Compatibility of the Fargues-Scholze and Gan-Takeda Local Langlands

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October 2021

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Notation

We let:

1. $\ell \neq p$ be distinct primes.
2. $G/\mathbb{Q}_p$ be a connected reductive group.
3. $W_{\mathbb{Q}_p}$ be the Weil group of $\mathbb{Q}_p$
4. $\hat{G}$ the Langlands dual group of G viewed as a reductive group over $\overline{\mathbb{Q}}_\ell$
5. ${}^L G := W_{\mathbb{Q}_p} \ltimes \hat{G}$
6. $i : \overline{\mathbb{Q}}_\ell \xrightarrow{\simeq} \mathbb{C}$ and $j : \overline{\mathbb{Q}}_p \xrightarrow{\simeq} \mathbb{C}$ be fixed isomorphisms.

Notation

We set:

1. $\Pi(G)$ to be isomorphism classes of smooth irreducible representations of $G(\mathbb{Q}_p)$.
2. $\Phi(G)$ to be the set of conjugacy classes of admissible homomorphisms:

$$\phi : W_{\mathbb{Q}_p} \times SL(2, \overline{\mathbb{Q}}_\ell) \to {}^LG(\overline{\mathbb{Q}}_\ell)$$

3. $\Phi^W(G)$ to be the set of conjugacy classes of continuous semisimple homomorphisms:

$$\phi : W_{\mathbb{Q}_p} \to {}^LG(\overline{\mathbb{Q}}_\ell)$$

Theorem (Harris-Taylor/Henniart/Scholze)

Let $G = GL_n$ then, for every $n \geq 1$, there exists a unique bijection:

$$\Pi(G) \xrightarrow{LLC_n} \Phi(G)$$

$$\pi \mapsto \phi_\pi$$

generalizing local class field theory and characterized by the preservation of character twists, L, ϵ, and γ factors in pairs of representations.

Theorem (Fargues-Scholze)

For any G, there exists a map:

$$\Pi(G) \xrightarrow{LLC_G^{FS}} \Phi^W(G)$$

$$\pi \mapsto \phi_\pi^{FS}$$

enjoying the following properties:

1. It is compatible with restriction of scalars, central characters, products, and isogenies.
2. It is is compatible with parabolic induction of representations.
3. It is compatible with the correspondence of Harris-Taylor/Henniart for $G = GL_n$ and its inner forms.

Key Question:

Can we show that the Fargues-Scholze Local Langlands correspondence is compatible with other instances of the correspondence? Namely, given a local Langlands correspondence:

$$\Pi(G) \xrightarrow{LLC_G} \Phi(G)$$

We expect a commutative diagram of the form:

$$\begin{array}{ccc} \Pi(G) & \xrightarrow{LLC_G} & \Phi(G) \\ & \searrow^{LLC_G^{FS}} & \downarrow \\ & & \Phi^W(G) \end{array}$$

where the right vertical arrow precomposes the map $\phi \in \Phi(G)$ with $g \in W_{\mathbb{Q}_p} \mapsto (g, \begin{pmatrix} |g|^{\frac{1}{2}} & 0 \\ 0 & |g|^{\frac{-1}{2}} \end{pmatrix}) \in W_{\mathbb{Q}_p} \times SL(2, \overline{\mathbb{Q}}_\ell)$

Theorem (Gan-Takeda/Gan-Tantono/Chan-Gan)

1. Let $L/\mathbb{Q}_p$ be a finite extension.
2. Let $G = Res_{L/\mathbb{Q}_p}GSp_4$ and $J = Res_{L/\mathbb{Q}_p}GU_2(D)$ be its unique non-split inner form, where D is the quaternion division algebra over L.
3. Then, for $H = G$ or J, up to the choice of the fixed isomorphism i, there exists a unique map:

$$LLC_H : \Pi(H) \to \Phi(H)$$

$$\pi \mapsto \{\phi_\pi : W_L \times SL(2, \overline{\mathbb{Q}}_\ell) \to \hat{H}(\overline{\mathbb{Q}}_\ell) = GSpin_5(\overline{\mathbb{Q}}_\ell) \simeq GSp_4(\overline{\mathbb{Q}}_\ell)\}$$

   characterized by preservation of character twists, L, ϵ, γ factors, and a condition on the Plancharel measure of a family of induced representations.
4. The L-packets satisfy the endoscopic character identities.

