Lecture 5 The minimal genus problem.

$$(X, s) \mapsto SW(X, s) \in \mathbb{Z}$$

Seiberg-Witten invt.

Q $\alpha \in H_2(X;\mathbb{Z})$. What's the minimal genus of a smooth embedded surface $\Sigma \hookrightarrow X$ s.t. $[\Sigma] = \alpha$?

One can use $SW(X,s)$ to provide lower bounds.

Thm (Adjunction ineq). Assume $b_2^+ \geq 2$.

self-int zero.

s spin^c str. s.t. $SW(X,s) \neq 0$.

Then, if Σ is a surface with $g(\Sigma) \geq 1$ and $[\Sigma]\cdot[\Sigma] = 0$

$$2g(\Sigma) - 2 \geq |c_1(s)\cdot[\Sigma]|.$$

Rmk 1 If $[\Sigma]\cdot[\Sigma] \geq 0$.

$$\Rightarrow 2g(\Sigma) - 2 \geq |c_1(s)\cdot[\Sigma]| + [\Sigma]\cdot[\Sigma].$$

Rmk 2 If $\Sigma \subset X$

$\mathbb{C}$-curve $\quad$ $\mathbb{C}$-surface

$$\Rightarrow 2g - 2 = K_X\cdot[\Sigma] + [\Sigma]\cdot[\Sigma].$$

(adjunction equality).

"$\mathbb{C}$-curves are the simplest"

Proof (mod technicalities)

↳ $SW(X,\mathfrak{s}) \neq 0$ ↳ unperturbed eq. for simplicity.

$\Rightarrow \forall g$, $\exists (A,\Phi)$ soln to SW equations.

Variational interpretation:

$$\int |F_{A^t}|^2 + \underbrace{4\int |D_A\Phi|^2 + \int \left(|\Phi|^2 + \frac{s}{2}\right)^2}_{\geq 0} + $$

$$- \int \frac{s^2}{4} - \int F_{A^t} \wedge F_{A^t} = 0 .$$

$$\Rightarrow \int |F_{A^t}|^2 \leq \int \frac{s^2}{4} + \int F_{A^t} \wedge F_{A^t}$$

$$= \int \frac{s^2}{4} - \underbrace{4\pi^2 \langle c_1^2(\mathfrak{s}), [X] \rangle}_{\text{Independent of } g!}$$

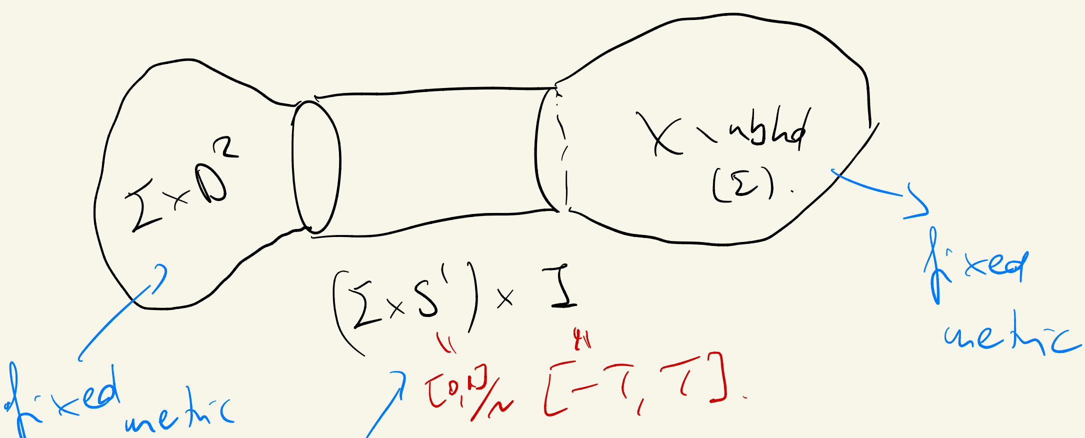
$\forall g$.

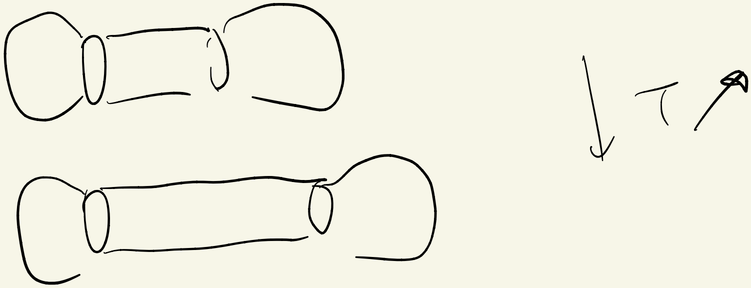
<u>Idea</u> construct an interesting family g_T of metric.

