ARTHUR'S TRUNCATED EISENSTEIN SERIES FOR $SL(2,\mathbf{Z})$ AND THE RIEMANN ZETA FUNCTION, A SURVEY

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ABSTRACT. Eienstein series are not in $\mathcal{L}^2$. Maass, and later Selberg, naively truncated the constant term of Eisenstein series (for symmetric spaces of rank 1) so that the resulting truncated Eisenstein series was square integrable. This procedure was generalized to Eisenstein series on higher rank groups by Langlands and Arthur. In this survey we focus on the deep connections between Eisenstein series for $SL(2,\mathbf{Z})$, truncation, and the Riemann zeta function. Applications to zero free regions for the Riemann zeta function and automorphic L-functions are elucidated.

1. INTRODUCTION

Let $\mathfrak{h} := \{x+iy \mid x \in \mathbf{R}, y > 0\}$ denote the upper half plane. Then, as is well known, the modular group $\Gamma = SL(2,\mathbf{Z})$ acts on $\mathfrak{h}$ as follows. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $z \in \mathfrak{h}$, the action of γ on z is given by $\gamma z := (az+b)/(cz+d)$.

The Hilbert space $\mathcal{L}^2(\Gamma\backslash\mathfrak{h})$ of smooth $\mathcal{L}^2$ automorphic functions $f : \mathfrak{h} \to \mathbf{C}$ satisfying

$$f\left(\frac{az+b}{cz+d}\right) = f(z), \qquad \text{for all } z \in \mathfrak{h},\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

with Petersson inner product

$$\langle f,\ g\rangle := \iint\limits_{\Gamma\backslash\mathfrak{h}} f(z)\cdot\overline{g(z)}\ \frac{dxdy}{y^2}, \qquad f, g \in \mathcal{L}^2(\mathfrak{h}/\Gamma),$$

has been extensively studied. Here $f \in \mathcal{L}^2$ means

$$\int\limits_{\Gamma\backslash\mathfrak{h}} |f(x+iy)|^2\,\frac{dxdy}{y^2} < \infty,$$

where $\Gamma\backslash\mathfrak{h}$ can be taken as the region $\mathcal{F} := \{z \in \mathfrak{h} \mid |z| \geq 1,\ |\Re(z)| \leq 1/2\}$ with congruent boundary points symmetric with respect to the imaginary axis.

One of the first examples of automorphic functions discovered is the Eisenstein series. For a complex variable s with $\Re(s) > 1$ and $z = x+iy \in \mathfrak{h}$, the Eisenstein series for the modular group $\Gamma = SL(2,\mathbf{Z})$ is defined to be

$$E(z,s) := \frac{1}{2}\sum_{\substack{c,\,d\,\in\,\mathbf{Z}\\(c,d)=1}} \frac{y^s}{|cz+d|^{2s}}.$$

It is not hard to show that (see [9])

$$E\left(\frac{az+b}{cz+d}\right) = E(z,s), \qquad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

In this survey, we will examine the intimate connection between Eisenstein series and the Riemann zeta function $\zeta(s)$, which is already apparent in the well known Fourier expansion [9]:

$$E(z,s) = y^s \ + \ \phi(s) y^{1-s} \ + \ \frac{2\pi^s \sqrt{y}}{\Gamma(s)\zeta(2s)} \sum_{n\neq 0} \sigma_{1-2s}(n) |n|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}\big(2\pi |n| y\big) e^{2\pi i n x}, \tag{1}$$

where

$$\phi(s) = \sqrt{\pi} \frac{\Gamma\left(s-\frac{1}{2}\right)\zeta(2s-1)}{\Gamma(s)\zeta(2s)}, \quad \sigma_s(n) = \sum_{\substack{d|n \\ d\geq 1}} d^s, \quad K_s(y) = \frac{1}{2}\int_0^\infty e^{-\frac{y}{2}\left(u+u^{-1}\right)} u^s \, \frac{du}{u}.$$

Here $K_s(y)$ is the modified Bessel function of the second kind.

Consider the completed Eisenstein series

$$E^*(z,s) := \zeta^*(2s) E(z,s),$$

where $\zeta^*(s) = \pi^{-\frac{s}{2}}\Gamma\left(s/2\right)\zeta(s) = \zeta^*(1-s)$. Then $E^*(z,s)$ has the Fourier expansion

$$E^*(z,s) = \zeta^*(2s) y^s + \zeta^*(2-2s) y^{1-s} + 2\sqrt{y} \sum_{n\neq 0} \sigma_{1-2s}(n) |n|^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi |n| y) e^{2\pi i n x}$$

from which one easily deduces that

$$E^*(z,s) = E^*(z,1-s)$$

since $K_s(y) = K_{-s}(y)$ for any $s \in \mathbf{C}$ and $y > 0$. The analytic continuation of the Riemann zeta function then gives the meromorphic continuation of the Eisenstein series to the whole complex plane.

