Variation of the Swan conductor of an $\mathbb{F}_\ell$-sheaf on a rigid disc

Amadou Bah

Paris-Saclay & IHÉS

December 18, 2020

$$D^{(t)} \subset D$$

$$t \mapsto Sw(\mathcal{F}|D^{(t)})$$

Constructed with the ramification theory of Abbes-Saito.

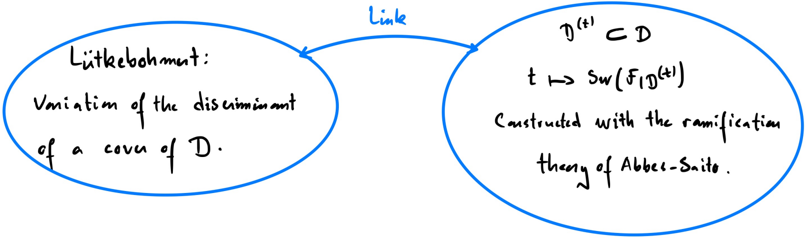
Lütkebohmert:

Variation of the discriminant of a cover of D.

$D^{(t)} \subset D$

$t \mapsto Sw(\mathcal{F}_{|D^{(t)}})$

Constructed with the ramification theory of Abbes-Saito.


Link
Lütkebohmert:
Variation of the discriminant
of a cover of D.
$D^{(t)} \subset D$
$t \mapsto Sw(\mathcal{F}_{|D^{(t)}})$
Constructed with the ramification
theory of Abbes-Saito.

Lütkebohmert:

Variation of the discriminant of a cover of D.

Link

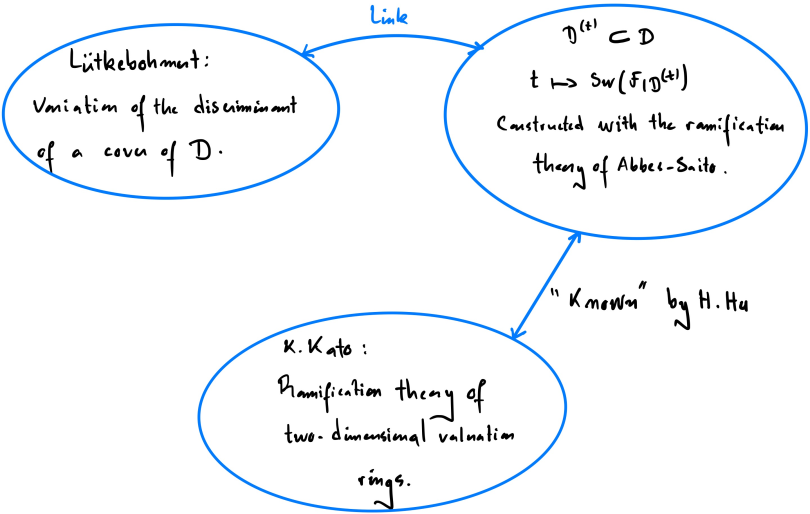
$D^{(t)} \subset D$

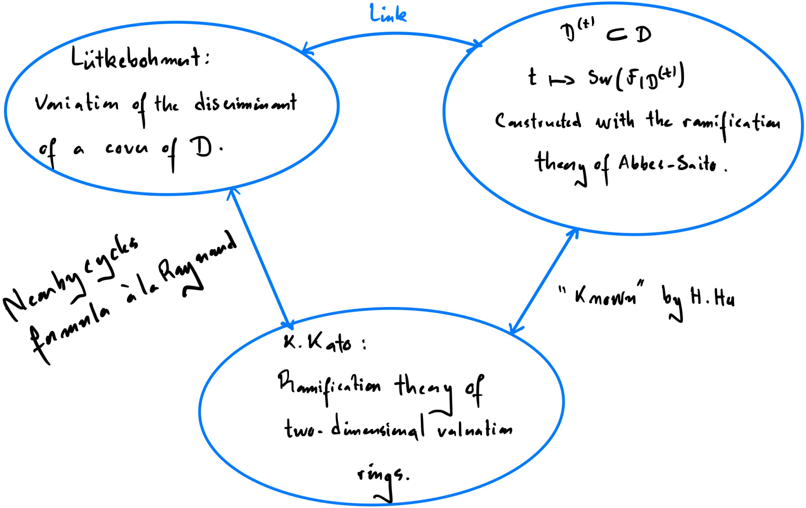
$t \mapsto Sw(\mathcal{F}_{|D^{(t)}})$

Constructed with the ramification theory of Abbes-Saito.

K. Kato:

Ramification theory of two-dimensional valuation rings.


Link
Lütkebohmert:
Variation of the discriminant of a cover of D.
$D^{(t)} \subset D$
$t \mapsto Sw(\mathcal{F}|D^{(t)})$
Constructed with the ramification theory of Abbes-Saito.
"Known" by H. Hu
K. Kato:
Ramification theory of two-dimensional valuation rings.


Link
Lütkebohmert:
Variation of the discriminant of a cover of D.
$D^{(t)} \subset D$
$t \mapsto Sw(\mathcal{F}_{|D^{(t)}})$
Constructed with the ramification theory of Abbes-Saito.
Nearby cycles formula à la Raynaud
"Known" by H. Hu
K. Kato:
Ramification theory of two-dimensional valuation rings.

Notation

We let

- K be a discrete valuation field

Notation

We let

- K be a discrete valuation field
- $\mathcal{O}_K$ its valuation ring, *henselian*

Notation

We let

- K be a discrete valuation field
- $\mathcal{O}_K$ its valuation ring, *henselian*
- k the residue field of characteristic $p > 0$

Notation

We let

- K be a discrete valuation field
- $\mathcal{O}_K$ its valuation ring, *henselian*
- k the residue field of characteristic $p > 0$
- π a uniformizer

Notation

We let

- K be a discrete valuation field
- $\mathcal{O}_K$ its valuation ring, *henselian*
- k the residue field of characteristic $p > 0$
- π a uniformizer
- $\overline{K}$ a separable closure, with residue field $\overline{k}$

Notation

We let

- K be a discrete valuation field
- $\mathcal{O}_K$ its valuation ring, *henselian*
- k the residue field of characteristic $p > 0$
- π a uniformizer
- $\overline{K}$ a separable closure, with residue field $\overline{k}$
- $G_K = \mathrm{Gal}(\overline{K}/K)$

Notation

We let

- K be a discrete valuation field
- $\mathcal{O}_K$ its valuation ring, *henselian*
- k the residue field of characteristic $p > 0$
- π a uniformizer
- $\overline{K}$ a separable closure, with residue field $\overline{k}$
- $G_K = \mathrm{Gal}(\overline{K}/K)$
- $v : \overline{K}^{\times} \to \mathbb{Q}$ the valuation map normalized by $v(\pi) = 1$.

Logarithmic ramification filtration

- When k is perfect, we have the *classical ramification theory*.

Logarithmic ramification filtration

- When k is perfect, we have the *classical ramification theory*.
- For general k, Abbes and Saito (~ 2000) defined the *logarithmic ramification filtration* $(G^r_{K,\log})_{r\in\mathbb{Q}_{\geq 0}}$ of G_K.

