratification of the module category $\longrightarrow$ "filtration" of the identity functor $\longrightarrow$ spectral sequence converging to any object.
   - In particular, we can apply this to simple objects.
   - Terms of this spectral sequence involve homological information, in the form of certain Ext groups.
   - Concentration of these Ext groups (cf. "Kostant modules") imply a "BGG resolution".
   - This idea is due to Gaitsgory, Ayala-Mazel-Gee-Rozenblyum [AMGR22], and Dhillon [Dhi19].
2. Second key idea: This homological information can be computed using Koszul methods.

Slogan

Koszulity of half of A is intimately connected to BGG resolutions.

- Then we can use a naive resolution to compute these Ext groups.
- The spectral sequence is a resolution for modules which are Koszul over half of A.

Thank you!

Thank you for coming to my talk!
Questions

Algebraic recollement

- The stratification will have recollements on each level. Details unimportant.
- By the $\mathsf{D}\,\mathsf{Mod}\,A/AeA \longrightarrow \mathsf{D}\,\mathsf{Mod}\,A \longrightarrow \mathsf{D}\,\mathsf{Mod}\,eAe$ setup of [CPS88], set $A = A^{\geq\theta}$ and $e = e^\theta$ to get:

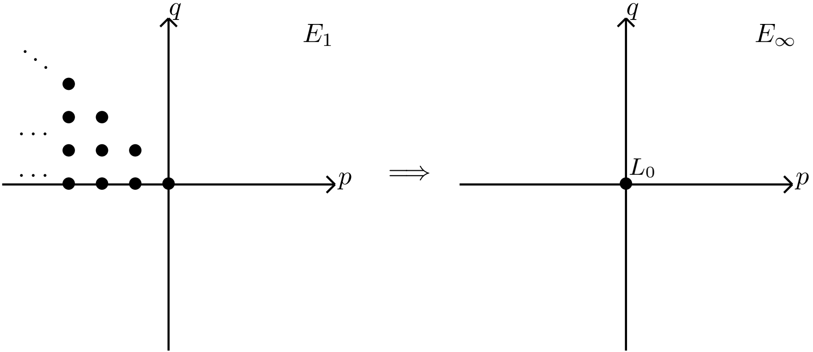
$$\begin{tikzcd}[column sep=huge]
\mathsf{D}^-\ \mathsf{Mod}\ A^{>\theta} \arrow[r, "\iota_\theta"] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, phantom, shift right=4, "\perp"] & \mathsf{D}^-\ \mathsf{Mod}\ A^{\geq\theta} \arrow[l, bend right=60, "\iota_\theta^* = A^{>\theta} \overset{\mathsf{L}}{\otimes}_{A^{\geq\theta}} \square"'] \arrow[l, bend left=60, "\iota_\theta^! = \bigoplus_i \mathrm{RHom}_{A^{\geq\theta}}(A^{>\theta} 1^i, \square)"] \arrow[r, "\jmath^\theta = e^\theta \square"] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, phantom, shift right=4, "\perp"] & \mathsf{D}^-\ \mathsf{Mod}\ A^\theta \arrow[l, bend right=60, "\jmath_!^\theta = A^{\geq\theta} e^\theta \otimes_{A^\theta} \square"'] \arrow[l, bend left=60, "\jmath_*^\theta = \bigoplus_i \mathrm{Hom}_{A^\theta}(e^\theta A^{\geq\theta} 1^i, \square)"]
\end{tikzcd}$$

Stratification and spectral sequences

- Let Θ be sufficiently nice.
- There is a spectral sequence (functorial in the input $\square$)

$$E_1^{p,q} = \bigoplus_{\ell(\theta)=-p} \Delta(\theta) \otimes_{A^\theta} \mathrm{Ext}_A^{-(p+q)}(\Delta(\theta), \square^\dagger)^* \implies E_\infty^{p,q} = \mathrm{gr}^{-p} H^{p+q}(\square),$$

where $\Delta(\theta) \coloneqq \bigoplus_{\lambda \in \theta} \Delta_\lambda^{\overline{l_\lambda(\theta)}} = A^{\geq\theta} e^\theta$. $\deg \mathrm{d}_r = (r, 1-r)$:



Remark: Cf. Koszul duality.

- To obtain a resolution, we need the Ext groups to be concentrated in certain degrees.
- Idea: Koszul objects have good Ext concentration properties.

