BGG resolutions for nil-Brauer

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Link

These slides can be found on my webpage:

math.columbia.edu/~fanzhou/files/beamer-AMS2025-NB.pdf

The paper can be found at

arxiv.org/abs/2402.06890

The ι-quantum group

- The split ι-quantum group of rank 1 $U_q^\iota(\mathfrak{sl}_2)$ is a coideal subalgebra of $U_q(\mathfrak{sl}_2)$.
- This e.g. appears in ι-Schur duality.
- As an algebra, it is isomorphic to $\mathbb{Q}(q)[B]$, where $B = F + qK^{-1}E$.
- We can consider $U_q^\iota(\mathfrak{sl}_2)_t$, satisfying $\dot{U}_q^\iota(\mathfrak{sl}_2) = \dot{U}_q^\iota(\mathfrak{sl}_2)1_0 \oplus \dot{U}_q^\iota(\mathfrak{sl}_2)1_1$.
- This has certain special bases.

Change of basis

Character formula (Brundan-Wang-Webster 2023, [BWW23a])

$$[L_n] = \sum_{k=0}^{\infty} (-1)^k \frac{q^{-k(1+2\delta_{n\not\equiv t})}}{(1-q^{-4})(1-q^{-8})\cdots(1-q^{-4k})} [\overline{\Delta}_{n+2k}],$$

where $[L_n]$ is the dual canonical basis and $[\overline{\Delta}_n]$ is the dual PBW basis.

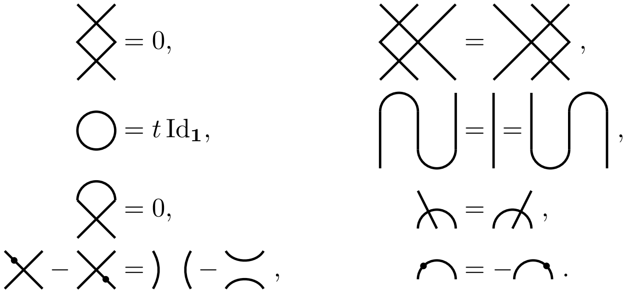
Spoilers: this was categorified into a statement in a Grothendieck group.

The nil-Brauer algebra

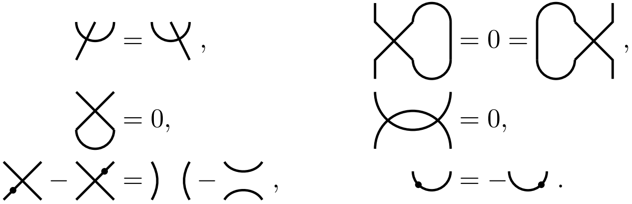
Defined by [BWW23b] and denoted $\mathrm{N}\mathcal{B}$, depending on $t = 0, 1$, it is generated by

degree 2 -2 0 0

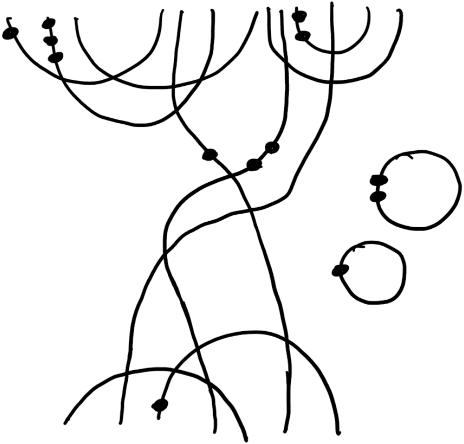
subject to conditions



One can then show the following are also satisfied



A typical example of an element of this algebra:



Triangular-based algebras

This is a circle of ideas including [Bru23], [BS21], [GRS23], etc..
A minimal setup:

- Let Θ be a poset of "weights".
- Let e^θ be a set of orthogonal homogeneous idempotents, labeled by $\theta \in \Theta$.

Example

In N$\mathcal{B}$, let $\Theta = \mathbb{N}$ under $0 < 1 < \cdots$, and let e^θ be the idempotent corresponding to θ strands:

$$\Big|\,\Big|\,\Big|\cdots\Big|$$

In the below, let ϖ be the identity map.

Definition

A is "graded triangular-based" if there are (homogeneous) sets $\mathrm{X}(i,\alpha) \subseteq 1^i A 1^\alpha$, $\mathrm{H}(\alpha,\beta) \subseteq 1^\alpha A 1^\beta$, $\mathrm{Y}(\beta,j) \subseteq 1^\beta A 1^j$ such that

1. products of these elements in these sets give a basis for A, i.e.