Remarks

1. The proof of compatibility will formally reduce to case of supercuspidal representations ($\implies$ discrete series). The L-parameters of such representations will be discrete (i.e ϕ does not factor through a Levi subgroup of GSp_4.)
2. Given such a parameter ϕ, the L-packets $\Pi_{\phi}(G) := LLC_{G}^{-1}(\phi)$ and $\Pi_{\phi}(J) := LLC_{J}^{-1}(\phi)$ have size 1 or 2, we say that ϕ is stable or endoscopic, respectively.
3. Let $std : GSp_4(\overline{\mathbb{Q}}_{\ell}) \to GL_4(\overline{\mathbb{Q}}_{\ell})$ be the standard embedding.
   - (stable) $std \circ \phi$ is irreducible.
   - (endoscopic) $std \circ \phi \simeq \phi_1 \oplus \phi_2$, with ϕ_1 and ϕ_2 distinct irreducible 2-dimensional reps of $W_L \times SL(2, \overline{\mathbb{Q}}_{\ell})$ such that $det(\phi_1) = det(\phi_2)$.

Remarks

We can further classify discrete parameters as follows:

- (supercuspidal) The SL_2-factor acts trivially. In this case, $\Pi_\phi(G)$ and $\Pi_\phi(J)$ consist only of supercuspidal representations.
- (mixed supercuspidal) The SL_2-factor acts non-trivially. In this case, $\Pi_\phi(G) = \{\pi_{disc}, \pi_{sc}\}$ and $\Pi_\phi(J)$ contain a mix of supercuspidal and (discrete series) non-supercuspidal representations. There are two cases:
  1. (Saito-Kurokawa Type) We have $std \circ \phi = \phi_0 \oplus \chi \boxtimes \nu(2)$ and $\Pi_\phi(J) = \{\rho_{disc}, \rho_{sc}\}$.
  2. (Howe-Piatetski–Shapiro Type) We have $std \circ \phi = \chi_1 \boxtimes \nu(2) \oplus \chi_2 \boxtimes \nu(2)$ and $\Pi_\phi(J) = \{\rho^1_{disc}, \rho^2_{disc}\}$.

The Main Theorem

Theorem (H)

The following is true.

1. For any $\pi \in \Pi(G)$ (resp. $\rho \in \Pi(J)$) such that the Gan-Takeda (resp. Gan-Tantono) parameter is not supercuspidal the Gan-Takeda (resp. Gan-Tantono) correspondence is compatible with the Fargues-Scholze correspondence.
2. If $L/\mathbb{Q}_p$ is unramified and $p > 2$, we have, for all $\pi \in \Pi(G)$ (resp. $\rho \in \Pi(J)$) such that the Gan-Takeda (resp. Gan-Tantono) parameter is supercuspidal the Gan-Takeda (resp. Gan-Tantono) correspondence is compatible with the Fargues-Scholze correspondence.

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Rappaport-Zink Spaces

Harris-Taylor realized the Local Langlands correspondence for $G = GL_n$ in the cohomology of the generic fibers of Rapoport-Zink spaces. The generic fiber of these spaces at infinite level can be reinterpreted as shtuka spaces, which in turn relate to the Fargues-Scholze LLC.

Definition

We set $k := \overline{\mathbb{F}}_p$, $\breve{\mathbb{Z}}_p := W(k)$. We consider $\mathbb{X}$ a p-divisible group over k of dimension d and height n. We consider the formal scheme:

$$\mathcal{M}_{\mathbb{X}}/Spf(\breve{\mathbb{Z}}_p)$$

parametrizing, for $S/Spf(\breve{\mathbb{Z}}_p)$, pairs (X, ρ), where X/S is a p-divisible group, and $\rho : X \times_S \overline{S} \to \mathbb{X} \times_k \overline{S}$ is a quasi-isogeny, where $\overline{S} = S \times_{Spf(\breve{\mathbb{Z}}_p)} Spec(k)$ is the special fiber.