Logarithmic ramification filtration

- When k is perfect, we have the *classical ramification theory*.
- For general k, Abbes and Saito (~ 2000) defined the *logarithmic ramification filtration* $(G^r_{K,\log})_{r\in\mathbb{Q}_{\geq 0}}$ of G_K. For $r \in \mathbb{Q}_{\geq 0}$, put

$$G^{r+}_{K,\log} = \overline{\cup_{s>r} G^s_{K,\log}}.$$

Logarithmic ramification filtration

- When k is perfect, we have the *classical ramification theory*.
- For general k, Abbes and Saito (~ 2000) defined the *logarithmic ramification filtration* $(G^r_{K,\log})_{r\in\mathbb{Q}_{\geq 0}}$ of G_K. For $r \in \mathbb{Q}_{\geq 0}$, put
$$G^{r+}_{K,\log} = \overline{\cup_{s>r} G^s_{K,\log}}.$$
  - $G^0_{K,\log} = I_K$ the inertia subgroup of G_K and $G^{0+}_{K,\log} = P_K \subset I_K$ the wild inertia subgroup.

Logarithmic ramification filtration

- When k is perfect, we have the *classical ramification theory*.
- For general k, Abbes and Saito (~ 2000) defined the *logarithmic ramification filtration* $(G^r_{K,\log})_{r\in\mathbb{Q}_{\geq 0}}$ of G_K. For $r \in \mathbb{Q}_{\geq 0}$, put

$$G^{r+}_{K,\log} = \overline{\cup_{s>r} G^s_{K,\log}}.$$

  - $G^0_{K,\log} = I_K$ the inertia subgroup of G_K and $G^{0+}_{K,\log} = P_K \subset I_K$ the wild inertia subgroup.
  - If K'/K is a finite extension, then $G^{er}_{K',\log} \subset G^r_{K,\log}$. It is an equality when K'/K is tame.

Logarithmic ramification filtration

- When k is perfect, we have the *classical ramification theory*.
- For general k, Abbes and Saito (~ 2000) defined the *logarithmic ramification filtration* $(G^r_{K,\log})_{r\in\mathbb{Q}_{\geq 0}}$ of G_K. For $r \in \mathbb{Q}_{\geq 0}$, put

$$G^{r+}_{K,\log} = \overline{\cup_{s>r} G^s_{K,\log}}.$$

  - $G^0_{K,\log} = I_K$ the inertia subgroup of G_K and $G^{0+}_{K,\log} = P_K \subset I_K$ the wild inertia subgroup.
  - If K'/K is a finite extension, then $G^{er}_{K',\log} \subset G^r_{K,\log}$. It is an equality when K'/K is tame.
  - When k is perfect, $(G^r_{K,\log})_{r\in\mathbb{Q}_{\geq 0}}$ coincide with the classical upper ramification filtration.

Logarithmic ramification filtration

- When k is perfect, we have the *classical ramification theory*.
- For general k, Abbes and Saito (~ 2000) defined the *logarithmic ramification filtration* $(G^r_{K,\log})_{r\in\mathbb{Q}_{\geq 0}}$ of G_K. For $r \in \mathbb{Q}_{\geq 0}$, put

$$G^{r+}_{K,\log} = \overline{\cup_{s>r} G^s_{K,\log}}.$$

  - $G^0_{K,\log} = I_K$ the inertia subgroup of G_K and $G^{0+}_{K,\log} = P_K \subset I_K$ the wild inertia subgroup.
  - If K'/K is a finite extension, then $G^{er}_{K',\log} \subset G^r_{K,\log}$. It is an equality when K'/K is tame.
  - When k is perfect, $(G^r_{K,\log})_{r\in\mathbb{Q}_{\geq 0}}$ coincide with the classical upper ramification filtration.
  - Graded quotient

$$\mathrm{Gr}^r_{\log} G_K = G^r_{K,\log}/G^{r+}_{K,\log} \quad (r > 0)$$

  is abelian and killed by p.

The refined Swan conductor

Theorem (Kato, Abbes-Saito, Saito)

Assume that k is of finite type over a perfect sub-field k_0. For every $r > 0$, there is an injective homomorphism, the refined Swan conductor

$$\mathrm{rsw} : \mathrm{Hom}(\mathrm{Gr}^r_{\log} G_K, \mathbb{F}_p) \to \mathrm{Hom}_{\overline{k}}(\mathfrak{m}^r_{\overline{K}}/\mathfrak{m}^{r+}_{\overline{K}}, \Omega^1_k(\log) \otimes_k \overline{k}).$$

The refined Swan conductor

Theorem (Kato, Abbes-Saito, Saito)

Assume that k is of finite type over a perfect sub-field k_0. For every $r > 0$, there is an injective homomorphism, the refined Swan conductor

$$\mathrm{rsw} : \mathrm{Hom}(\mathrm{Gr}^r_{\log} G_K, \mathbb{F}_p) \to \mathrm{Hom}_{\overline{k}}(\mathfrak{m}^r_{\overline{K}}/\mathfrak{m}^{r+}_{\overline{K}}, \Omega^1_k(\log) \otimes_k \overline{k}).$$

For $r \in \mathbb{Q}$, $\mathfrak{m}^r_{\overline{K}}$ (resp. $\mathfrak{m}^{r+}_{\overline{K}}$) is the set of elements x of $\overline{K}$ satisfying $v(x) \geq r$ (resp. $v(x) > r$).

The refined Swan conductor

Theorem (Kato, Abbes-Saito, Saito)

Assume that k is of finite type over a perfect sub-field k_0. For every $r > 0$, there is an injective homomorphism, the refined Swan conductor

$$\mathrm{rsw} : \mathrm{Hom}(\mathrm{Gr}^r_{\log} G_K, \mathbb{F}_p) \to \mathrm{Hom}_{\overline{k}}(\mathfrak{m}^r_{\overline{K}}/\mathfrak{m}^{r+}_{\overline{K}}, \Omega^1_k(\log) \otimes_k \overline{k}).$$

For $r \in \mathbb{Q}$, $\mathfrak{m}^r_{\overline{K}}$ (resp. $\mathfrak{m}^{r+}_{\overline{K}}$) is the set of elements x of $\overline{K}$ satisfying $v(x) \geq r$ (resp. $v(x) > r$). The k-vector space of logarithmic differential 1-forms is

$$\Omega^1_k(\log) = (\Omega^1_{k/k_0} \oplus (k \otimes_{\mathbb{Z}} K^\times))/(\mathrm{d}\overline{a} - \overline{a} \otimes a, a \in \mathcal{O}_K^\times).$$

The Swan conductor

Let Λ be a finite field of char. $\ell \neq p$, $L \subset \overline{K}$ a finite Galois ext. of K of group G and $\rho : G \to \mathrm{Aut}_\Lambda(M)$ a finite dim. cont. rep.

The Swan conductor

Let Λ be a finite field of char. $\ell \neq p$, $L \subset \overline{K}$ a finite Galois ext. of K of group G and $\rho : G \to \mathrm{Aut}_\Lambda(M)$ a finite dim. cont. rep.
M has a unique *slope decomposition* into G_K-stable sub-mod.

$$M = \bigoplus_{r \in \mathbb{Q}_{\geq 0}} M^{(r)}$$

$M^{(0)} = M^{P_K}$, $(M^{(r)})^{G^r_{K,\log}} = 0$ and $(M^{(r)})^{G^{r+}_{K,\log}} = M^{(r)}$ $(r > 0)$.