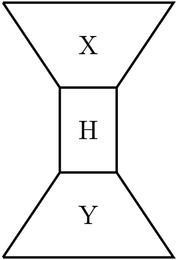
$$\left\{ xhy : (x,h,y) \in \bigcup_{i,j,\alpha,\beta} \mathrm{X}(i,\alpha) \times \mathrm{H}(\alpha,\beta) \times \mathrm{Y}(\beta,j) \right\}$$

   forms a basis of A;
2. $\mathrm{X}(\alpha,\alpha) = \mathrm{Y}(\alpha,\alpha) = \{1^\alpha\}$;
3. for $\alpha \neq \beta$,

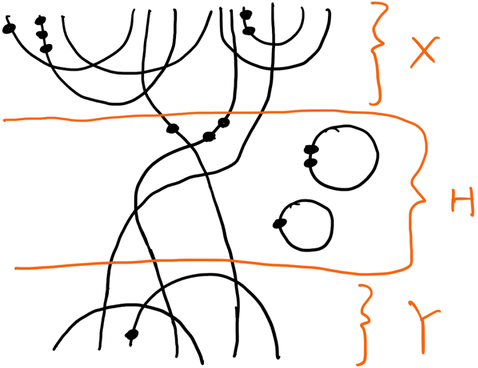
$$\begin{aligned} \mathrm{X}(\alpha,\beta) \neq \emptyset &\implies \varpi(\alpha) > \varpi(\beta), \\ \mathrm{H}(\alpha,\beta) \neq \emptyset &\implies \varpi(\alpha) = \varpi(\beta), \\ \mathrm{Y}(\alpha,\beta) \neq \emptyset &\implies \varpi(\alpha) < \varpi(\beta); \end{aligned}$$

Example

$N\mathcal{B}$ is graded-triangular-based by setting $\Theta = \mathbb{N}$, with e^n being the idempotent for n strands.


X
H
Y

Same typical example from earlier:



Remark: $\mathrm{Y}(\alpha, \alpha) = \{1^\alpha\}$ means the straight lines is a Y-diagram.

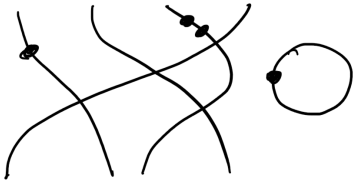
Cartan algebras

We can define

$$A^{\geq\theta} := A \big/ \langle e^\phi : \phi \not\geq \theta \rangle.$$

The "Cartan algebras" are defined as

$$A^\theta = e^\theta A^{\geq\theta} e^\theta.$$



Let Λ_θ label the simples $L_\lambda(\theta)$ and projectives $P_\lambda(\theta)$ of A^θ, and let $\Lambda = \bigsqcup_\theta \Lambda_\theta$.

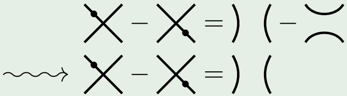
Fact (Brundan 2023)

The simples of A are also labeled by Λ.

For the nil-Brauer

Example

Quotienting out by $e^{\phi} : \phi < \theta$:



$N\mathcal{B}^{\theta}$ is isomorphic to the nil-Hecke algebra on θ strands (over the ring Γ of "Schur q-functions", isomorphic to the bubbles).
This algebra has (up to grading shift) exactly one simple, so
$\Lambda = \Theta = \mathbb{N}$.

More generally

- This theory is able to handle much more, e.g. even when the idempotents corresponding to strands don't have an obvious ordering.

Standardization/costandardization

- Given a module $A^{\geq\theta} \circlearrowright M$, we can consider the functor

$$\begin{aligned} \jmath^\theta \colon \mathsf{Mod}\, A^{\geq\theta} &\longrightarrow \mathsf{Mod}\, A^\theta \\ M &\longmapsto e^\theta M. \end{aligned}$$

- This lands in $\mathsf{Mod}\, A^\theta$, since $e^\theta A^{\geq\theta} e^\theta \circlearrowright e^\theta M$.
- So the "Cartan" is acting on the "weight space".
- This functor has a left and a right adjoint:

$$\jmath_!^\theta \dashv \jmath^\theta \dashv \jmath_*^\theta.$$

- $\jmath_!^\theta = A^{\geq\theta} e^\theta \otimes_{A^\theta} \square$.