The modified Riemann hypothesis for a function $Z(s)$ asserts that all its zeros are either on the line $\Re(s) = 1$ or on the real axis in the interval $(0,1)$. Hejhal [11] proved, in 1990, that the constant term

$$\zeta^*(2s) y^s + \zeta^*(2-2s) y^{1-s}$$

of the completed Eisenstein series satisfies the modified Riemann hypothesis if $y \geq 1$. His proof used the Maass-Selberg relation. This result was later extended in [13].

For fixed $s \in \mathbf{C}$, the Bessel function $K_s(y)$ has exponential decay in y for $y \to \infty$. This implies that for large $y > 0$, we have $E(z,s) \ \sim \ y^s \ + \ \phi(s) y^{1-s}$. Consequently

$$\int\limits_{\Gamma\backslash\mathfrak{h}} \left| y^s \ + \ \phi(s) y^{1-s} \right|^2 \ \frac{dxdy}{y^2} \ > \int\limits_{x=-\frac{1}{2}}^{\frac{1}{2}} \int\limits_{y=0}^{\infty} \left| y^s \ + \ \phi(s) y^{1-s} \right|^2 \ \frac{dxdy}{y^2} = \infty$$

for any value of $s \in \mathbf{C}$. It follows that the Eisenstein series is not in $\mathcal{L}^2(\Gamma\backslash\mathfrak{h})$. To overcome the fact that $E(z,s)$ is not in $\mathcal{L}^2$ and faciliate the computation of integrals involving Eisenstein series, Maass [16], in 1949, introduced the naive truncated Eisenstein series $E^T(z,s)$. Let $T > 0$. Then for $z \in \mathfrak{h}$, $s \in \mathbb{C}$,

$$E^T(z,s) := \begin{cases} E(z,s) & \text{if } y < T, \\ E(z,s) - y^s - \phi(s) y^{-s} & \text{if } y > T. \end{cases}$$

This function is not continuous in z but will lie in $\mathcal{L}^2$. It is also not automorphic, i.e.,

$$E^T\left(\frac{az+b}{cz+d},\ s\right) \neq E^T(z,s)$$

for most $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

In 1962, Selberg [19] extended Maass' work on naive truncated Eisenstein series to the more general situation of symmetric spaces of rank 1. The inner product of two truncated Eisenstein series (Maass-Selberg relations), which was based on Green's theorem, played a crucial role in the computation of the Selberg trace formula [18]. The more general Arthur-Selberg trace formula (see [15]) has been central for the advancement of modern number theory.

One of the most basic tools in the theory of automorphic forms on higher rank groups is the truncation operator which was first introduced by Langlands [14] in 1966. In 1978-1980 Arthur, [1], [2], made an important advance by extending and making much more explicit Langlands' truncation operator, especially when applied to Eisenstein series. In particular, Arthur's truncation involves a summation $\sum_F (-1)^{\mathrm{codim}(F)} \mathcal{E}_F$ for a certain characteristic function $\mathcal{E}_F$ over the faces F of a certain convex polyhedron in a Euclidean space V.

The key property we want to point out for this survey is that Arthur's truncated Eisenstein series are automorphic which was not the case with Maass and Selberg's naively truncated Eisenstein series. Arthur's truncated Eisenstein series greatly simplified the computation of the Maass-Selberg relations, especially in the situation of higher rank groups. The use of differential operators (via Green's theorem) was avoided and all proofs of the Maass-Selberg relations were reduced to algebraic computations as in the Rankin-Selberg method [9].

In this survey we focus on Arthur's truncated Eisenstein series for the modular group Γ and its connections with the Riemann zeta function. The reader may find it useful to concurrently look at other surveys on truncation such as [4], [5], [6].

2. Arthur's truncated Eisenstein series

In order to define Arthur's truncated Eisenstein series for Γ we require some preliminary definitions. A function $F : \mathfrak{h} \to \mathbf{C}$ is said to be periodic if $F(z+1) = F(z)$ for all $z \in \mathfrak{h}$.

Definition (Constant term): *For a piecewise continuous periodic function $F : \mathfrak{h} \to \mathbf{C}$ define*

$$CF(z) := \int_0^1 F(z+u)\, du, \qquad \text{(for } z \in \mathfrak{h}\text{)},$$

to be the constant term of F.

Definition (Truncated Constant term): *Let $T > 0$ and $z = x + iy \in \mathfrak{h}$. For an integrable periodic function $F : \mathfrak{h} \to \mathbf{C}$ define*

$$C^T F(z) := \begin{cases} 0 & \text{if } 0 \le y \le T, \\ CF(z) & \text{if } y > T, \end{cases}$$

to be the truncated constant term of F.

Definition (Arthur's Truncation Operator): *Let $T > 0$ and $z \in \mathfrak{h}$. For an integrable periodic function $F : \mathfrak{h} \to \mathbf{C}$ define*

$$\Lambda^T F(z) \;:=\; F(z) \;-\; \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} C^T F(\gamma z),$$

to be the truncation of the function $F(z)$ with respect to T. Here

$$\Gamma_\infty := \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, \quad m \in \mathbf{Z} \right\}$$

is the stabilizer of ∞ under the action of Γ.