The Swan conductor

Let Λ be a finite field of char. $\ell \neq p$, $L \subset \overline{K}$ a finite Galois ext. of K of group G and $\rho : G \to \mathrm{Aut}_\Lambda(M)$ a finite dim. cont. rep. M has a unique *slope decomposition* into G_K-stable sub-mod.

$$M = \bigoplus_{r \in \mathbb{Q}_{\geq 0}} M^{(r)}$$

$M^{(0)} = M^{P_K}$, $(M^{(r)})^{G^r_{K,\log}} = 0$ and $(M^{(r)})^{G^{r+}_{K,\log}} = M^{(r)}$ $(r > 0)$.

Definition

The (*logarithmic*) *Swan conductor* of M is

$$\mathrm{sw}_G(M) = \sum_{r \in \mathbb{Q}_{\geq 0}} r \cdot \dim_\Lambda M^{(r)}.$$

The Swan conductor

Let Λ be a finite field of char. $\ell \neq p$, $L \subset \overline{K}$ a finite Galois ext. of K of group G and $\rho : G \to \mathrm{Aut}_\Lambda(M)$ a finite dim. cont. rep. M has a unique *slope decomposition* into G_K-stable sub-mod.

$$M = \bigoplus_{r \in \mathbb{Q}_{\geq 0}} M^{(r)}$$

$M^{(0)} = M^{P_K}$, $(M^{(r)})^{G^r_{K,\log}} = 0$ and $(M^{(r)})^{G^{r+}_{K,\log}} = M^{(r)}$ $(r > 0)$.

Definition

The (*logarithmic*) *Swan conductor* of M is

$$\mathrm{sw}_G(M) = \sum_{r \in \mathbb{Q}_{\geq 0}} r \cdot \dim_\Lambda M^{(r)}.$$

$\mathrm{sw}_G(M) = 0 \Leftrightarrow M^{P_K} = M.$

The characteristic cycle

Let $\psi : \mathbb{F}_p \to \Lambda^\times$ be a nontrivial character. For $r > 0$, $M^{(r)} \neq 0$ has a *central character decomposition*

$$M^{(r)} = \bigoplus_{\chi} M_\chi^{(r)}$$

indexed by a finite number of characters $\chi : \mathrm{Gr}^r_{\log} G_K \to \Lambda^\times$.

The characteristic cycle

Let $\psi : \mathbb{F}_p \to \Lambda^\times$ be a nontrivial character. For $r > 0$, $M^{(r)} \neq 0$ has a *central character decomposition*

$$M^{(r)} = \bigoplus_{\chi} M_\chi^{(r)}$$

indexed by a finite number of characters $\chi : \mathrm{Gr}^r_{\log} G_K \to \Lambda^\times$. Each χ factors as $\mathrm{Gr}^r_{\log} G_K \xrightarrow{\overline{\chi}} \mathbb{F}_p \xrightarrow{\psi} \Lambda^\times$.

The characteristic cycle

Let $\psi : \mathbb{F}_p \to \Lambda^\times$ be a nontrivial character. For $r > 0$, $M^{(r)} \neq 0$ has a *central character decomposition*

$$M^{(r)} = \bigoplus_{\chi} M_\chi^{(r)}$$

indexed by a finite number of characters $\chi : \mathrm{Gr}^r_{\log} G_K \to \Lambda^\times$. Each χ factors as $\mathrm{Gr}^r_{\log} G_K \xrightarrow{\overline{\chi}} \mathbb{F}_p \xrightarrow{\psi} \Lambda^\times$.

Definition

The *Characteristic cycle* of M is

$$\mathrm{CC}_\psi(M) = \bigotimes_{r \in \mathbb{Q}_{>0}} \bigotimes_{\chi} (\mathrm{rsw}(\overline{\chi})(\pi^r))^{\otimes(\dim_\Lambda M_\chi^{(r)})} \in (\Omega_k^1(\log) \otimes_k \overline{k})^{\otimes m}$$

where $m = \dim_\Lambda M/M^{(0)}$.

Theorem (H. Hu, 2015)

If L/K is of type (II), i.e. $\mathcal{O}_L/\mathcal{O}_K$ is monogenic with purely inseparable residue extension, then

$$\mathrm{CC}_\psi(M) \in (\Omega^1_k)^m.$$

Lisse sheaf on unit disc

- Assume that K is *complete* and k is *algebraically closed*.

Lisse sheaf on unit disc

- Assume that K is *complete* and k is *algebraically closed*.
- Let $D = \{x \in \overline{K} \mid v(x) \geq 0\}$ be the closed rigid unit disc and $\mathcal{F}$ a lisse sheaf of Λ-modules on D.

Lisse sheaf on unit disc

- Assume that K is *complete* and k is *algebraically closed*.
- Let $D = \{x \in \overline{K} \mid v(x) \geq 0\}$ be the closed rigid unit disc and $\mathcal{F}$ a lisse sheaf of Λ-modules on D. By de Jong,

$$\mathcal{F} \longleftrightarrow [f : X \to D \ + \ \Lambda - \text{rep. } \rho_{\mathcal{F}} \text{ of } G = \text{Aut}(X/D)].$$

We consider the Cartesian diagram ($t \in \mathbb{Q}_{\geq 0}$)

$$\begin{array}{ccc} X^{(t)} & \longrightarrow & X \\ \downarrow & \square & \downarrow f \\ D^{(t)} & \longrightarrow & D, \end{array}$$

$D^{(t)} = \{x \in \overline{K} \mid v(x) \geq t\}$.

We consider the Cartesian diagram ($t \in \mathbb{Q}_{\geq 0}$)

$$\begin{array}{ccccc} \mathfrak{X}_{K'}^{(t)} & \leftarrow\cdots & X^{(t)} & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow f \\ \mathfrak{D}_{K'}^{(t)} & \leftarrow\cdots & D^{(t)} & \longrightarrow & D, \end{array}$$

$D^{(t)} = \{x \in \overline{K} \mid v(x) \geq t\}$, $\mathfrak{D}_{K'}^{(t)} = \mathrm{Spf}(\mathcal{O}^{\circ}(D_{K'}^{(t)}))$, $\mathfrak{X}_{K'}^{(t)} = \mathrm{Spf}(\mathcal{O}^{\circ}(X_{K'}^{(t)})$

We consider the Cartesian diagram ($t \in \mathbb{Q}_{\geq 0}$)

$$\begin{array}{ccccccc} \overline{\mathfrak{q}}^{(t)} & \longrightarrow & \mathfrak{X}^{(t)}_{K'} & \longleftarrow & X^{(t)} & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow f \\ \overline{\mathfrak{p}}^{(t)} & \longrightarrow & \mathfrak{D}^{(t)}_{K'} & \longleftarrow & D^{(t)} & \longrightarrow & D, \end{array}$$

$D^{(t)} = \{x \in \overline{K} \mid v(x) \geq t\}$, $\mathfrak{D}^{(t)}_{K'} = \mathrm{Spf}(\mathcal{O}^\circ(D^{(t)}_{K'}))$, $\mathfrak{X}^{(t)}_{K'} = \mathrm{Spf}(\mathcal{O}^\circ(X^{(t)}_{K'}))$

