BGG resolutions, Koszulity, and stratifications: the nil-Brauer algebra

Fan Zhou Columbia University, Department of Mathematics fz2326@columbia.edu

1 Introduction

The "split $\imath$-quantum group of rank 1", or $U_q^\imath(\mathfrak{sl}_2)$, is a coideal subalgebra of $U_q(\mathfrak{sl}_2)$. It is for example the object appearing in $\imath$-Schur-Weyl duality, replacing the quantum group when the Hecke of type A is replaced with the Hecke of type B. We can consider $U_q^\imath(\mathfrak{sl}_2)_t$, satisfying $\dot{U}_q^\imath(\mathfrak{sl}_2) = \dot{U}_q^\imath(\mathfrak{sl}_2)1_0 \oplus \dot{U}_q^\imath(\mathfrak{sl}_2)1_1$, which is a subalgebra of the summand of

$$\dot{U}_q(\mathfrak{sl}_2) = \bigoplus_{\lambda,\mu \equiv 0 \pmod 2} 1_\lambda \dot{U}_q^\imath(\mathfrak{sl}_2)1_\mu \oplus \bigoplus_{\lambda,\mu \equiv 1 \pmod 2} 1_\lambda \dot{U}_q^\imath(\mathfrak{sl}_2)1_\mu$$

consisting of weights of parity t. This $\imath$-quantum group has certain special bases, namely the canonical basis and the PBW basis, as well as their appropriate duals. There is a change-of-basis formula between these bases, for example ([5])

$$[L_n] = \sum_{k=0}^{\infty} (-1)^k \frac{q^{-k(1+2\delta_{n\neq t})}}{(1-q^{-4})(1-q^{-8})\cdots(1-q^{-4k})}[\overline{\Delta}_{n+2k}], \qquad \text{(BWW)}$$

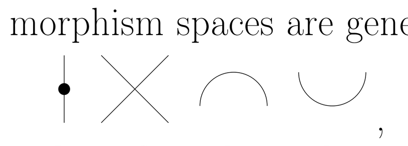
where $[L_n]$ is the dual canonical basis and $[\overline{\Delta}_n]$ is the dual PBW basis (notation provocatively chosen).

In 2023, Brundan-Wang-Webster ([5],[4]) defined the nil-Brauer algebra $\mathrm{N}\mathcal{B} = \mathrm{N}\mathcal{B}_t$, which is locally unital but not unital, proved that it is a (graded) triangular-based algebra in the sense of [2], and showed that its representation theory categorified $U_q^\imath(\mathfrak{sl}_2)$.

In this paper [8], we will categorify this change-of-basis formula into a BGG resolution and prove that half of $\mathrm{N}\mathcal{B}$ is Koszul.

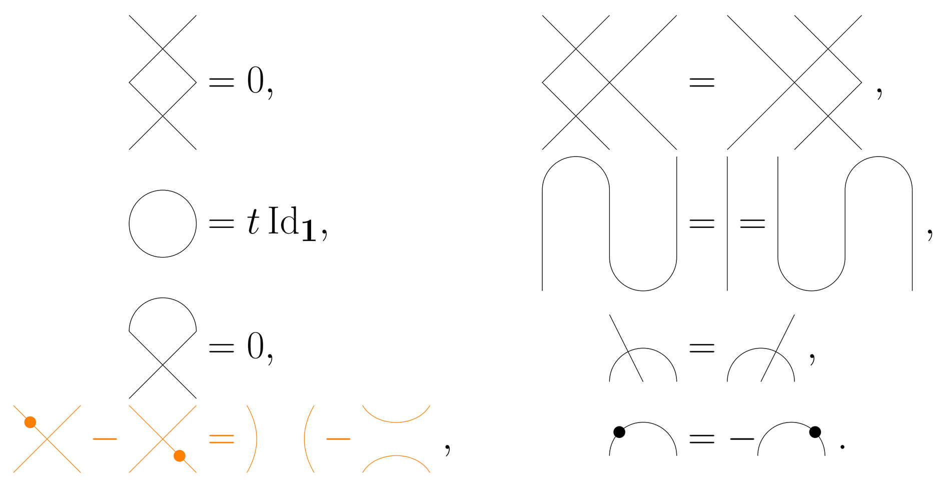
The nil-Brauer

Let us briefly define this nil-Brauer algebra. Defined by [5] and denoted $\mathrm{N}\mathcal{B}_t$, depending on $t = 0, 1$, it is defined as the path algebra of the nil-Brauer category, and is hence only locally unital. The nil-Brauer category, also denoted $\mathrm{N}\mathcal{B}_t$ in an abuse of notation, is a monoidal $\Bbbk$-linear category generated by the object B, diagrammatically represented as an upward string, and its morphism spaces are generated by