Definition (Polynomial Growth): A periodic function $\phi : \mathfrak{h} \to \mathbf{C}$ has polynomial growth if there exist constants $C, B > 0$ such that

$$|\phi(z)| \;\le\; C\,\mathrm{Im}(z)^B, \qquad (\mathrm{Im}(z) \to \infty).$$

The following theorem summarizes the main properties of the truncation operator Λ^T.

Theorem 2.1. (Properties of Λ^T) *Let $T > 0$. The truncation operator Λ^T satisfies the following properties.*

- *Let $\phi \in \mathcal{L}^2(\Gamma\backslash\mathfrak{h})$ be a cusp form (the constant term of ϕ is zero) then $\Lambda^T \phi = \phi$.*
- *Let $\phi : \Gamma\backslash\mathfrak{h} \to \mathbf{C}$ be integrable. Assume $T \ge 1$ and $z \in \mathcal{F}$. Then $\Lambda^T\phi(z) = \phi - C^T\phi(z)$.*
- *Assume $\phi : \Gamma\backslash\mathfrak{h} \to \mathbf{C}$ is smooth, is an eigenfunction of the invariant Laplacian, and has polynomial growth. Then $\Lambda^T\phi \in \mathcal{L}^2(\Gamma\backslash\mathfrak{h})$.*
- *Let $\eta, \phi \in \mathcal{L}^2(\Gamma\backslash\mathfrak{h})$. Then $\langle \Lambda^T\eta,\ \phi\rangle = \langle \eta, \Lambda^T\phi\rangle$.*

Proof of Theorem 2.1:

It is clear that if ϕ is a cusp form then truncation has no effect since the constant term of ϕ is zero. Next, assume z lies in the fundamental domain $\mathcal{F}$. Then for $\gamma \in \Gamma/\Gamma_\infty$, with $\gamma \notin \Gamma_\infty$, the complex number γz must lie outside of $\mathcal{F}$ and not in a translate $m + \mathcal{F}$ with $m \in \mathbf{Z}$. It is easy to see that in this situation, we must have

$$\mathrm{Im}(\gamma z) < 1.$$

The second bulleted item immediately follows from this. The third bulleted item is much more difficult to prove, so we do not give the details. The last bulleted item is easily proved with a change of variables $z \mapsto \gamma^{-1} z$ for $\gamma \in \Gamma/\Gamma_\infty$. □

We now carefully examine the action of the truncation operator on the Eisenstein series $E(z,s)$ with Fourier expansion given in (1). Then

$$CE(z,s) = y^s \;+\; \phi(s) y^{1-s},$$

and for $T > 0$,

$$C^T E(z,s) = \left(y^s \;+\; \phi(s) y^{1-s}\right) \cdot \mathfrak{char}_{[T,\,\infty]}(y),$$

where

$$\mathfrak{char}_{[\alpha,\,\beta]}(y) = \begin{cases} 1 & \text{if } \alpha \le y \le \beta, \\ 0 & \text{otherwise.} \end{cases}$$

It immediately follows that for $T > 0$, Arthur's truncated Eisenstein series $\Lambda^T E(z,s)$ is given by

$$\Lambda^T E(z,s) = E(z,s) \;-\; \sum_{\substack{\gamma\in\Gamma_\infty\backslash\Gamma \\ \mathrm{Im}(\gamma z)\ge T}} \Big(\mathrm{Im}(\gamma z)^s \;+\; \phi(s)\,\mathrm{Im}(\gamma z)^{1-s}\Big). \tag{2}$$

3. Fourier expansion of Arthur's truncated Eisenstein series

We now compute the Fourier expansion of $\Lambda^T E(z,s)$. The proof of the Fourier expansion requires the following lemma.

Lemma 3.1. *Let $v \in \mathbf{C}$ with $\Re(v) > 0$. Then*

$$\frac{1}{2\pi i}\int\limits_{2-i\infty}^{2+i\infty} \frac{Y^w}{w^{1+v}}\,dw \;=\; \begin{cases} \frac{(\log Y)^v}{\Gamma(1+v)} & \text{if } 1 \le Y, \\ 0 & \text{if } 0 \le Y < 1. \end{cases}$$

Proof of Lemma 3.1:

For $\Re(w) > 0$ and $\Re(v) > 0$ we have

$$w^{-v} = \frac{1}{\Gamma(v)}\int_0^\infty e^{-wx}\,x^v\,\frac{dx}{x}.$$

It follows that

$$\begin{aligned}\frac{1}{2\pi i}\int\limits_{2-i\infty}^{2+i\infty} \frac{Y^w}{w^{1+v}}\,dw \;&=\; \frac{1}{2\pi i}\int\limits_{2-i\infty}^{2+i\infty} \frac{Y^w}{w}\cdot\left(\frac{1}{\Gamma(v)}\int\limits_0^\infty e^{-wx}\,x^v\,\frac{dx}{x}\right)\,dw \\ &=\; \frac{1}{\Gamma(v)}\int\limits_0^\infty x^v\cdot\frac{1}{2\pi i}\int\limits_{2-i\infty}^{2+i\infty}\left(\frac{Y}{e^x}\right)^w\,\frac{dw}{w}\,\frac{dx}{x} \\ &=\; \frac{1}{\Gamma(v)}\int\limits_0^{\log Y} x^v\,\frac{dx}{x} \;=\; \frac{(\log Y)^v}{\Gamma(1+v)}. \quad \square\end{aligned}$$