Fact (Brundan 2023)

$\jmath_!^\theta$ and $\jmath_*^\theta$ are exact due to the triangular-based nature of A.

“Vermas”

Definition

The “(proper) (co)standard modules” are defined by

standard module (big Verma)	$\Delta_\lambda = \jmath_!^\theta P_\lambda(\theta)$
proper standard module (small Verma)	$\overline{\Delta}_\lambda = \jmath_!^\theta L_\lambda(\theta)$
costandard module (big coVerma)	$\nabla_\lambda = \jmath_*^\theta Q_\lambda(\theta)$
proper costandard module (small coVerma)	$\overline{\nabla}_\lambda = \jmath_*^\theta L_\lambda(\theta)$

These “(big/small) Verma modules” form an analogue of “highest-weight theory”.

Categorification

Theorem (Brundan-Wang-Webster 2023)

There is an isomorphism between the Grothendieck group of $\mathrm{N}\mathcal{B}_t$ and (an integral form of) $U_q^\iota(\mathfrak{sl}_2)_t$, under which

- P_λ goes to the canonical basis;
- Δ_λ goes to the PBW basis;
- $\overline{\Delta}_\lambda$ goes to the dual PBW basis;
- L_λ goes to the dual canonical basis.

So the formula

$$[L_n] = \sum_{k=0}^{\infty} (-1)^k \frac{q^{-k(1+2\delta_{n\not\equiv t})}}{(1-q^{-4})(1-q^{-8})\cdots(1-q^{-4k})} [\overline{\Delta}_{n+2k}]$$

becomes a statement in the Grothendieck group of representations.

Question

Can this formula be further categorified into a resolution?

The BGG resolution

Theorem (Z. 2024)

At parameter $t = 0$, the 1-dimensional simple L_0 has a BGG resolution

$$\cdots \to C^{-n}_{\mathrm{BGG}}(L_0) \longrightarrow C^{-(n-1)}_{\mathrm{BGG}}(L_0) \longrightarrow \cdots \longrightarrow C^{0}_{\mathrm{BGG}}(L_0) \longrightarrow L_0 \longrightarrow 0$$

where the terms have character

$$\chi(C^{-n}_{\mathrm{BGG}}(L_0)) = \frac{q^{-n}}{(1-q^{-4})(1-q^{-8})\cdots(1-q^{-4n})}\chi(\overline{\Delta}_{2n})$$

and admit filtrations $C^{-n}_{\mathrm{BGG}}(L_0) = \mathrm{F}^0_{\mathrm{BGG}} \supset \mathrm{F}^1_{\mathrm{BGG}} \supset \cdots$ such that

$$\mathrm{gr}^k C^{-n}_{\mathrm{BGG}}(L_0) = \overline{\Delta}_{2n} \otimes_{\mathbb{C}} q^{-n}\mathbb{C}[p_2, p_4, \cdots, p_{2n}]_{\deg_{\mathrm{sym}}=k},$$

where $\deg_{\mathrm{sym}} p_i = 1$.

For other simples, we instead have a spectral sequence categorifying the character formula.

Koszulity

A graded algebra is "quadratic" if it is generated in degree 1 and relations are generated in degree 2.

Definition

A quadratic graded algebra A ($A_0 = \Bbbk$) is "Koszul" if $\mathrm{Ext}_A(\Bbbk, \Bbbk)$ in nonzero only when the homological degree agrees with the Koszul degree.

The nilalgebra is Koszul

Theorem (Z. 2024)

There is a "lower-half subalgebra" $\mathrm{N}\mathcal{B}^-$ of $\mathrm{N}\mathcal{B}$ which is Koszul. This algebra is defined as

$$\mathrm{N}\mathcal{B}^- = \bigoplus_{\psi \leq \theta} e^{\psi} \mathbb{C}\mathrm{Y} e^{\theta}.$$

Here the Koszul grading is coming from the weight theory – it is given by the number of caps.

This theorem is key to proving the BGG resolution.

Key ideas

1. First key idea: “weight theory” $\longrightarrow$ stratification of the module category $\longrightarrow$ “filtration” of the identity functor $\longrightarrow$ spectral sequence converging to any object.
    - In particular, we can apply this to simple objects.
    - Terms of this spectral sequence capture homological information, in the form of certain Ext groups.
    - Concentration of these Ext groups (cf. “Kostant modules”) imply a “BGG resolution”.
    - This idea is due to Gaitsgory, Ayala-Mazel-Gee-Rozenblyum [AMGR22], and Dhillon [Dhi19].
2. Second key idea: This homological information can be computed using Koszul methods.