We consider the Cartesian diagram ($t \in \mathbb{Q}_{\geq 0}$)

$$\begin{array}{ccccccc}
\overline{\mathfrak{q}}^{(t)} & \longrightarrow & \mathfrak{X}^{(t)}_{K'} & \leftarrow\cdots\cdots & X^{(t)} & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow f \\
\overline{\mathfrak{p}}^{(t)} & \longrightarrow & \mathfrak{D}^{(t)}_{K'} & \leftarrow\cdots\cdots & D^{(t)} & \longrightarrow & D,
\end{array}$$

$$D^{(t)} = \{x \in \overline{K} \mid v(x) \geq t\},\ \mathfrak{D}^{(t)}_{K'} = \mathrm{Spf}(\mathcal{O}^\circ(D^{(t)}_{K'})),\ \mathfrak{X}^{(t)}_{K'} = \mathrm{Spf}(\mathcal{O}^\circ(X^{(t)}_{K'}))$$

The group G acts transitively on $\{\overline{\mathfrak{q}}^{(t)}\}$. The stabilizer $G_{\overline{\mathfrak{q}}^{(t)}}$ is the Galois group of a finite extension of type (II) of a discrete valuation field with imperfect residue field.

We consider the Cartesian diagram ($t \in \mathbb{Q}_{\geq 0}$)

$$\begin{array}{ccccccc}
\overline{\mathfrak{q}}^{(t)} & \longrightarrow & \mathfrak{X}^{(t)}_{K'} & \leftarrow\cdots\cdots & X^{(t)} & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow f \\
\overline{\mathfrak{p}}^{(t)} & \longrightarrow & \mathfrak{D}^{(t)}_{K'} & \leftarrow\cdots\cdots & D^{(t)} & \longrightarrow & D,
\end{array}$$

$$D^{(t)} = \{x \in \overline{K} \mid v(x) \geq t\},\ \mathfrak{D}^{(t)}_{K'} = \mathrm{Spf}(\mathcal{O}^\circ(D^{(t)}_{K'})),\ \mathfrak{X}^{(t)}_{K'} = \mathrm{Spf}(\mathcal{O}^\circ(X^{(t)}_{K'}))$$

The group G acts transitively on $\{\overline{\mathfrak{q}}^{(t)}\}$. The stabilizer $G_{\overline{\mathfrak{q}}^{(t)}}$ is the Galois group of a finite extension of type (II) of a discrete valuation field with imperfect residue field.

$$\rho_{\mathcal{F}} \quad \rightsquigarrow \quad \rho_{\overline{\mathfrak{q}}^{(t)}} : G_{\overline{\mathfrak{q}}^{(t)}} \to \mathrm{Aut}_\Lambda(M_{\overline{\mathfrak{q}}^{(t)}}).$$

We consider the Cartesian diagram ($t \in \mathbb{Q}_{\geq 0}$)

$$\begin{array}{ccccccc}
\overline{\mathfrak{q}}^{(t)} & \longrightarrow & \mathfrak{X}_{K'}^{(t)} & \leftarrow\cdots\cdots & X^{(t)} & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow f \\
\overline{\mathfrak{p}}^{(t)} & \longrightarrow & \mathfrak{D}_{K'}^{(t)} & \leftarrow\cdots\cdots & D^{(t)} & \longrightarrow & D,
\end{array}$$

$$D^{(t)} = \{x \in \overline{K} \mid v(x) \geq t\},\ \mathfrak{D}_{K'}^{(t)} = \mathrm{Spf}(\mathcal{O}^\circ(D_{K'}^{(t)})),\ \mathfrak{X}_{K'}^{(t)} = \mathrm{Spf}(\mathcal{O}^\circ(X_{K'}^{(t)})$$

The group G acts transitively on $\{\overline{\mathfrak{q}}^{(t)}\}$. The stabilizer $G_{\overline{\mathfrak{q}}^{(t)}}$ is the Galois group of a finite extension of type (II) of a discrete valuation field with imperfect residue field.

$$\rho_{\mathcal{F}} \quad \rightsquigarrow \quad \rho_{\overline{\mathfrak{q}}^{(t)}} : G_{\overline{\mathfrak{q}}^{(t)}} \to \mathrm{Aut}_\Lambda(M_{\overline{\mathfrak{q}}^{(t)}}).$$

$$\rightsquigarrow \quad \mathrm{sw}_{G_{\overline{\mathfrak{q}}^{(t)}}}(M_{\overline{\mathfrak{q}}^{(t)}}) \quad \text{and} \quad \mathrm{CC}_\psi(M_{\overline{\mathfrak{q}}^{(t)}}).$$

Main theorem

Theorem

The function $\mathrm{sw}(\mathcal{F},\cdot):\mathbb{Q}_{\geq 0}\to\mathbb{Q},\quad t\mapsto \mathrm{sw}_{G_{\overline{\mathfrak{q}}(t)}}(M_{\overline{\mathfrak{q}}(t)})$ *is continuous and piecewise linear, with finitely many slopes which are all integers.*

Main theorem

Theorem

The function $\mathrm{sw}(\mathcal{F},\cdot):\mathbb{Q}_{\geq 0}\to\mathbb{Q},\quad t\mapsto \mathrm{sw}_{G_{\overline{\mathfrak{q}}^{(t)}}}(M_{\overline{\mathfrak{q}}^{(t)}})$ *is continuous and piecewise linear, with finitely many slopes which are all integers. Its right derivative is the locally constant function*

$$\varphi_s(\mathcal{F},\cdot): t\mapsto -\mathrm{ord}_{\overline{\mathfrak{p}}^{(t)}}(\mathrm{CC}_\psi(M_{\overline{\mathfrak{q}}^{(t)}}))+\dim_\Lambda(M_{\overline{\mathfrak{q}}^{(t)}}/M^{(0)}_{\overline{\mathfrak{q}}^{(t)}}),$$

where $M^{(0)}_{\overline{\mathfrak{q}}^{(t)}}$ *is the tame part of* $M_{\overline{\mathfrak{q}}^{(t)}}$ *and* $\mathrm{ord}_{\overline{\mathfrak{p}}^{(t)}}$ *is the extension to* $\Omega^1_{\kappa(\overline{\mathfrak{p}}^{(t)})}$ *of the normalized discrete valuation on the residue field* $\kappa(\overline{\mathfrak{p}}^{(t)})$*, which is the field of fraction of* $\mathcal{O}_{\mathfrak{D}^{(t)}_{s'},\overline{\mathfrak{p}}^{(t)}}$.

Remarks

(1) $\varphi_s(\mathcal{F}, t)$ is the dimension of the space of nearby cycles $\Psi_0(\mathcal{F}_{|D^{(t)}})$ (Deligne-Kato formula).

Remarks

(1) $\varphi_s(\mathcal{F}, t)$ is the dimension of the space of nearby cycles $\Psi_0(\mathcal{F}_{|D^{(t)}})$ (Deligne-Kato formula).

(2) The function $\mathrm{sw}(\mathcal{F}, \cdot)$ should be convex.

Remarks

(1) $\varphi_s(\mathcal{F}, t)$ is the dimension of the space of nearby cycles $\Psi_0(\mathcal{F}_{|D^{(t)}})$ (Deligne-Kato formula).