Definition

A quadratic graded algebra A ($A_0 = \Bbbk$) is "Koszul" if $\mathrm{Ext}_A(\Bbbk, \Bbbk)$ in nonzero only when the homological degree agrees with the Koszul degree.

Example: category $\mathcal{O}$ for $\mathfrak{sl}_2$

- It is classical that L_0 has concentrated Ext groups:

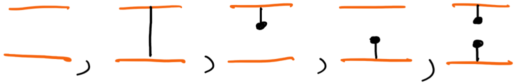
$$\mathrm{Ext}^0(\Delta_0, L_0) = \mathbb{C}, \qquad \mathrm{Ext}^0(\Delta_{-2}, L_0) = 0$$

$$\mathrm{Ext}^1(\Delta_0, L_0) = 0, \qquad \mathrm{Ext}^1(\Delta_{-2}, L_0) = \mathbb{C}.$$

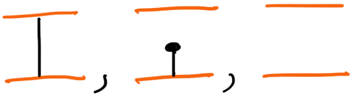
- Then the spectral sequence above exactly recovers the BGG resolution.
- Remark: It can also recover the standard filtration of projectives.

The algebra controlling this

- Recall that the principal block of $\mathcal{O}(\mathfrak{sl}_2)$ is Morita equivalent to the 5-dimensional algebra $A_{\mathfrak{sl}_2}$ spanned by



- The subalgebra $A^-_{\mathfrak{sl}_2}$ of this spanned by



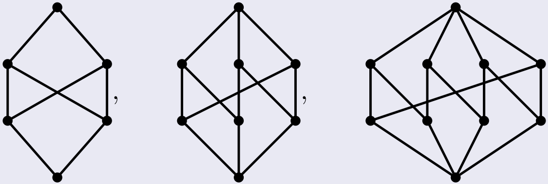
is Koszul.

Fact #1

The proof of Koszulness of $\mathrm{End}(\bigoplus_w \Delta_w)$ boils down to

Theorem (Jantzen?)

In the Bruhat graph of S_n, the only intervals of length 3 which can appear are either 2-crowns, 3-crowns, or 4-crowns:



(In fact this is true for any Weyl group.)

Key Fact #2

Compare the fact that the constant term of every $P_{u,w}$ is 1 to

Theorem (Phillip Hall)

In a poset, if $\mu(u, w)$ is the Mobius function, then

$$\mu(u, w) = \sum_{i \geq 0} (-1)^i (\text{number of chains of length } i \text{ between } u, w),$$

where $1 < s$ is a chain of length 1.

Theorem

The Mobius function of the symmetric group is $\mu(u, w) = (-1)^{\ell(w) - \ell(u)}$.

Thank you!

Thank you for coming to my talk!
Questions

Non-split decomposition

- Lauda's $\mathcal{U}_q(\mathfrak{sl}_2)$ is triangular-based.
- However, it does not have a subalgebra like $\mathcal{U}_q(\mathfrak{sl}_2)^-$.
- Instead, one needs to work with a bigger object $\mathcal{U}_q(\mathfrak{sl}_2)^\flat$, which actually is a subalgebra.
- Categorification: projectives categorify Lusztig's canonical basis.

Question

What do the standard modules correspond to?

Stratifications within stratifications

- The affine oriented Brauer category is triangular-based.
- However, this structure alone does not utilize the obvious ordering on the simples of each Cartan.
- By using the stratification of $\widehat{\mathcal{H}}_n$ above, we should obtain finer stratifications.

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Koszul duality

We are using the *inverse* Koszul duality of [BGS96], and we flip the axes and swap the roles of A and $A^!$.

$$\mathcal{K}_{A^+} \colon \mathsf{D}^{\nwarrow} \operatorname{\mathsf{Mod}} A^+ \longrightarrow \mathsf{D}^{\searrow} \operatorname{\mathsf{Mod}} A^{+,!},$$

where

$$\mathcal{K}_{A^+} = \operatorname{sh}(\mathbb{K} \overset{\mathsf{L}}{\otimes}_{A^+} \operatorname{refl} \square) = \operatorname{sh} \operatorname{\mathsf{RHom}}_{A^-}(\mathbb{K}, \operatorname{refl} \square^{\dagger})^*.$$

Here $\operatorname{sh} M = M[n]$ if M is concentrated in Koszul degree n, and $\operatorname{refl}(M)_j = M_{-j}$.
Then the spectral sequence looks like

$$\Delta \otimes_{A^\circ} \mathcal{K}_{A^+}(\square).$$