It immediately follows from (2) and lemma 3.1 that

$$\Lambda^T E(z,s) = E(z,s) \;-\; E^T(z,s) - \phi(s)E^T(z,1-s),$$

where we have defined

$$E^T(z,s) := \lim_{v\to 0}\frac{1}{2\pi i}\int\limits_{2-i\infty}^{2+i\infty}\frac{T^{-w}}{w^{1+v}}E(z,s+w)\,dw.$$

Warning: The function $E^T(z,s)$ is not the naive truncation of the Eisenstein series that we considered earlier in the introduction.

Theorem 3.2. (Fourier Expansion of E^T) *Let $T>0$, and $x+iy\in\mathfrak{h}$. Let $\mu(n)$ denote the Mobius function. Then*

$$\int_0^1 E^T(x+iy,s)e^{-2\pi inx}\,dx$$

$$=\begin{cases} y^s\cdot\mathfrak{char}_{[T,\,\infty]}(y)\ +\ \frac{\mathfrak{char}_{[0,\,1/T]}(y)}{2\pi i}\int\limits_{2-i\infty}^{2+i\infty}\frac{T^{-w}}{w}\phi(s+w)\,y^{1-s-w}\,dw & \text{if } n=0,\\[2ex] 0 & \text{if } yT\geq 1 \text{ and } n\neq 0,\\[2ex] 2y^{1-s}\sum\limits_{\substack{d\mid n\\ yTd^2\ell^2\,\leq 1\\ d\geq 1}}\sum\limits_{\ell=1}^{\infty} d^{1-2s}\,\frac{\mu(\ell)}{\ell^{2s}}\int\limits_0^{\sqrt{\frac{1}{Tyd^2\ell^2}-1}}\frac{\cos(2\pi nyt)}{(t^2+1)^s}\,dt & \text{otherwise.}\end{cases}$$

Theorem 3.3. Constant Term of $\Lambda^T E(z,s)$) *Let $T>0$ and $z=x+iy\in\mathfrak{h}$. Then the constant term of $\Lambda^T E(z,s)$ is given by*

$$\int\limits_0^1 \Lambda^T E(z+u,s)\,du = \big(y^s+\phi(s)y^{1-s}\big)\cdot\Big(1-\mathfrak{char}_{[T,\,\infty]}(y)\Big)$$

$$-\ \mathfrak{char}_{[0,\,1/T]}(y)\cdot\frac{1}{2\pi i}\int\limits_{2-i\infty}^{2+i\infty}\frac{T^{-w}}{w}\Big(\phi(s+w)y^{1-s-w}\ +\ \phi(s)\phi(1-s+w)y^{-s+w}\Big)\,dw.$$

Proof of Theorem 3.2:

With the Fourier expansion (1), we compute

$$\int_0^1 E^T(x+iy,s)e^{-2\pi inx}\,dx$$

$$=\begin{cases}\lim\limits_{v\to 0}\frac{1}{2\pi i}\int\limits_{2-i\infty}^{2+i\infty}\frac{T^{-w}}{w^{1+v}}\Big(y^{s+w}+\phi(s+w)y^{1-s-w}\Big)\,dw & n=0,\\[2ex] \lim\limits_{v\to 0}\frac{1}{2\pi i}\int\limits_{2-i\infty}^{2+i\infty}\frac{T^{-w}}{w^{1+v}}\frac{2\pi^{s+w}\sqrt{y}}{\Gamma(s+w)\zeta(2s+2w)}\sum\limits_{n\neq 0}\sigma_{1-2s-2w}(n)|n|^{s+w-\frac12}K_{s+w-\frac12}\big(2\pi|n|y\big)\,dw & n\neq 0.\end{cases}$$

We evaluate the above integrals individually. It follows from lemma 3.1 that

$$\lim_{v\to 0}\frac{1}{2\pi i}\int\limits_{2-i\infty}^{2+i\infty}\frac{T^{-w}}{w^{1+v}}\Big(y^{s+w}+\phi(s+w)y^{1-s-w}\Big)\,dw$$

$$=\ y^s\cdot\mathfrak{char}_{[T,\,\infty]}(y)\ +\ \left(\frac{1}{2\pi i}\int\limits_{2-i\infty}^{2+i\infty}\frac{T^{-w}}{w}\phi(s+w)\,y^{1-s-w}\,dw\right)\cdot\mathfrak{char}_{[0,\,1/T]}(y).$$

Note that if $yT > 1$, then the integral on the right above is easily seen to be zero by shifting the line of integration to the right. This is the reason why we can multiply the integral by $\mathfrak{char}_{[0,\,1/T]}(y)$. This establishes the first case of theorem 3.2.