Slogan

Koszulity of half of A is intimately connected to BGG resolutions.

- Then we can use a naive resolution to compute these Ext groups.
- The spectral sequence is a resolution for modules which are Koszul over half of A.

- These ideas are not specific to nil-Brauer.

Algebraic recollement

- The stratification will have recollements on each level. Details unimportant.
- By the $\mathsf{D}\,\mathsf{Mod}\, A/AeA \longrightarrow \mathsf{D}\,\mathsf{Mod}\, A \longrightarrow \mathsf{D}\,\mathsf{Mod}\, eAe$ setup of [CPS88], set $A = A^{\geq\theta}$ and $e = e^\theta$ to get:

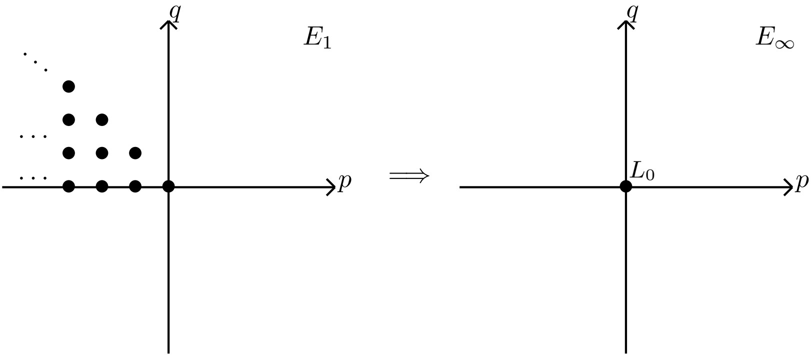
$$
\begin{tikzcd}[column sep=huge]
\mathsf{D}^-\,\mathsf{Mod}\, A^{>\theta} \arrow[r, "\iota_\theta"] \arrow[r, phantom, shift left=5ex, "\perp"] \arrow[r, phantom, shift right=5ex, "\perp"] & \mathsf{D}^-\,\mathsf{Mod}\, A^{\geq\theta} \arrow[l, bend right=60, "\iota_\theta^* = A^{>\theta} \overset{\mathsf{L}}{\otimes}_{A^{\geq\theta}} \square"'] \arrow[l, bend left=60, "\iota_\theta^! = \bigoplus_i \mathrm{RHom}_{A^{\geq\theta}}(A^{>\theta}1^i, \square)"] \arrow[r, "\jmath^\theta = e^\theta \square"] \arrow[r, phantom, shift left=5ex, "\perp"] \arrow[r, phantom, shift right=5ex, "\perp"] & \mathsf{D}^-\,\mathsf{Mod}\, A^\theta \arrow[l, bend right=60, "\jmath_!^\theta = A^{\geq\theta} e^\theta \otimes_{A^\theta} \square"'] \arrow[l, bend left=60, "\jmath_*^\theta = \bigoplus_i \mathrm{Hom}_{A^\theta}(e^\theta A^{\geq\theta} 1^i, \square)"]
\end{tikzcd}
$$

Stratification and spectral sequences

- Let Θ be sufficiently nice.
- There is a spectral sequence (functorial in the input $\square$)

$$E_1^{p,q} = \bigoplus_{\ell(\theta)=-p} \Delta(\theta) \otimes_{A^\theta} \mathrm{Ext}_A^{-(p+q)}(\Delta(\theta), \square^\dagger)^* \implies E_\infty^{p,q} = \mathrm{gr}^{-p} H^{p+q}(\square),$$

where $\Delta(\theta) := \bigoplus_{\lambda \in \theta} \Delta_\lambda^{\overline{l_\lambda(\theta)}} = A^{\geq \theta} e^\theta$. $\deg \mathrm{d}_r = (r, 1-r)$:



Remark: Cf. Koszul duality.

- To obtain a resolution, we need the Ext groups to be concentrated in certain degrees.
- Idea: Koszul objects have good Ext concentration properties.

A graded algebra is "quadratic" if it is generated in degree 1 and relations are generated in degree 2.

Definition

A quadratic graded algebra A ($A_0 = \Bbbk$) is "Koszul" if $\mathrm{Ext}_A(\Bbbk, \Bbbk)$ in nonzero only when the homological degree agrees with the Koszul degree.