Lubin-Tate Space

Example

In the case, that $d = 1$, we recover the Lubin-Tate space parametrizing $\mathbb{Z}$ disjoint copies of the space of deformations of a 1-dimensional formal group with height n:

$$LT_n := \mathcal{M}_{\mathbb{X}} \simeq \sqcup_{\mathbb{Z}} Spf(\breve{\mathbb{Z}}_p[|T_1, \ldots, T_{n-1}|])$$

Let $\mathcal{M}_{\mathbb{X},\breve{\mathbb{Q}}_p}$, be the adic generic fiber of this moduli space, and $\mathcal{M}_{\mathbb{X},\infty,\breve{\mathbb{Q}}_p}$ be the moduli space at infinite level. Scholze-Weinstein provide a moduli interpretation of this space in terms of shtuka spaces on the Fargues-Fontaine curve.

Definition

1. Let $F = \lim_{x \mapsto x^p} \mathbb{C}_p$ be the tilt of the completed algebraic closure of $\mathbb{Q}_p$.
2. We set X to be the (schematic) Fargues-Fontaine curve of F.

Remark

X is a Dedekind scheme. Its closed points correspond to characteristic 0 untilts of F.

Theorem (Fargues)

- G-bundles on X correspond to elements of the Kottwitz set $B(G) := G(\breve{\mathbb{Q}}_p)/(b \sim gb\sigma(g)^{-1})$, where σ is the Frobenius on $\breve{\mathbb{Q}}_p$. In other words, we have an isomorphism:

$$B(G) \xrightarrow{\simeq} |Bun_G|$$

$$b \mapsto |Bun_G^b| = *$$

Remark

Elements of $B(G)$ parametrize G-isocrystals over k

Definition

1. Let $\mu \in X_*(G_{\overline{\mathbb{Q}}_p})^+$ be a minuscule dominant cocharacter with reflex field E.
2. Let $b \in B(G, \mu) \subset B(G)$ be the unique basic element. We call the triple (G, b, μ) a basic local Shimura datum.
3. $\breve{E} = E\breve{\mathbb{Q}}_p$
4. Let $\mathcal{E}_b$ be the bundle on X corresponding to $b \in B(G)$.
5. We define the diamond $Sht(G, b, \mu)_\infty \to Spd(\breve{E})$ to be the space parametrizing modifications

$$\mathcal{E}_b \dashrightarrow \mathcal{E}_0$$

with meromorphy bounded by μ. A local Shimura variety at infinite level.

Remark

The space $Sht(G, b, \mu)_\infty$ has commuting actions by $G(\mathbb{Q}_p) = Aut(\mathcal{E}_0)$ and $J_b(\mathbb{Q}_p) = Aut(\mathcal{E}_b)$, where J_b is the σ-centralizer of $b \in G(\breve{\mathbb{Q}}_p)$. We have similar actions on $\mathcal{M}_{\mathbb{X},\infty,\breve{\mathbb{Q}}_p}$.

Theorem (Scholze-Weinstein)

For the fixed p-divisible group $\mathbb{X}/k$ of dimension d and height n as before, let $\mu = (1, \dots, 1, 0, \dots, 0)$ be the minisicule cocharacter of GL_n with d 1s and $b \in B(G)$ be the element corresponding to the isogeny class of $\mathbb{X}/k$. Then we have a $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$-equivariant isomorphism of diamonds over $Spd(\breve{\mathbb{Q}}_p)$:

$$\mathcal{M}^{\diamond}_{\mathbb{X},\infty,\breve{\mathbb{Q}}_p} \simeq Sht(G, \mu, b)_\infty$$

The Cohomology of the Lubin-Tate Tower

Example

Suppose that $\mathbb{X}$ has dimension 1 and height n, so that $\mu = (1, 0 \ldots, 0)$, then $LT^{\diamond}_{n,\infty} \simeq Sht(GL_n, \mu, b)_{\infty}$ parametrizes injections of the form:

$$\mathcal{O}(-\frac{1}{n}) \hookrightarrow \mathcal{O}_X^n$$

with cokernel of length 1 supported at a single closed point of X, where $\mathcal{O}(-\frac{1}{n})$ is the rank n bundle on X of slope $-\frac{1}{n}$. In this case, $J_b(\mathbb{Q}_p) = D^*_{1/n}$, where $D_{1/n}$ is the division algebra of invariant $1/n$.