Next, we consider the integral involving the K-Bessel function. To simplify the computation we make use of the following integral representation:

$$K_{s+w-\frac{1}{2}}(2\pi|n|y) = \frac{\Gamma\left(s+w\right)\left(4\pi|n|y\right)^{s+w-\frac{1}{2}}}{\sqrt{\pi}} \int\limits_0^\infty \frac{\cos(t)}{\left(t^2+4\pi^2n^2y^2\right)^{s+w}}\, dt.$$

It follows that

$$\lim_{v\to 0} \frac{1}{2\pi i} \int\limits_{2-i\infty}^{2+i\infty} \frac{T^{-w}}{w^{1+v}} \frac{2\pi^{s+w}\sqrt{y}}{\Gamma(s+w)\zeta(2s+2w)} \sigma_{1-2s-2w}(n)|n|^{s+w-\frac{1}{2}} K_{s+w-\frac{1}{2}}\big(2\pi|n|y\big)\, dw$$

$$= \lim_{v\to 0} \frac{1}{2\pi i} \int\limits_{2-i\infty}^{2+i\infty} \frac{T^{-w}}{w^{1+v}} \frac{2^{2s+2w}\pi^{2s+2w-1}y^{s+w}}{\zeta(2s+2w)} \sum_{d|n} d^{1-2s-2w}\,|n|^{2s+2w-1} \int\limits_0^\infty \frac{\cos(t)}{\left(t^2+4\pi^2n^2y^2\right)^{s+w}}\, dt\, dw$$

$$= \lim_{v\to 0}\; 2(2\pi|n|)^{2s-1}y^s \sum_{\ell=1}^\infty \frac{\mu(\ell)}{\ell^{2s}} \sum_{d|n} d^{1-2s} \int\limits_{t=0}^\infty \frac{\cos(t)}{\left(t^2+4\pi^2n^2y^2\right)^s}$$
$$\cdot \frac{1}{2\pi i} \int\limits_{2-i\infty}^{2+i\infty} \left(\frac{4\pi^2n^2y}{T\ell^2d^2\left(t^2+4\pi^2n^2y^2\right)}\right)^w \frac{dw}{w^{1+v}}\, dt.$$

Now,

$$\lim_{v\to 0} \frac{1}{2\pi i} \int\limits_{2-i\infty}^{2+i\infty} \left(\frac{4\pi^2n^2y}{T\ell^2d^2\left(t^2+4\pi^2n^2y^2\right)}\right)^w \frac{dw}{w^{1+v}} = \begin{cases} 1 & \text{if } \frac{4\pi^2n^2y}{T\ell^2d^2\left(t^2+4\pi^2n^2y^2\right)} > 1, \\ 0 & \text{otherwise.} \end{cases}$$

The condition

$$\frac{2\pi^2n^2y}{T\ell^2d^2\left(t^2+4\pi^2n^2y^2\right)} > 1$$

is equivalent to

$$yT\ell^2d^2 \le 1 \qquad \text{and} \qquad 0 \le t^2 \le 4\pi^2n^2y^2\left(\frac{1}{yT\ell^2d^2} - 1\right).$$

It follows that

$$\lim_{v\to 0}\frac{1}{2\pi i}\int\limits_{2-i\infty}^{2+i\infty}\frac{T^{-w}}{w^{1+v}}\frac{2\pi^{s+w}\sqrt{y}}{\Gamma(s+w)\zeta(2s+2w)}\sigma_{1-2s-2w}(n)|n|^{s+w-\frac12}K_{s+w-\frac12}\big(2\pi|n|y\big)\,dw$$

$$= 2(2\pi)^{2s-1}y^s\sum_{\substack{d|n\\ yT\ell^2d^2\le 1}}\sum_{\ell=1}^{\infty}|n|^{2s-1}\,\frac{\mu(\ell)}{\ell^{2s}}d^{1-2s}\int\limits_{t=0}^{2\pi|n|y\sqrt{\frac{1}{yT\ell^2d^2}-1}}\frac{\cos(t)}{\big(t^2+4\pi^2n^2y^2\big)^s}\,dt$$

$$= 2y^{1-s}\sum_{\substack{d|n\\ yTd^2\ell^2\ \le 1}}\sum_{\ell=1}^{\infty}d^{1-2s}\,\frac{\mu(\ell)}{\ell^{2s}}\int\limits_{0}^{\sqrt{\frac{1}{Tyd^2\ell^2}-1}}\frac{\cos(2\pi nyt)}{(t^2+1)^s}\;dt.$$

If $n\neq 0$ and $yT>1$ it is not possible to have integers $d,\ell\geq 1$ such that $yTd^2\ell^2\leq 1$. Therefore the above integral vanishes and the second case of theorem 3.2 has been proved. The third case of theorem 3.2 follows immediately from the above. □

Remarks: In [2], Arthur defined truncated Eisenstein series in great generality. The explicit computation of the Fourier expansion of Arthur's truncated Eisenstein series for higher rank groups, such as $SL(n,\mathbf{Z})$ with $n>2$, seems out of reach at present. This is not the case for $SL(2,\mathbf{Z})$ because of the existence of so many special known integral representations for Bessel functions. Our knowledge of higher rank Whittaker functions is much more limited at present.