Example: category $\mathcal{O}$ for $\mathfrak{sl}_2$

- It is classical that L_0 has concentrated Ext groups:

$$\mathrm{Ext}^0(\Delta_0, L_0) = \mathbb{C}, \qquad \mathrm{Ext}^0(\Delta_{-2}, L_0) = 0$$

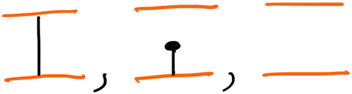
$$\mathrm{Ext}^1(\Delta_0, L_0) = 0, \qquad \mathrm{Ext}^1(\Delta_{-2}, L_0) = \mathbb{C}.$$

- Then the spectral sequence above exactly recovers the BGG resolution.
- Remark: It can also recover the standard filtration of projectives.

The algebra controlling this

- Recall that the principal block of $\mathcal{O}(\mathfrak{sl}_2)$ is Morita equivalent to the 5-dimensional algebra $A_{\mathfrak{sl}_2}$ spanned by

- The subalgebra $A^-_{\mathfrak{sl}_2}$ of this spanned by



is Koszul.

Lie algebras

- We wish to mimic the story of Lie algebra cohomology $H^\bullet(\mathfrak{n}^+ : \square)$.
- Recall

$$\begin{aligned} \mathsf{R}\mathrm{Hom}_{\mathcal{O}}(\Delta_\lambda, \square) &= \mathsf{R}\mathrm{Hom}_{\mathcal{O}}(U\mathfrak{g} \otimes_{U\mathfrak{b}^+} \mathbb{C}_\lambda, \square) \\ &= \mathsf{R}\mathrm{Hom}_{\mathfrak{b}^+}(\mathbb{C}_\lambda, \square) \\ &= (\mathsf{R}(\square^{\mathfrak{n}^+}))^\lambda =: H^\bullet(\mathfrak{n}^+ : \square)^\lambda. \end{aligned}$$

- This can be computed with the Chevalley-Eilenberg complex, which is finite in length.

Nilcohomology

- To this end, try to define

$$A^- := \bigoplus_{\psi \leq \theta} e^\psi \mathbb{C}\mathrm{Y} e^\theta = \mathbb{K} \oplus I,$$

where $\mathbb{K} := \bigoplus_\theta \Bbbk e^\theta$, $I = \bigoplus_{\psi < \theta} e^\psi \mathbb{C}\mathrm{Y}_+ e^\theta$.

- Subalgebra-ness needs to be checked. This is true for nil-Brauer.
- Note

$$\mathrm{Hom}_A(A^{\geq\theta} e^\theta, \square) = \mathrm{Hom}_A(A \otimes_{A^-} \Bbbk e^\theta, \square) = \mathrm{Hom}_{A^-}(\Bbbk e^\theta, \square).$$

- This can be identified with $e^{\theta}M^{A^-} = \{v \in M^{\theta} : I \cdot v = 0\}$. Its derived functor deserves to be called

$$H^{\bullet}(A^- : \square)^{\theta} = \mathsf{R}\mathrm{Hom}_A(A^{\geq \theta}e^{\theta}, \square).$$

- This can be computed with the complex

$$\cdots \to A^- \otimes I^{\otimes k} e^{\theta} \to \cdots \to A^- \otimes I \otimes I e^{\theta} \to A^- \otimes I e^{\theta} \to A^- e^{\theta} \to \Bbbk e^{\theta}.$$

- Remark: This can be made more uniform by defining

$$H^{\bullet}(A^- : \square) \coloneqq \mathsf{R}\mathrm{Hom}_{A^-}(\mathbb{K}, \square).$$

- In particular, for nil-Brauer, the trivial module $(t = 0)$ L_0 is $\Bbbk e^0$.
- The homological information we need is then

$$\mathrm{Ext}_A(A^{\geq\theta}e^\theta, L_0) = \mathrm{Ext}_{A^-}(\Bbbk e^\theta, \Bbbk e^0).$$

- This is subsumed by

$$H^\bullet(A^- : \mathbb{K}) = \mathsf{R}\mathrm{Hom}_{A^-}(\mathbb{K}, \mathbb{K}).$$

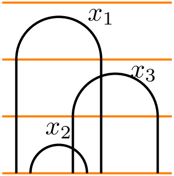
- Hence we wish to show $\mathrm{N}\mathcal{B}^-$ is Koszul.