(2) The function $\mathrm{sw}(\mathcal{F}, \cdot)$ should be convex.

(3) The theorem should also hold when $\mathcal{F}$ has "horizontal ramification".

Remarks

(1) $\varphi_s(\mathcal{F}, t)$ is the dimension of the space of nearby cycles $\Psi_0(\mathcal{F}_{|D^{(t)}})$ (Deligne-Kato formula).

(2) The function $\mathrm{sw}(\mathcal{F}, \cdot)$ should be convex.

(3) The theorem should also hold when $\mathcal{F}$ has "horizontal ramification".

(4) Analogous result by Ramero. Baldassarri, Pulita, Poineau-Pulita, Kedlaya proved an analogue for p-adic differential equations.

The discriminant of a rigid morphism

- X/K smooth affinoid space and $f : X \to D$ finite flat, étale over an admissible open subset of D containing 0.

The discriminant of a rigid morphism

- X/K smooth affinoid space and $f : X \to D$ finite flat, étale over an admissible open subset of D containing 0.
- $$\mathfrak{d}_f(t) = |\mathfrak{d}_{\mathcal{O}^\circ(X^{(t)}_{K'})/\mathcal{O}^\circ(D^{(t)}_{K'})}|_{\sup} = |\pi|^{\partial^\alpha_f(t)} \quad (t \in \mathbb{Q}_{\geq 0}).$$

The discriminant of a rigid morphism

- X/K smooth affinoid space and $f : X \to D$ finite flat, étale over an admissible open subset of D containing 0.
- $\mathfrak{d}_f(t) = |\mathfrak{d}_{\mathcal{O}^\circ(X^{(t)}_{K'})/\mathcal{O}^\circ(D^{(t)}_{K'})}|_{\sup} = |\pi|^{\partial^\alpha_f(t)} \quad (t \in \mathbb{Q}_{\geq 0})$.
- Weierstrass preparation theorem: an ivertible function on $A(\rho, \rho') = \{x \in \overline{K} \mid \rho \geq v(x) \geq \rho'\}$ $(\rho, \rho' \in \mathbb{Q})$ can be written in the form

$$\xi \mapsto c\xi^d(1 + h(\xi)), \quad \text{with} \quad h(\xi) = \sum_{i \in \mathbb{Z} - \{0\}} h_i \xi^i,$$

where $c \in K^\times$, $d \in \mathbb{Z}$ (the *order* of the function) and h such that $|h(\xi)|_{\sup} < 1$.

- When $X = A(r/d, r'/d)$ $(r \geq r' \geq 0)$, and $f : A(r/d, r'/d) \to A(r, r') \subset D$ finite étale of order d, Lütkebohmert computes ∂_f^α explicitly and observes that it is affine and is

$$\frac{d}{dt}\partial_f^\alpha(t+) = \sigma - d + 1, \quad t \in [r', r[\cap \mathbb{Q},$$

where σ is the order of f'.

- When $X = A(r/d, r'/d)$ $(r \geq r' \geq 0)$, and $f : A(r/d, r'/d) \to A(r, r') \subset D$ finite étale of order d, Lütkebohmert computes ∂_f^α explicitly and observes that it is affine and is

$$\frac{d}{dt}\partial_f^\alpha(t+) = \sigma - d + 1, \quad t \in [r', r[\cap \mathbb{Q},$$

where σ is the order of f'.

- More generally, by the semi-stable reduction theorem, ∂_f^α is continuous and piecewise linear with finitely many slopes (integers) given by

$$\frac{d}{dt}\partial_f^\alpha(t+) = \sigma_i - d + \delta_f(i),$$

for some partition $r_{n+1} = 0 < r_n < \cdots < r_0 = +\infty$ et $t \in [r_i, r_{i-1}[$.

Ramification of $\mathbb{Z}^2$-valuation rings

$$\begin{array}{ccccc}
\overline{\mathfrak{q}}^{(t)} & \rightsquigarrow & \overline{x}^{(t)} & \longrightarrow & \mathfrak{X}_{K'}^{(t)} \\
\downarrow & & \downarrow & & \downarrow \\
\overline{\mathfrak{p}}^{(t)} & \rightsquigarrow & \overline{0}^{(t)} & \longrightarrow & \mathfrak{D}_{K'}^{(t)}.
\end{array}$$

Ramification of $\mathbb{Z}^2$-valuation rings

$$\begin{array}{ccccc} \overline{\mathfrak{q}}^{(t)} & \rightsquigarrow & \overline{x}^{(t)} & \longrightarrow & \mathfrak{X}^{(t)}_{K'} \\ \downarrow & & \downarrow & & \downarrow \\ \overline{\mathfrak{p}}^{(t)} & \rightsquigarrow & \overline{0}^{(t)} & \longrightarrow & \mathfrak{D}^{(t)}_{K'}. \end{array}$$

- $\mathcal{O}_{\mathfrak{D}^{(t)}_{K'},\overline{0}^{(t)}} = A \subset V_t \subset A_{\mathfrak{p}^{(t)}} \quad \rightsquigarrow \quad V_t^h.$

Ramification of $\mathbb{Z}^2$-valuation rings

$$\begin{array}{ccccc} \overline{\mathfrak{q}}^{(t)} & \rightsquigarrow & \overline{x}^{(t)} & \longrightarrow & \mathfrak{X}^{(t)}_{K'} \\ \downarrow & & \downarrow & & \downarrow \\ \overline{\mathfrak{p}}^{(t)} & \rightsquigarrow & \overline{0}^{(t)} & \longrightarrow & \mathfrak{D}^{(t)}_{K'}. \end{array}$$

- $\mathcal{O}_{\mathfrak{D}^{(t)}_{K'},\overline{0}^{(t)}} = A \subset V_t \subset A_{\mathfrak{p}^{(t)}} \quad \rightsquigarrow \quad V_t^h.$
- K. Kato: ramification theory for monogenic extensions of V_t^h.

Ramification of $\mathbb{Z}^2$-valuation rings

$$
\begin{array}{ccccc}
\overline{\mathfrak{q}}^{(t)} & \rightsquigarrow & \overline{x}^{(t)} & \longrightarrow & \mathfrak{X}_{K'}^{(t)} \\
\downarrow & & \downarrow & & \downarrow \\
\overline{\mathfrak{p}}^{(t)} & \rightsquigarrow & \overline{0}^{(t)} & \longrightarrow & \mathfrak{D}_{K'}^{(t)}.
\end{array}
$$

- $\mathcal{O}_{\mathfrak{D}_{K'}^{(t)},\overline{0}^{(t)}} = A \subset V_t \subset A_{\mathfrak{p}^{(t)}} \quad \rightsquigarrow \quad V_t^h.$
- K. Kato: ramification theory for monogenic extensions of V_t^h.
- $\mathcal{O}_{\mathfrak{X}_{K'}^{(t)},\overline{x}^{(t)}} = B \subset W_t \subset B_{\mathfrak{q}^{(t)}} \quad \rightsquigarrow \quad V_t^h \to W_t^h.$