$[\Sigma]\cdot[\Sigma] = 0 \Rightarrow \mathrm{nbhd}(\Sigma) \cong \Sigma \times D^2$.

$\partial\, \mathrm{nbhd}(\Sigma) = \Sigma \times S^1$

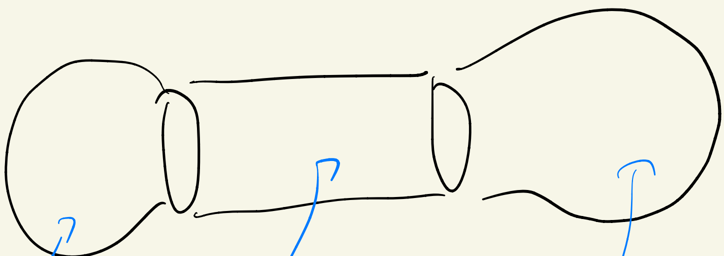
$X \to 3$ pieces





On Σ, pick constant curvature metric of area 1.

(by Gauss Bonnet $\Rightarrow s(\Sigma) = 8\pi(1-g)$.)



$s = 8\pi(1-g)$

s independent of T.

for $\delta\tau$,

$$\int_X \frac{s^2}{4} = C + \int_{\Sigma\times S^1\times[-T,T]} \frac{s^2}{4} =$$

$$= C + 32\pi^2 T\cdot(1-\delta)^2.$$

For F_{A^t}, $\left[\frac{i}{2\pi}F_{A^t}\right] = c_1(S)$.

By Chern-Weil

$$\Rightarrow \int_{\Sigma\times\{\theta\}\times\{t\}} \frac{i}{2\pi}F_{A^t} = \langle c_1(S), [\Sigma]\rangle$$

$$\Rightarrow \int_{\Sigma\times\{\theta\}\times\{t\}} |F_{A^t}|^2 \geq 4\pi^2 |\langle c_1(S), [\Sigma]\rangle|^2$$

(CS + Area$(\Sigma)=1$).

$$\Rightarrow \int_X |F_{A^\epsilon}|^2 \geq \int_{\Sigma \times S^1 \times [-T, T]} |F_{A^\epsilon}|^2 \geq 8\pi^2 T |\langle c_1(s), [\Sigma] \rangle|^2$$

$$\Rightarrow \forall T$$

$$8\pi^2 T |\langle c_1(s), [\Sigma] \rangle|^2 \leq C' + 32\pi^2 T (1-g)^2$$

$$4(1-g)^2 \geq |\langle c_1(s), [\Sigma] \rangle|^2.$$

$$\Downarrow \ (g \geq 1).$$

$$2g - 2 \geq |\langle c_1(s), [\Sigma] \rangle|$$

<u>Applicability</u>: need $SW(X, s) \neq 0$!

(Laura's class).

Today use almost this for <u>Thom conjecture</u>.

$\Sigma \subset \mathbb{C}P^2$ smooth algebraic curve

of degree d

$(\to [\Sigma] = d[\mathbb{C}P^1] \in H_2(\mathbb{C}P^2))$.

$k = -3[\mathbb{C}P^1]$

$$g(\Sigma) = \frac{d^2-3d+2}{2} \qquad \left(\begin{array}{ll} d=1,2 & g=0 \\ d=3 & g=1 \end{array}\right)$$

<u>Thom conj</u> if $\Sigma \subset \mathbb{C}P^2$ represents

$d[\mathbb{C}P^1]$, then $g(\Sigma) \geq \dfrac{d^2-3d+2}{2}$.

$\hookrightarrow$ Proved by Kronheimer – Mrowka

early triumphs of SW invariants.

<u>Strategy</u> $\Sigma \subset \mathbb{C}P^2$, $[\Sigma] = d[\mathbb{C}P^1]$.

$\leadsto [\Sigma]\cdot[\Sigma] = d^2$

blow up at d^2 points on Σ,

$\tilde{\Sigma} \subseteq \mathbb{C}P^2 \# d^2\,\overline{\mathbb{C}P}^2$ $\rightsquigarrow E_i$ exceptional divisor

proper transform

$$[\tilde{\Sigma}] = \underset{[\Sigma]}{d[\mathbb{C}P^1]} - \underset{d^2 \text{ elmts.}}{\sum E_i}$$

$\Rightarrow [\tilde{\Sigma}]\cdot[\tilde{\Sigma}] = 0$

$g(\tilde{\Sigma}) = g(\Sigma)$.