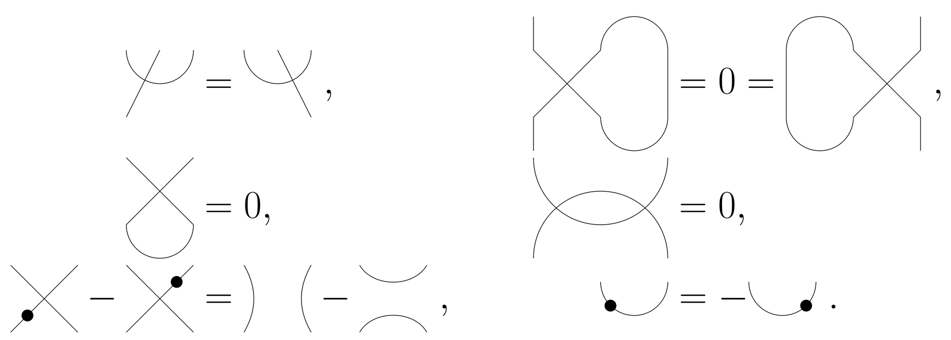


degree 2 −2 0 0

subject to the relations (note well the orange one)



One can then show that the following relations are also satisfied:



Triangularly-based algebras

One of the main points of Brundan-Wang-Webster is that $\mathrm{N}\mathcal{B} = \mathrm{N}\mathcal{B}_t$ is an example of an algebra with a "graded triangular basis". This notion belongs to a circle of ideas including [7], [6], [3], and surely others; we take the point of view of the most recent of these, [2], and present a minimal definition.

For an (locally unital graded) algebra A, let Θ be a poset of weights and let $\{e^\theta : \theta \in \Theta\}$ be a set of orthogonal homogeneous idempotents.

Definition 1.1. Let $i, j, \alpha, \beta \in \Theta$. A is "graded triangular-based" if there are (homogeneous) sets $\mathrm{X}(i,\alpha) \subseteq e^i A e^\alpha$, $\mathrm{H}(\alpha,\beta) \subseteq e^\alpha A e^\beta$, $\mathrm{Y}(\beta,j) \subseteq e^\beta A e^j$ such that

1. products of these elements in these sets give a basis for A, i.e.
$$\left\{ xhy : (x,h,y) \in \bigcup_{i,j,\alpha,\beta} \mathrm{X}(i,\alpha) \times \mathrm{H}(\alpha,\beta) \times \mathrm{Y}(\beta,j) \right\}$$
forms a basis of A;
2. $\mathrm{X}(\alpha,\alpha) = \mathrm{Y}(\alpha,\alpha) = \{e^\alpha\}$;
3. for $\alpha \neq \beta$,
$$\begin{aligned} \mathrm{X}(\alpha,\beta) \neq \emptyset &\implies \alpha > \beta, \\ \mathrm{H}(\alpha,\beta) \neq \emptyset &\implies \alpha = \beta, \\ \mathrm{Y}(\alpha,\beta) \neq \emptyset &\implies \alpha < \beta. \end{aligned}$$

One can then define

$$A^{\geq\theta} := A \big/ \langle e^\phi : \phi \ngeq \theta \rangle$$

and let the "Cartan subalgebra" be defined as the sandwich

$$A^\theta := e^\theta A^{\geq\theta} e^\theta.$$

Let Λ_θ label the simples $L_\lambda(\theta)$ and projectives $P_\lambda(\theta)$ of A^θ, and let $\Lambda := \bigsqcup_\theta \Lambda_\theta$.

Given a module $A^{\geq\theta} \circlearrowright M$, we can consider the functor

$$\begin{aligned} j^\theta \colon \mathsf{Mod}\, A^{\geq\theta} &\longrightarrow \mathsf{Mod}\, A^\theta \\ M &\longmapsto e^\theta M. \end{aligned}$$

This functor has both a left and a right adjoint

$$j_!^\theta \dashv j^\theta \dashv j_*^\theta,$$

both of which can be described explicitly, e.g. $j_!^\theta = A^{\geq\theta} e^\theta \otimes_{A^\theta} \square$.