Ramification of $\mathbb{Z}^2$-valuation rings

$$\begin{array}{ccccc}
\overline{\mathfrak{q}}^{(t)} & \rightsquigarrow & \overline{x}^{(t)} & \longrightarrow & \mathfrak{X}^{(t)}_{K'} \\
\downarrow & & \downarrow & & \downarrow \\
\overline{\mathfrak{p}}^{(t)} & \rightsquigarrow & \overline{0}^{(t)} & \longrightarrow & \mathfrak{D}^{(t)}_{K'}.
\end{array}$$

- $\mathcal{O}_{\mathfrak{D}^{(t)}_{K'},\overline{0}^{(t)}} = A \subset V_t \subset A_{\mathfrak{p}^{(t)}} \quad \rightsquigarrow \quad V_t^h.$
- K. Kato: ramification theory for monogenic extensions of V_t^h.
- $\mathcal{O}_{\mathfrak{X}^{(t)}_{K'},\overline{x}^{(t)}} = B \subset W_t \subset B_{\mathfrak{q}^{(t)}} \quad \rightsquigarrow \quad V_t^h \to W_t^h.$

$\Rightarrow$ Ramification filtration of $\mathrm{Gal}(\mathbb{L}_t^h/\mathbb{K}_t^h) \subset G$ indexed by the value group of V_t^h (isomorphic to $\mathbb{Z}^2$).

Ramification of $\mathbb{Z}^2$-valuation rings

$$\begin{array}{ccccc} \overline{\mathfrak{q}}^{(t)} & \rightsquigarrow & \overline{x}^{(t)} & \longrightarrow & \mathfrak{X}_{K'}^{(t)} \\ \downarrow & & \downarrow & & \downarrow \\ \overline{\mathfrak{p}}^{(t)} & \rightsquigarrow & \overline{0}^{(t)} & \longrightarrow & \mathfrak{D}_{K'}^{(t)}. \end{array}$$

- $\mathcal{O}_{\mathfrak{D}_{K'}^{(t)},\overline{0}^{(t)}} = A \subset V_t \subset A_{\mathfrak{p}^{(t)}} \quad \rightsquigarrow \quad V_t^h$.
- K. Kato: ramification theory for monogenic extensions of V_t^h.
- $\mathcal{O}_{\mathfrak{X}_{K'}^{(t)},\overline{x}^{(t)}} = B \subset W_t \subset B_{\mathfrak{q}^{(t)}} \quad \rightsquigarrow \quad V_t^h \to W_t^h$.

$\Rightarrow$ Ramification filtration of $\mathrm{Gal}(\mathbb{L}_t^h/\mathbb{K}_t^h) \subset G$ indexed by the value group of V_t^h (isomorphic to $\mathbb{Z}^2$).

$$\widetilde{a}_f^\alpha(t) : G = \mathrm{Aut}(X/D) \to \mathbb{Q} \quad \text{and} \quad \widetilde{\mathrm{sw}}_f^\beta(t) : G \to \mathbb{Z}.$$

The link

Proposition

Assume $K_X \simeq \mathcal{O}_X$. Then, we have the identity

$$\partial_f^\alpha(t) = \langle \widetilde{a}_f^\alpha(t), r_G \rangle, \tag{14.1}$$

where $\langle \cdot, \cdot \rangle$ is the usual pairing for class functions and r_G is the character of the regular representation of G.

The link

Proposition

Assume $K_X \simeq \mathcal{O}_X$*. Then, we have the identity*

$$\partial_f^\alpha(t) = \langle \widetilde{a}_f^\alpha(t), r_G \rangle, \tag{15.1}$$

where $\langle \cdot, \cdot \rangle$ *is the usual pairing for class functions and* r_G *is the character of the regular representation of* G*. The right derivative of* ∂_f^α *at* $t \in [r_i, r_{i-1}[$ *is*

$$\frac{d}{dt}\partial_f^\alpha(t+) = \sigma_i - d + \delta_f(i) = \langle \widetilde{\mathrm{sw}}_f^\beta(t), r_G \rangle. \tag{15.2}$$

- 14.1 is an incarnation of the classical equality of the valuation of the different with the value of the Artin character at 1.
- 14.2 is deduced from a formula à la Raynaud for the dimension of some nearby cycle involving σ and $\delta_f(i)$.

A nearby cycles formula

$$\begin{array}{ccc} \overline{x}^{(t)} & \longrightarrow & \mathfrak{X}_{K'}^{(t)} \\ \downarrow & & \downarrow \\ \overline{0}^{(t)} & \longrightarrow & \mathfrak{D}_{K'}^{(t)}. \end{array}$$

$$\begin{array}{ccc} \overline{x}_j^{(t)} & \longrightarrow & \mathfrak{X}_{K'}^{(t)} \\ \downarrow & & \downarrow \\ \overline{0}^{(t)} & \longrightarrow & \mathfrak{D}_{K'}^{(t)}. \end{array} \quad \rightsquigarrow \quad \mathcal{O}_{\mathfrak{D}_{K'}^{(t)},\overline{0}^{(t)}} = A^{(t)} \to B_j^{(t)} = \mathcal{O}_{\mathfrak{X}_{K'}^{(t)},\overline{x}_j^{(t)}}.$$

$$\begin{array}{ccc} \overline{x}_j^{(t)} & \longrightarrow & \mathfrak{X}_{K'}^{(t)} \\ \downarrow & & \downarrow \\ \overline{0}^{(t)} & \longrightarrow & \mathfrak{D}_{K'}^{(t)}. \end{array} \quad \rightsquigarrow \quad \mathcal{O}_{\mathfrak{D}_{K'}^{(t)}, \overline{0}^{(t)}} = A^{(t)} \to B_j^{(t)} = \mathcal{O}_{\mathfrak{X}_{K'}^{(t)}, \overline{x}_j^{(t)}}.$$

- $P_j^{(t)}$ = set of height 1 prime ideals of $B_j^{(t)}$

$$\begin{array}{ccc} \overline{x}_j^{(t)} & \longrightarrow & \mathfrak{X}_{K'}^{(t)} \\ \downarrow & & \downarrow \\ \overline{0}^{(t)} & \longrightarrow & \mathfrak{D}_{K'}^{(t)}. \end{array} \quad \rightsquigarrow \quad \mathcal{O}_{\mathfrak{D}_{K'}^{(t)},\overline{0}^{(t)}} = A^{(t)} \to B_j^{(t)} = \mathcal{O}_{\mathfrak{X}_{K'}^{(t)},\overline{x}_j^{(t)}}.$$

- $P_j^{(t)}$ = set of height 1 prime ideals of $B_j^{(t)}$
- $B_{j,s}^{(t)} = B_j^{(t)}/\mathfrak{m}_K B_j^{(t)}$ is reduced
  $\widetilde{B_{j,0}^{(t)}}$ normalization of $B_{j,0}^{(t)}$
  $\delta_j^{(t)} = \dim_k(\widetilde{B_{j,0}^{(t)}}/B_{j,0}^{(t)})$.

$$\begin{array}{ccc} \overline{x}_j^{(t)} & \longrightarrow & \mathfrak{X}_{K'}^{(t)} \\ \downarrow & & \downarrow \\ \overline{0}^{(t)} & \longrightarrow & \mathfrak{D}_{K'}^{(t)}. \end{array} \quad \rightsquigarrow \quad \mathcal{O}_{\mathfrak{D}_{K'}^{(t)},\overline{0}^{(t)}} = A^{(t)} \to B_j^{(t)} = \mathcal{O}_{\mathfrak{X}_{K'}^{(t)},\overline{x}_j^{(t)}}.$$