$\boxed{\text{IF}}$ adjunction ineq on $\mathbb{C}P^2$ holds

for the class $c_1(s) = -3[\mathbb{C}P^1] + \sum E_i$ "canonical" K

$$\Rightarrow 2g(\Sigma) - 2 = 2g(\tilde{\Sigma}) - 2 \geq$$

$$\geq \left| \left\langle \underbrace{-3[\mathbb{C}P^1] + \sum E_i}_{PD(c_1(s))} \, , \, \overbrace{d[\mathbb{C}P^1] - \sum E_i}^{[\tilde{\Sigma}]} \right\rangle \right|$$

$$= d^2 - 3d$$

$$\Rightarrow g(\Sigma) \geq \frac{d^2 - 3d + 2}{2}$$

Problems

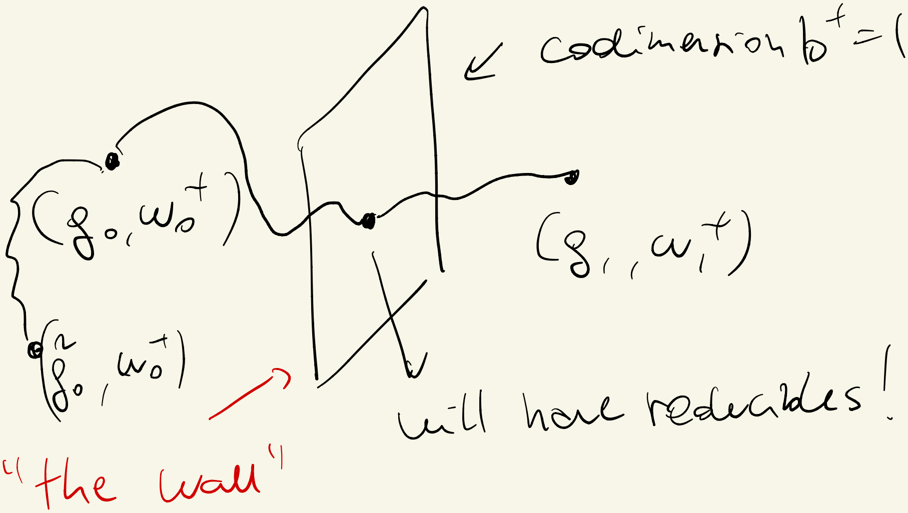
- $b^+(\mathbb{C}P^2 \# d^2 \overline{\mathbb{C}P^2}) = 1$

  $\rightarrow SW(X, s)$ not well-defined.

- how to show $SW(X, s) \neq 0$?

We'll kill two birds with one stone!

by thinking harder about $b^+ = 1$.



$$[\mathcal{M}_{(g_0,\omega_0)}] = [\mathcal{M}_{(\tilde{g}_0,\tilde{\omega}_0)}]$$

$\Rightarrow$ when $b^+=1$, we get 2 invts, one for each side of the wall!

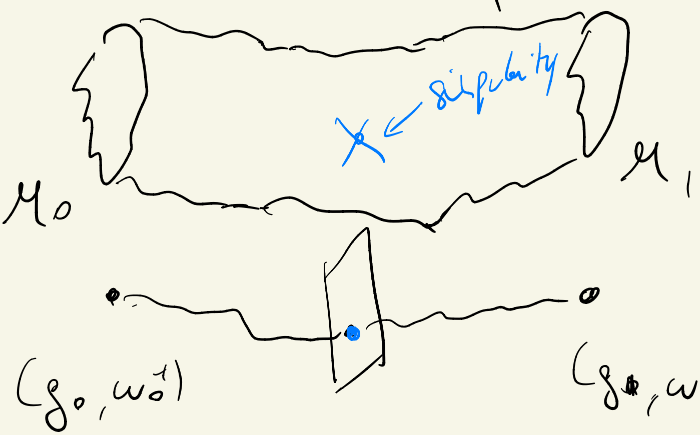
<u>Q</u> what's the relation between the two sides?

Assume $b_1 = 0$. Ex $X = \mathbb{C}P^2 \# n\overline{\mathbb{C}P}^2$

(or in general any mfld with $b_1 = 0$, $b^+ = 1$, $b^- = n$).