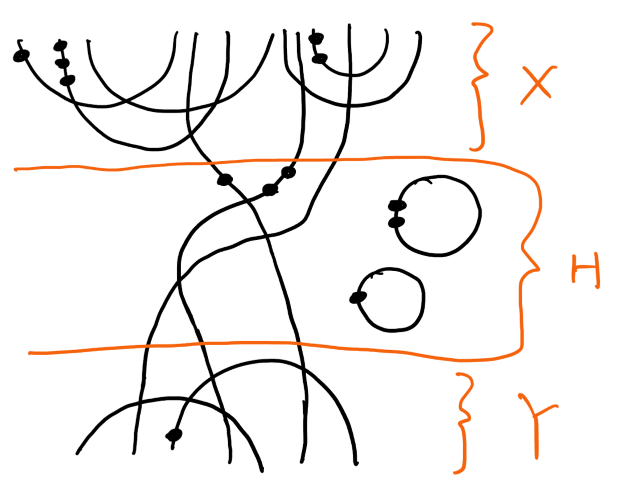
Then we may define "(co)standard modules" and "proper (co)standard modules" by

standard module (big Verma)	$\Delta_\lambda = j_!^\theta P_\lambda(\theta)$
proper standard module (small Verma)	$\overline{\Delta}_\lambda = j_!^\theta L_\lambda(\theta)$
costandard module (big coVerma)	$\nabla_\lambda = j_*^\theta Q_\lambda(\theta)$
proper costandard module (small coVerma)	$\overline{\nabla}_\lambda = j_*^\theta L_\lambda(\theta)$

These modules give a close analogue of highest-weight theory, with many nice homological properties; see [2] for more.

Fitting the nil-Brauer

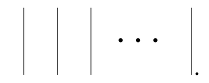
A typical example of an element (in black) of $\mathrm{N}\mathcal{B}$ is



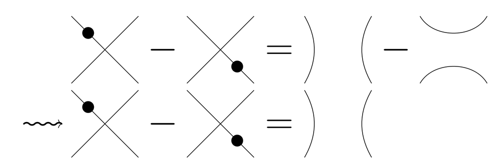
Here we have marked out in orange how this element can be written as a basis element, xhy. In short:

- The x's are cups, possibly intersecting (but not more than once) and possibly carrying dots (in appointed locations);
- The h's are propagating strings, possibly crossing (but not more than once) and possibly carrying dots (in appointed locations); as well as bubbles;
- The y's are caps, possibly intersecting (but not more than once) and possibly carrying dots (in appointed locations).

From this example it is intuitively clear (but very hard to prove; this is the main theorem of [5]) that $\mathrm{N}\mathcal{B}$ is graded-triangularly-based by setting $I = \Phi = \Theta = \mathbb{N}$, with e^n being the idempotent for n strands:



When forming the Cartan for $\mathrm{N}\mathcal{B}$, quotienting out by $e^\phi : \phi < \theta$ turns the orange relation from earlier into the nil-Hecke relation:



Hence the θ-th Cartan for nil-Brauer, $\mathrm{N}\mathcal{B}^\theta$, is isomorphic to the nil-Hecke algebra on θ strands (technically not over $\mathbb{C}$ but over the ring Γ of "Schur q-functions", isomorphic to the ring of bubbles, but this is a technicality). This Cartan algebra has (up to grading shift) exactly one simple, so $\Lambda = \Theta = \mathbb{N}$.

Categorification

It is the main theorem of [4] that

Theorem 1.2 (Brundan-Wang-Webster 2023). There is an isomorphism between the Grothendieck group of $\mathrm{N}\mathcal{B}_t$ and (an integral form of) $U_q^\imath(\mathfrak{sl}_2)_t$, under which

- P_λ goes to the canonical basis;
- Δ_λ goes to the PBW basis;
- $\overline{\Delta}_\lambda$ goes to the dual PBW basis;
- L_λ goes to the dual canonical basis.

Hence equation BWW can be interpreted as a statement in the Grothendieck group of representations over $\mathrm{N}\mathcal{B}$. One can then ask the very natural question: Can this formula be further categorified into a resolution of modules?

2 Main results

We answer this question in the positive.