$\mathrm{N}\mathcal{B}^-$ is Koszul

The proof is technical.
Somehow the fact $\mathrm{N}\mathcal{B}^-$ is Koszul boils down to algebras like

$$\mathbb{C}\langle x_1, x_2, x_3\rangle \big/ \langle x_i^2 = 0,\ x_1x_3 = x_3x_1,\ x_2x_3 = x_3x_2,\ x_2x_1 = 0\rangle$$

being Koszul.



Back to nilcohomology

- We can then use the naive complex to compute the dimensions of these Ext groups.

Theorem

Let $t = 0$ and $\theta = 2n$. Then the dimension of the Ext groups are

$$\dim_q \operatorname{Ext}^n_{\mathrm{N}\mathcal{B}}(\Delta(\theta), L_0)^* = \frac{q^{-n}[2n]!}{(1-q^{-4})\cdots(1-q^{-4n})}$$

Moreover, the $\mathrm{N}\mathcal{B}^\theta$-module $\operatorname{Ext}^n_{\mathrm{N}\mathcal{B}}(\Delta(\theta), L_0)^*$ is isomorphic to a quotient of the polynomial ring,

$$\operatorname{Ext}^n_{\mathrm{N}\mathcal{B}}(\Delta(\theta), L_0)^* \cong \mathbb{C}[X_1, \cdots, X_\theta] \big/ \langle p_1, p_3, \cdots, p_{2n-1} \rangle.$$

This proves the theorem:

Theorem (Z. 2024)

At parameter $t = 0$, the 1-dimensional simple L_0 has a BGG resolution

$$\cdots \to C^{-n}_{\mathrm{BGG}}(L_0) \longrightarrow C^{-(n-1)}_{\mathrm{BGG}}(L_0) \longrightarrow \cdots \longrightarrow C^{0}_{\mathrm{BGG}}(L_0) \longrightarrow L_0 \longrightarrow 0$$

where the terms have character

$$\chi(C^{-n}_{\mathrm{BGG}}(L_0)) = \frac{q^{-n}}{(1-q^{-4})(1-q^{-8})\cdots(1-q^{-4n})}\chi(\overline{\Delta}_{2n})$$

and admit filtrations $C^{-n}_{\mathrm{BGG}}(L_0) = \mathrm{F}^0_{\mathrm{BGG}} \supset \mathrm{F}^1_{\mathrm{BGG}} \supset \cdots$ such that

$$\mathrm{gr}^k\, C^{-n}_{\mathrm{BGG}}(L_0) = \overline{\Delta}_{2n} \otimes_{\mathbb{C}} q^{-n}\mathbb{C}[p_2, p_4, \cdots, p_{2n}]_{\deg_{\mathrm{sym}}=k},$$

where $\deg_{\mathrm{sym}} p_i = 1$.

Thank you!

Thank you for coming to my talk!
Questions

Non-split decomposition

- Lauda's $\mathcal{U}_q(\mathfrak{sl}_2)$ is triangular-based.
- However, it does not have a subalgebra like $\mathcal{U}_q(\mathfrak{sl}_2)^-$.
- Instead, one needs to work with a bigger object $\mathcal{U}_q(\mathfrak{sl}_2)^\flat$, which actually is a subalgebra.
- Categorification: projectives categorify Lusztig's canonical basis.

Question

What do the standard modules correspond to?

Stratifications within stratifications

- The affine oriented Brauer category is triangular-based.
- However, this structure alone does not utilize the obvious ordering on the simples of each Cartan.
- By using the stratification of $\widehat{\mathcal{H}}_n$ above, we should obtain finer stratifications.

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Koszul duality

We are using the *inverse* Koszul duality of [BGS96], and we flip the axes and swap the roles of A and $A^!$.

$$\mathcal{K}_{A^+} \colon \mathsf{D}^{\nwarrow} \mathsf{Mod}\, A^+ \longrightarrow \mathsf{D}^{\searrow} \mathsf{Mod}\, A^{+,!},$$

where

$$\mathcal{K}_{A^+} = \mathrm{sh}(\mathbb{K} \overset{\mathsf{L}}{\otimes}_{A^+} \mathrm{refl}\, \square) = \mathrm{sh}\, \mathsf{RHom}_{A^-}(\mathbb{K}, \mathrm{refl}\, \square^{\dagger})^*.$$

Here $\mathrm{sh}\, M = M[n]$ if M is concentrated in Koszul degree n, and $\mathrm{refl}(M)_j = M_{-j}$.
Then the spectral sequence looks like

$$\Delta \otimes_{A^\circ} \mathcal{K}_{A^+}(\square).$$