4. Maass-Selberg relation for $SL(2,\mathbf{Z})$

In this section we compute the inner product of two truncated Eisenstein series which is called the Maass-Selberg relation. Arthur's proof of the Maass Selberg relation just uses the Rankin-Selberg method [9]. It is an algebraic proof based on the fact that if a group G acts on a topological space X then the union of all translates $g\cdot(G\backslash X)$ (with $g\in G$) is just the space X. More generally for a subgroup $H\leq G$, we also have

$$\bigcup_{g\in H\backslash G} g\cdot(G\backslash X)\;=\;H\backslash X.$$

Arthur's algebraic proof stands in stark contrast to the original analytic proof of Maass and Selberg which required differential operators and Green's theorem.

Theorem 4.1. Maass-Selberg Relation *Let $s,s'\in\mathbf{C}$ and $T\geq 1$. Then*

$$\Big\langle\Lambda^T E(*,s),\;E(*,s')\Big\rangle=\frac{T^{s+\overline{s'}-1}}{s+\overline{s'}-1}+\overline{\phi(s')}\frac{T^{s-\overline{s'}}}{s-\overline{s'}}+\phi(s)\frac{T^{\overline{s'}-s}}{\overline{s'}-s}+\phi(s)\overline{\phi(s')}\frac{T^{1-s-\overline{s'}}}{1-s-\overline{s'}}.$$

Proof of Theorem 4.1: It is clear that

$$\Big\langle \Lambda^T E(*,s),\ \Lambda^T(E(*,s')\Big\rangle = \iint\limits_{\Gamma\backslash\mathfrak{h}} \Lambda^T E(z,s)\cdot\Big(\overline{E(z,s)\ -\ E^T(z,s)-\phi(s)E^T(z,1-s)}\Big)\ \frac{dxdy}{y^2}. \tag{3}$$

To simplify the computation of (3) we require a lemma.

Lemma 4.2. *We have* $\Big\langle \Lambda^T E(*,s),\ \Lambda^T(E(*,s')\Big\rangle = \Big\langle \Lambda^T E(*,s),\ (E(*,s')\Big\rangle.$

Proof of Lemma 4.2:

By (3), it is enough to prove that

$$\iint\limits_{\Gamma\backslash\mathfrak{h}} \Lambda^T E(z,s)\cdot\Big(\overline{E^T(z,s')+\phi(s')E^T(z,1-s')}\Big)\ \frac{dxdy}{y^2}\ =\ 0.$$

This can be shown as follows:

$$\begin{aligned}
&\iint\limits_{\Gamma\backslash\mathfrak{h}} \Lambda^T E(z,s)\cdot\Big(\overline{E^T(z,s')+\phi(s')E^T(z,1-s')}\Big)\ \frac{dxdy}{y^2}\\
&= \iint\limits_{\Gamma\backslash\mathfrak{h}} \Lambda^T E(z,s) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\bigg(\overline{\Big(\,\mathrm{Im}(\gamma z)^{s'}+\phi(s')\mathrm{Im}(\gamma z)^{1-s'}\Big)\cdot \mathfrak{char}_{[T,\,\infty]}(\mathrm{Im}(\gamma z))}\bigg)\frac{dxdy}{y^2}\\
&\qquad = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\ \iint\limits_{\gamma^{-1}(\Gamma\backslash\mathfrak{h})} \Lambda^T E(z,s)\bigg(\overline{y^{s'}+\phi(s')y^{1-s'}\cdot\mathfrak{char}_{[T,\,\infty]}(y)}\bigg)\frac{dxdy}{y^2}\\
&\qquad = \iint\limits_{\Gamma_\infty\backslash\mathfrak{h}} \Lambda^T E(z,s)\overline{\Big(\ y^{s'}+\phi(s')y^{1-s'}\Big)}\cdot\mathfrak{char}_{[T,\,\infty]}(y)\frac{dxdy}{y^2}\\
&\qquad = \int_0^1\int_0^\infty \Lambda^T E(z,s)\Big(\ y^{\overline{s'}}+\overline{\phi(s')}y^{1-\overline{s'}}\Big)\cdot\mathfrak{char}_{[T,\,\infty]}(y)\frac{dxdy}{y^2}.
\end{aligned}$$

This procedure is termed: *"unravelling* $\overline{E^T(z,s)+\phi(s)E^T(z,1-s)}$*".* The x-integral above picks off the constant term of $\Lambda^T E(z,s)$ which is given in theorem 3.3 and equals

$$\begin{aligned}
&\Big(y^s+\phi(s)y^{1-s}\Big)\cdot\Big(1-\mathfrak{char}_{[T,\,\infty]}(y)\Big)\\
&\qquad -\mathfrak{char}_{[0,\,1/T]}(y)\cdot\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{T^{-w}}{w}\Big(\phi(s+w)y^{1-s-w}\ +\ \phi(s)\phi(1-s+w)y^{-s+w}\Big)\,dw.
\end{aligned}$$