- $P_j^{(t)}$ = set of height 1 prime ideals of $B_j^{(t)}$
- $B_{j,s}^{(t)} = B_j^{(t)}/\mathfrak{m}_K B_j^{(t)}$ is reduced
  $\widetilde{B_{j,0}^{(t)}}$ normalization of $B_{j,0}^{(t)}$
  $\delta_j^{(t)} = \dim_k(\widetilde{B_{j,0}^{(t)}}/B_{j,0}^{(t)})$.
- $A_{K'}^{(t)} = A^{(t)} \otimes_{\mathcal{O}_{K'}} K' \to B_{j,K'}^{(t)} = A^{(t)} \otimes_{\mathcal{O}_{K'}} K'$.
  Bilinear trace map $B_{j,K'} \times B_{j,K'} \to A_{K'}^{(t)}$ well-defined
  $\rightsquigarrow$ K'-linear determinant homomorphism $T_j^{(t)}$
  $d_j^{(t)} = \dim_{K'}(\mathrm{Coker}(T_j^{(t)}))$

Proposition

For each $i = 1, \ldots, n$ *and each* $t \in]r_i, r_{i-1}[\cap \mathbb{Q}$, *we have*

$$\sum_j (d_j^{(t)} - 2\delta_j^{(t)} + |P_j|) = \sigma_i + \delta_f(i). \tag{16.1}$$

Proposition

For each $i = 1, \ldots, n$ *and each* $t \in]r_i, r_{i-1}[\cap \mathbb{Q}$, *we have*

$$\sum_j (d_j^{(t)} - 2\delta_j^{(t)} + |P_j|) = \sigma_i + \delta_f(i). \tag{18.1}$$

Remark

Imagine $Y^{(t)} = \mathfrak{X}_{K'}^{(t)}$ were a scheme over $S' = \mathrm{Spec}(\mathcal{O}_{K'})$. Then, Kato proved that

$$2\delta_j^{(t)} - |P_j^{(t)}| + 1 = \dim_\Lambda H^1_{\text{ét}}(Y_{(\overline{x}'_j)}^{(t)} \times \overline{\eta}, \Lambda). \tag{19.1}$$

Sketch of proof

- $$\begin{array}{ccc} X^{[t]} & \hookrightarrow & X^{(t)} \\ \downarrow & \square & \downarrow f^{(t)} \\ D^{[t]} & \hookrightarrow & D^{(t)} \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} \mathfrak{X}^{[t]}_{K'} & \hookrightarrow & \mathfrak{X}^{(t)}_{K'} \\ \downarrow & \square & \downarrow \widehat{f}^{(t)} \\ \mathfrak{D}^{[t]}_{K'} & \hookrightarrow & \mathfrak{D}^{(t)}_{K'} \end{array}$$

Sketch of proof

- $$\begin{array}{ccc} X^{[t]} & \hookrightarrow & X^{(t)} \\ \downarrow & \square & \downarrow f^{(t)} \\ D^{[t]} & \hookrightarrow & D^{(t)} \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} \mathfrak{X}^{[t]}_{K'} & \hookrightarrow & \mathfrak{X}^{(t)}_{K'} \\ \downarrow & \square & \downarrow \widehat{f}^{(t)} \\ \mathfrak{D}^{[t]}_{K'} & \hookrightarrow & \mathfrak{D}^{(t)}_{K'} \end{array}$$

  $$X^{[t]} = \coprod_{j=1}^{\delta_f(i)} D^{[t/d_{ij}]} \quad \rightsquigarrow \quad \mathfrak{X}^{[t]}_{K'} = \coprod_{j=1}^{\delta_f(i)} \mathfrak{D}^{[t/d_{ij}]}$$

Sketch of proof

- $$\begin{array}{ccc} X^{[t]} & \hookrightarrow & X^{(t)} \\ \downarrow & \square & \downarrow f^{(t)} \\ D^{[t]} & \hookrightarrow & D^{(t)} \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} \mathfrak{X}^{[t]}_{K'} & \hookrightarrow & \mathfrak{X}^{(t)}_{K'} \\ \downarrow & \square & \downarrow \widehat{f}^{(t)} \\ \mathfrak{D}^{[t]}_{K'} & \hookrightarrow & \mathfrak{D}^{(t)}_{K'} \end{array}$$

  $$X^{[t]} = \coprod_{j=1}^{\delta_f(i)} D^{[t/d_{ij}]} \quad \rightsquigarrow \quad \mathfrak{X}^{[t]}_{K'} = \coprod_{j=1}^{\delta_f(i)} \mathfrak{D}^{[t/d_{ij}]}$$

  $$\mathfrak{Y}^{(t)}_{K'} = (\mathfrak{X}^{(t)}_{K'} \cup (\coprod_{j}^{\delta_f(i)} \mathfrak{D}^{(t)}_{ij})/\mathfrak{D}^{[t/d_{ij}]} \sim \mathfrak{D}^{[t]}_{ij} \to \mathrm{Spf}(\mathcal{O}_{K'}).$$

  is a formal relative curve, normal, proper (flat); smooth rigid generic fiber $\mathfrak{Y}^{(t)}_{\eta'}$ and $\mathrm{Sing}(\mathfrak{Y}^{(t)}_{s'}) \subset \mathfrak{X}^{(t)}_{s'} - \mathfrak{X}^{[t]}_{s'} = \widehat{f}^{(t)\,-1}_{s'}(0^{(t)})$.

- $\mathfrak{Y}_{K'}^{(t)}$ proper flat formal curve $\Rightarrow$ algebraizable (Grothendieck) : there exists $Y_{K'}^{(t)}$ normal, proper flat over $S' = \mathrm{Spec}(\mathcal{O}_{K'})$, with smooth generic fiber, such that $\widehat{Y_{K'}^{(t)}} \cong \mathfrak{Y}_{K'}^{(t)}$.

- $\mathfrak{Y}_{K'}^{(t)}$ proper flat formal curve $\Rightarrow$ algebraizable (Grothendieck) : there exists $Y_{K'}^{(t)}$ normal, proper flat over $S' = \mathrm{Spec}(\mathcal{O}_{K'})$, with smooth generic fiber, such that $\widehat{Y_{K'}^{(t)}} \cong \mathfrak{Y}_{K'}^{(t)}$.
- Approximation of $f^{(t)}$: rigid Runge theorem (Raynaud) $\Rightarrow \quad \exists \quad g^{(t)} : Y_{K'}^{(t)} \to \mathbb{P}^1_{S'}$ s.t. $\widehat{g}_{\eta'}^{(t)}$ is close enough to $f^{(t)}$ on $D_{ij}^{[t]}$ that $df^{(t)}$ and $dg_{\eta'}^{(t)}$ have the same zeros with same orders of vanishing on $D_{ij}^{[t]}$.