Theorem 2.1 (Z. 2024). At parameter $t = 0$, the 1-dimensional simple L_0 has a BGG resolution

$$\cdots \longrightarrow C_{\mathrm{BGG}}^{-n}(L_0) \longrightarrow \cdots \longrightarrow C_{\mathrm{BGG}}^0(L_0) \longrightarrow L_0 \longrightarrow 0$$

where the terms have character

$$\chi(C_{\mathrm{BGG}}^{-n}(L_0)) = \frac{q^{-n}}{(1-q^{-4})(1-q^{-8})\cdots(1-q^{-4n})}\chi(\overline{\Delta}_{2n})$$

and admit filtrations $C_{\mathrm{BGG}}^{-n}(L_0) = \mathrm{F}_{\mathrm{BGG}}^0 \supset \mathrm{F}_{\mathrm{BGG}}^1 \supset \cdots$ such that

$$\mathrm{gr}^k\, C_{\mathrm{BGG}}^{-n}(L_0) = \overline{\Delta}_{2n} \otimes_{\mathbb{C}} q^{-n}\mathbb{C}[\mathrm{p}_2, \mathrm{p}_4, \cdots, \mathrm{p}_{2n}]_{\deg_{\mathrm{sym}}=k},$$

where $\deg_{\mathrm{sym}} \mathrm{p}_i = 1$.

For other simples, we instead have a spectral sequence categorifying the character formula.

The key to proving this theorem is our second main result,

Theorem 2.2 (Z. 2024). The subalgebra of $\mathrm{N}\mathcal{B}$

$$\mathrm{N}\mathcal{B}^- := \bigoplus_{\psi \leq \theta} e^\psi \mathbb{K}\mathrm{Y} e^\theta,$$

which deserves the name "lower-half nilalgebra", is Koszul with respect to the "cap grading", given by the number of caps.

Indeed, our slogan is:

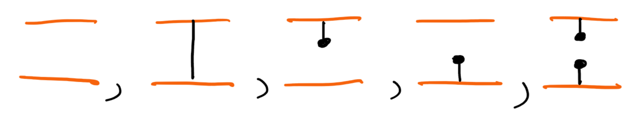
Koszulity of half of A is intimately connected to BGG resolutions.

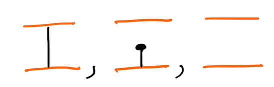
The key idea is that we can use the concentration properties afforded by Koszul theory in tandem with the "reconstruction-from-stratification" machine of [1] to produce BGG spectral sequences/resolutions. In fact, this BGG spectral sequence is also in some sense given by a Koszul duality functor with respect to A^+; this is a resolution when the input module is Koszul over A^+. In fact, the differential maps of the resolution are baked into the computation of the Koszul dual.

3 Future work/other examples

For us, the main appeal of this paper is that the key ideas and techniques are very general. For example one can do this for category $\mathcal{O}$, for Temperley-Lieb, for Khovanov-Sazdanovic's categorifications of Chebyshev and Hermite polynomials, for KLR, and many, many more. In forthcoming work we plan to tackle each of these algebras/categories.

To demonstrate the ideas of this paper with a minimalist example, let us consider the principal block of category $\mathcal{O}$ for $\mathfrak{sl}_2$. Recall that this is equivalent to modules over the 5-dimensional algebra A with basis



where the barbell is set to zero. Then the subalgebra with basis  might be called A^-, where the middle element is declared to have degree 1 and the other two elements constitute a locally unital ground field. Morally A^- is the same as $\mathbb{C}[x]/x^2$, which is famously Koszul. Computing Koszul duality (and tensoring with the Verma)

$$\Delta \otimes_{A^\circ} \mathcal{K}_{A^+} \square$$

with respect to this locally unital algebra (or rather, A^+, defined by using the first three of the five basis elements) and plugging in L_0 will reveal the usual BGG resolution.

References

[1] David Ayala, Aaron Mazel-Gee, and Nick Rozenblyum. Stratified noncommutative geometry, 2022. arXiv:1910.14602.

[2] Jonathan Brundan. Graded triangular bases, 2023. arXiv:2305.05122.

[3] Jonathan Brundan and Catharina Stroppel. Semi-infinite highest weight categories, 2021. arXiv:1808.08022.

[4] Jonathan Brundan, Weiqiang Wang, and Ben Webster. Nil-Brauer categorifies the split ı-quantum group of rank one, 2023. arXiv:2305.05877.

[5] Jonathan Brundan, Weiqiang Wang, and Ben Webster. The nil-Brauer category, 2023. arXiv:2305.03876.

[6] Mengmeng Gao, Hebing Rui, and Linliang Song. Representations of weakly triangular categories. Journal of Algebra, 614:481–534, 2023. arXiv:2012.02945.

[7] Steven V. Sam and Andrew Snowden. The representation theory of brauer categories i: Triangular categories. Applied Categorical Structures, 30:1203 – 1256, 2020. arXiv:2006.04328.

[8] Fan Zhou. Bgg resolutions, koszulity, and stratifications, part i: the nilbrauer algebra, 2024. arXiv:2402.06890.