Since

$$\Big(1-\mathfrak{char}_{[T,\,\infty]}(y)\Big)\cdot\mathfrak{char}_{[T,\,\infty]}(y)\ =\mathfrak{char}_{[0,\,1/T]}(y)\cdot\mathfrak{char}_{[T,\,\infty]}(y)\ =\ 0$$

for $T\ge 1$ the proof of the lemma follows immediately. □

To complete the proof of the Maass-Selberg relation we apply lemma 4.2 and unravel $\Lambda^T E(z,s)$. We assume for the moment that $\Re(s+s')>1$ and $\Re(s)>\Re(s')$. It follows that

$$\begin{aligned}
\Big\langle \Lambda^T E(*,s),\ \Lambda^T(E(*,s')\Big\rangle &= \Big\langle \Lambda^T E(*,s),\ (E(*,s')\Big\rangle \\
&= \int_0^1\int_0^\infty \Big[y^s - \big(y^s+\phi(s)y^{1-s}\big)\,\mathfrak{char}_{[T,\,\infty]}(y)\Big]\cdot\overline{E(z,s')}\,\frac{dxdy}{y^2} \\
&= \int_0^\infty \Big[y^s - \big(y^s+\phi(s)y^{1-s}\big)\,\mathfrak{char}_{[T,\,\infty]}(y)\Big]\cdot\Big(y^{\overline{s'}}+\overline{\phi(s')}y^{1-\overline{s'}}\Big)\ \frac{dy}{y^2} \\
&= \int_0^\infty y^s\Big(y^{\overline{s'}}+\overline{\phi(s')}y^{1-\overline{s'}}\Big)\ \Big(1-\mathfrak{char}_{[T,\,\infty]}(y)\Big)\frac{dy}{y^2} \\
&\qquad - \int_0^\infty \phi(s)y^{1-s}\Big(y^{\overline{s'}}+\overline{\phi(s')}y^{1-\overline{s'}}\Big)\ \mathfrak{char}_{[T,\,\infty]}(y)\ \frac{dy}{y^2} \\
&= \int_0^T \Big(y^{s+\overline{s'}-2}+\overline{\phi(s')}y^{s-\overline{s'}-1}\Big)\,dy - \int_T^\infty \phi(s)\Big(y^{-s+\overline{s'}-1}+\overline{\phi(s')}y^{-s-\overline{s'}}\Big)\,dy \\
&= \frac{T^{s+\overline{s'}-1}}{s+\overline{s'}-1}+\overline{\phi(s')}\frac{T^{s-\overline{s'}}}{s-\overline{s'}}+\phi(s)\frac{T^{\overline{s'}-s}}{\overline{s'}-s}+\phi(s)\overline{\phi(s')}\frac{T^{1-s-\overline{s'}}}{1-s-\overline{s'}}.
\end{aligned}$$

This identity then extends by analytic continuation in s, s' to all $s, s' \in \mathbf{C}$.

□

5. A zero free region for the Riemann zeta function

It was proved independently by Hadamard and de la Vallée Poussin in 1896 that the Riemann zeta function $\zeta(s)$ has no zeros on the line $\Re(s)=1$ (see [20]). Only a few different proofs of this are now known, but the most intriguing is the proof, first discovered by Selberg (unpublished), that is based on the Eisenstein series $E(z,s)$.

Recall the completed Eisenstein series

$$E^*(z,s) := \zeta^*(2s)E(z,s),$$

with $\zeta^*(s)=\pi^{-\frac{s}{2}}\Gamma(s/2)\,\zeta(s)=\zeta^*(1-s)$, where $E^*(z,s)$ has the Fourier expansion

$$E^*(z,s)=\zeta^*(2s)y^s+\zeta^*(2-2s)y^{1-s}+2\sqrt{y}\sum_{n\neq 0}\sigma_{1-2s}(n)|n|^{s-\frac12}K_{s-\frac12}(2\pi|n|y)e^{2\pi inx}.$$

Now if $\zeta(1+it_0)=0$ for some real number t_0 this implies that

$$E^*\big(z,\ (1+it_0)/2\big)=2\sqrt{y}\sum_{n\neq 0}\sigma_{-it_0}(n)|n|^{it_0}K_{it_0/2}(2\pi|n|y)e^{2\pi inx}$$

which would be a (not identically zero) square integrable non-holomorphic cusp form for $SL(2,Z)$. This is not possible! A simple way to see that it is not possible is to consider the inner product $\big\langle E^*\big(\,*,(1+2it_0)/2\big),\ E^*\big(\,*,(1+2it_0)/2\big)\big\rangle$ which is a convergent integral (because of the rapid decay of the K-Bessel function) which must have a positive value > 0. This is a contradiction because by analytic continuation in the variable w it follows from the Rankin-Selberg unfolding method that

$$\left\langle E^*\big(\,*,(1+it_0)/2\big),\ E^*\big(\,*,\overline{w}\big)\right\rangle = \int_0^1\int_0^\infty E^*(z,\ (1+it_0)/2)\cdot\zeta^*(2w)\,y^w\,\frac{dxdy}{y^2} \;=\; 0$$

which vanishes for all $w \in \mathbf{C}$ because the constant term of $E^*(z,\ (1+it_0)/2)$ is zero.