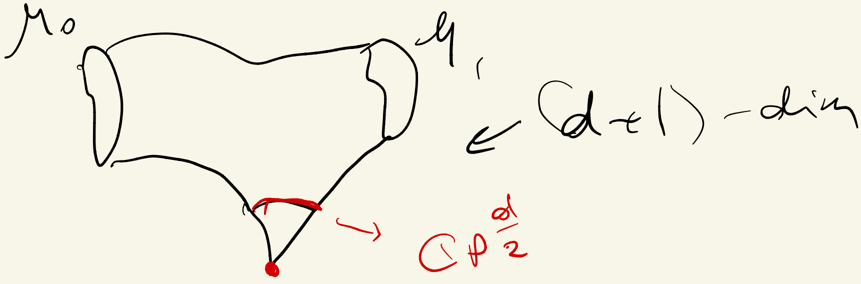
$$\dim \mathcal{M}_{(g_i,\omega_i)}(X,s) = \frac{c_1^2 - 2\chi - 3\sigma}{4} = \frac{c_1^2 - 9 + b_2^-}{4} = d(s)$$

even integer!



If $d(s) \geq 0$, singularity is

a cone over $\mathbb{CP}^{\frac{d}{2}} \subseteq$



s.t

$$[\mathbb{CP}^{\frac{d}{2}}] = U^d \in H_d(\mathbb{CP}^\infty; \mathbb{Z}).$$

$\Rightarrow$ cobordism between M_0, M_1, & $\mathbb{CP}^{\frac{d}{2}}$.

<u>Thm</u> (Wall-crossing formula). If $d(s) \geq 0$, $b_1 = 0$,

(g_0, ω_0), (g_1, ω_1) are on opposite sides of the wall, then

$SW_{(g_0,\omega_0)}(X,s)$ & $SW_{(g_1,\omega_1)}(X,s)$ differ by one!

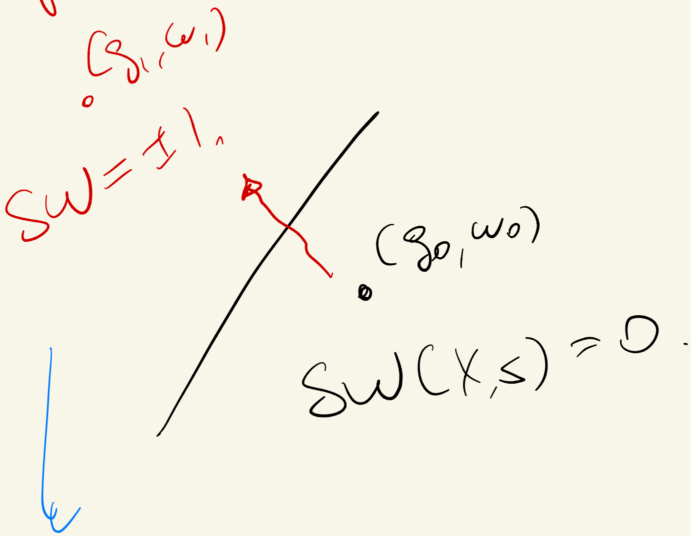
<u>Now</u> $\mathbb{C}P^2 \# n\overline{\mathbb{C}P}^2$ admits a metric g_0 with PSC!

(# Fubini-Study metrics).

$\Rightarrow$ for small perturbation $\omega_0 \sim 0$,

$$SW_{(g_0,\omega_0)}(X,s) \equiv 0.$$

if $d(s) \geq 0$



On this side, there are always solns to (perturbed) SW!

$\Rightarrow$ adjunction ineq.

<u>Technical part</u>: in the argument for Thom, we work on the other side of the wall!

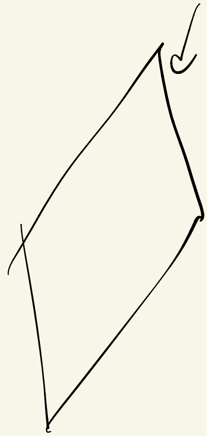
Q <u>Which side of the wall?</u> (fix orientation of $\mathcal{H}^+_g$)

$b^+ = 1 \rightsquigarrow$ pick $k \in \mathcal{H}^+_g \cong \mathbb{R}$.

Recall that the wall is

$\omega \in i\Omega^+$ s.t

$$4i\int \omega\wedge k = 2\pi\left(c_1(S^+)\cup[k]\right)[X].$$



← reducibles.

$\pm$ side of the wall

$$4i\int \omega\wedge k \gtrless 2\pi\left(c_1(S^+)\cup[k]\right)[X].$$

$$\Rightarrow SW^{\pm}(X,s)$$