- $\mathfrak{Y}_{K'}^{(t)}$ proper flat formal curve $\Rightarrow$ algebraizable (Grothendieck) : there exists $Y_{K'}^{(t)}$ normal, proper flat over $S' = \mathrm{Spec}(\mathcal{O}_{K'})$, with smooth generic fiber, such that $\widehat{Y_{K'}^{(t)}} \cong \mathfrak{Y}_{K'}^{(t)}$.
- Approximation of $f^{(t)}$: rigid Runge theorem (Raynaud) $\Rightarrow \quad \exists \quad g^{(t)} : Y_{K'}^{(t)} \to \mathbb{P}^1_{S'}$ s.t. $\widehat{g}_{\eta'}^{(t)}$ is close enough to $f^{(t)}$ on $D_{ij}^{[t]}$ that $df^{(t)}$ and $dg_{\eta'}^{(t)}$ have the same zeros with same orders of vanishing on $D_{ij}^{[t]}$.
  Then,
$$2g(Y_{\overline{\eta}}^{(t)}) - 2|\pi_0(Y_{\overline{\eta}}^{(t)})| = \deg(\mathrm{div}(dg_{\eta'}^{(t)})).$$

- $\mathfrak{Y}_{K'}^{(t)}$ proper flat formal curve $\Rightarrow$ algebraizable (Grothendieck) : there exists $Y_{K'}^{(t)}$ normal, proper flat over $S' = \mathrm{Spec}(\mathcal{O}_{K'})$, with smooth generic fiber, such that $\widehat{Y_{K'}^{(t)}} \cong \mathfrak{Y}_{K'}^{(t)}$.
- Approximation of $f^{(t)}$: rigid Runge theorem (Raynaud) $\Rightarrow \quad \exists \quad g^{(t)} : Y_{K'}^{(t)} \to \mathbb{P}^1_{S'}$ s.t. $\widehat{g}_{\eta'}^{(t)}$ is close enough to $f^{(t)}$ on $D_{ij}^{[t]}$ that $df^{(t)}$ and $dg_{\eta'}^{(t)}$ have the same zeros with same orders of vanishing on $D_{ij}^{[t]}$.

  Then,
$$2g(Y_{\overline{\eta}}^{(t)}) - 2|\pi_0(Y_{\overline{\eta}}^{(t)})| = \deg(\mathrm{div}(dg_{\eta'}^{(t)})).$$
- $\deg(\mathrm{div}(dg_{\eta'}^{(t)})) = \sum_{j=1}^{N} d_j^{(t)} - \sigma_i - 2\delta_f(i).$

- $$2|\pi_0(Y_{\overline{\eta}}^{(t)})| - 2g(Y_{\overline{\eta}}^{(t)}) = \chi(Y_{\overline{\eta}}^{(t)}, \Lambda) = \chi(Y_{s'}^{(t)}, R\Psi_{Y_{K'}^{(t)}/S'}(\Lambda))$$

- $$2|\pi_0(Y_{\overline{\eta}}^{(t)})|-2g(Y_{\overline{\eta}}^{(t)}) = \chi(Y_{\overline{\eta}}^{(t)}, \Lambda) = \chi(Y_{s'}^{(t)}, R\Psi_{Y_{K'}^{(t)}/S'}(\Lambda))$$

$$\chi(Y_{s'}^{(t)}, R\Psi_{Y_{K'}^{(t)}/S'}(\Lambda)) = N+\delta_f(i)-\sum_{j=1}^{N} \dim_\Lambda H^1_{\text{ét}}(Y_{(\overline{x}'_j)}^{(t)} \times \overline{\eta}, \Lambda)$$

- $$2|\pi_0(Y_{\overline{\eta}}^{(t)})|-2g(Y_{\overline{\eta}}^{(t)}) = \chi(Y_{\overline{\eta}}^{(t)},\Lambda) = \chi(Y_{s'}^{(t)}, R\Psi_{Y_{K'}^{(t)}/S'}(\Lambda))$$

$$\chi(Y_{s'}^{(t)}, R\Psi_{Y_{K'}^{(t)}/S'}(\Lambda)) = N+\delta_f(i)-\sum_{j=1}^{N} \dim_\Lambda H^1_{\text{ét}}(Y_{(\overline{x}'_j)}^{(t)}\times\overline{\eta},\Lambda)$$

$$\dim_\Lambda H^1_{\text{ét}}(Y_{(\overline{x}'_j)}^{(t)}\times\overline{\eta},\Lambda) = 2\delta_j^{(t)} - |P_j^{(t)}|+1 \quad \text{(Kato)}.$$

- $$2|\pi_0(Y_{\overline{\eta}}^{(t)})|-2g(Y_{\overline{\eta}}^{(t)}) = \chi(Y_{\overline{\eta}}^{(t)},\Lambda) = \chi(Y_{s'}^{(t)}, R\Psi_{Y_{K'}^{(t)}/S'}(\Lambda))$$

$$\chi(Y_{s'}^{(t)}, R\Psi_{Y_{K'}^{(t)}/S'}(\Lambda)) = N+\delta_f(i)-\sum_{j=1}^{N}\dim_\Lambda H^1_{\text{ét}}(Y_{(\overline{x}'_j)}^{(t)}\times\overline{\eta},\Lambda)$$

$$\dim_\Lambda H^1_{\text{ét}}(Y_{(\overline{x}'_j)}^{(t)}\times\overline{\eta},\Lambda) = 2\delta_j^{(t)} - |P_j^{(t)}|+1 \quad \text{(Kato)}.$$

Put together $\Rightarrow$ QED.

Theorem

Let $\chi \in R_\Lambda(G)$*. Then, the map*

$$\widetilde{a}_f^\alpha(\chi, \cdot) : \mathbb{Q}_{\geq 0} \to \mathbb{Q}, \quad t \mapsto \langle \widetilde{a}_f^\alpha(t), \chi \rangle_G \tag{20.1}$$

is continuous and piecewise linear, with finitely many slopes which are all integers. Its right derivative at $t \in \mathbb{Q}_{\geq 0}$ *is*

$$\frac{d}{dt}\widetilde{a}_f^\alpha(\chi, t+) = \langle \widetilde{\mathrm{sw}}_f^\beta(t), \chi \rangle_G . \tag{20.2}$$

Conclusion

Proposition

Let M be a Λ-valued representation of G. Then, we have the identities

$$\langle \widetilde{a}_f^{\alpha}, \chi_M \rangle = \mathrm{sw}_G(M), \tag{21.1}$$

$$\langle \widetilde{\mathrm{sw}}_f^{\beta}, \chi_M \rangle = -\operatorname{ord}_{\mathfrak{p}}(\mathrm{CC}_{\psi}(M)) + \dim_{\Lambda}(M/M^{(0)}). \tag{21.2}$$

Conclusion

Proposition

Let M be a Λ-valued representation of G. Then, we have the identities

$$\langle \widetilde{a}_f^\alpha, \chi_M \rangle = \mathrm{sw}_G(M), \tag{22.1}$$

$$\langle \widetilde{\mathrm{sw}}_f^\beta, \chi_M \rangle = -\operatorname{ord}_{\mathfrak{p}}(\mathrm{CC}_\psi(M)) + \dim_\Lambda(M/M^{(0)}). \tag{22.2}$$

Deduced from a comparison theorem of H. Hu:

$$\mathrm{CC}_\psi(M) = \mathrm{KCC}_{\psi(1)}(\chi_M).$$

Thank you !