This method was later generalized by Jacquet and Shalika [12] where they prove that all automorphic L-functions of the group $GL(n)$ do not vanish on the line $\Re(s) = 1$. The Eisenstein approach is very powerful. It is possible to obtain standard zero free regions of the form

$$\Re(s) > 1 - \frac{c}{\log(2+\mathrm{Im}(s))^B},$$

with explicit constants $B, c > 0$, for many families of L-functions by use of the Maass-Selberg relations following an approach developed by Sarnak [17] in 2004. In this regard see [3], [7], [8], [10]. All these recent results specifically require the theory of Arthur's truncated Eisenstein series for higher rank groups.

To complete this survey, we present a short sketch of Sarnak's 2004 method in the setting of Arthur's truncated Eisenstein series. The following theorem (proved in [17]) is equivalent to a standard zero free region for $\zeta(s)$.

Theorem 5.1. *There exists an effectively computable constant $C > 0$ such that*

$$|\zeta(1+2it)| \;>\; \frac{C}{\log\big(2+|t|\big)^3}$$

for all $t \in \mathbf{R}$.

Proof of Theorem 5.1:

Let $T > 1$, $0 < \eta < 1$ be fixed and $t \in \mathbf{R}$ with $|t|$ large. The main idea of the proof is to evaluate the following integral

$$\mathcal{I}^T(t,\eta) := |\zeta(1+2it)|^2 \int_\eta^\infty\int_0^1 \left|\Lambda^T E(z,\ 1/2+it)\right|^2\ \frac{dxdy}{y^2}$$

in two different ways.

First computation of $\mathcal{I}^T(t,\eta)$ (Upper Bound):

We will prove the upper bound

$$\boxed{\mathcal{I}^T(t,\eta) \;\ll\; \frac{|\zeta(1+2it)}{\eta}\cdot\big(\log(2+|t|)^2 + 2\log(T)\big).} \tag{4}$$

To prove (4) note that

$$\mathcal{I}^T(t,\eta) = |\zeta(1+2it)|^2 \iint\limits_{\Gamma\backslash\mathfrak{h}} N(z,\eta)\left|\Lambda^T E(z,\ 1/2+it)\right|^2\ \frac{dxdy}{y^2}$$

where

$$N(z,\eta) := \#\{\gamma \in \Gamma_\infty\backslash\Gamma \mid \mathrm{Im}(\gamma z) > \eta\}.$$

Since $0 < \eta < 1$, we have the bound $N(z,\eta) \ll \frac{1}{\eta}$ which may be combined with the Maass-Selberg relation

$$\iint\limits_{\Gamma\backslash\mathfrak{h}} \left|\Lambda^T E(z,\ 1/2+it)\right|^2\ \frac{dxdy}{y^2} = 2\log(T) - \frac{\phi'}{\phi}(1/2+it) + \frac{\overline{\phi(1/2+it)}T^{2it} + \phi(1/2+it)T^{-2it}}{2it}.$$

This gives the upper bound (4) .

Second computation of $\mathcal{I}^T(t,\eta)$ (Lower Bound):

Assume $\eta = T^{-1}$. We will prove the lower bound

$$\boxed{\mathcal{I}^T(t,\eta)\ \gg\ \frac{1}{\eta}\cdot\frac{1}{\log(2+|t|)}.} \tag{5}$$

It follows from theorem 3.2 that if $y \ge T^{-1}$ then

$$\int_0^1 E^T(x+iy)e^{-2\pi inx}\ dx = 0.$$

Recall that

$$\Lambda^T E(z,s) = E(z,s)\ -\ E^T(z,s) - \phi(s)E^T(z,1-s).$$

It follows that if $\eta = T^{-1}$ then the nonzero Fourier coefficients of $E(z,s)$ coincide with the nonzero Fourier coefficients of $\Lambda^T E(z,s)$ for $y \ge \eta$. Hence

$$\begin{aligned}\mathcal{I}^T(t,\eta) &\gg \sum_{m=1}^\infty |\sigma_{-2it}(m)|^2 \int_{\eta m}^\infty \left|\frac{K_{it}(2\pi y)}{\Gamma\left(\frac{1}{2}+it\right)}\right|^2 \frac{dy}{y} \\ &\gg \frac{1}{|t|}\sum_{m\le\frac{|t|}{4\eta}} |\sigma_{-2it}(m)|^2 \\ &\gg\gg \frac{1}{|t|}\sum_{p\le\frac{|t|}{4\eta}} |\sigma_{-2it}(p)|^2\end{aligned}$$

Using sieve theory one may show that $|\sigma_{-2it}(p)| \gg 1$ for a positive proportion of primes. We thus obtain the lower bound (5). There is a loss of one logarithm in the lower bound because (for large x) the number of primes $\le x$ is asymptotic to $x/\log x$ by the prime number theorem.

Theorem 5.1 immediately follows by comparing the lower bound (5) with the upper bound (4).

□

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