The Cohomology of the Lubin-Tate Tower

Definition

1. Let (G, b, μ) be a local Shimura datum.
2. Let $\pi \in \Pi(G)$ and $\rho \in \Pi(J_b)$.
3. Let $\mathcal{H}(G) := C_c^{\infty}(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ and $\mathcal{H}(J_b) := C_c^{\infty}(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ denote the usual smooth Hecke algebras.
4. Set $R\Gamma_c(G, b, \mu) := \mathrm{colim}_{K \to \{1\}} R\Gamma_c(Sht(G, b, \mu)_\infty / \underline{K}, \overline{\mathbb{Q}}_\ell)$.
5. Set $R\Gamma_c(G, b, \mu)[\pi] := R\Gamma_c(G, b, \mu) \otimes^{\mathbb{L}}_{\mathcal{H}(G)} \pi$ and $R\Gamma_c(G, b, \mu)[\rho] := R\Gamma_c(G, b, \mu) \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \rho$.

The Cohomology of the Lubin-Tate Tower

Theorem (Harris-Taylor/Dat)

1. Let $(G, b, \mu) = (GL_n, b, (1, 0, \dots, 0))$.
2. Let π be a supercuspidal representation of $G(\mathbb{Q}_p)$ and $\rho := JL^{-1}(\pi) \in \Pi(D^*_{\frac{1}{n}})$ with associated Weil parameter ϕ.
3. Then $R\Gamma_c(G, b, \mu)[\pi]$ (resp. $R\Gamma_c(G, b, \mu)[\rho]$) is concentrated in middle degree $n - 1$, and its cohomology decomposes as a $J_b(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$ (resp. $G(\mathbb{Q}_p) \times W_{\mathbb{Q}_p}$) representation as:

$$\rho \boxtimes \phi^\vee \otimes |\cdot|^{(1-n)/2}$$

and

$$\pi \boxtimes \phi \otimes |\cdot|^{(1-n)/2}$$

respectively.

Remarks

1. The idea behind the proof of this Theorem is to relate this complex to the cohomology of a global Shimura variety associated to an outer form of $GL_n/\mathbb{Q}$, via basic uniformization, and appeal to Global Results.
2. We claim that this result implies compatibility of the Fargues-Scholze correspondence with the Harris-Taylor one.

Function-Sheaf Dictionary

To a locally spatial diamond or Artin v-stack X, Fargues-Scholze define a triangulated category $D(X) := D_{lis}(X, \overline{\mathbb{Q}}_\ell)$

Proposition (Fargues-Scholze)

1. For a connected reductive group $G/\mathbb{Q}_p$, define the v-stack $B\underline{G(\mathbb{Q}_p)} := [*/\underline{G(\mathbb{Q}_p)}]$.
2. We have an equivalence of categories:

$$D(B(\underline{G(\mathbb{Q}_p)})) \xrightarrow{\simeq} D(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$$

where the RHS is the unbounded derived category of smooth $\overline{\mathbb{Q}}_\ell$-representations of $G(\mathbb{Q}_p)$ and Verdier duality corresponds to smooth duality.

Function-Sheaf Dictionary

- We have an open immersion:

$$j : B\underline{G(\mathbb{Q}_p)} \simeq Bun_G^{\mathbf{1}} \hookrightarrow Bun_G$$

  given by the inclusion of the HN-strata of Bun_G corresponding to the trivial bundle.

- Given $\pi \in \Pi(G)$, we can use the previous proposition to construct a sheaf:

$$j_!(\mathcal{F}_\pi) \in D(Bun_G)$$

- Following V.Lafforgue, Fargues-Scholze construct an excursion algebra acting on $D(Bun_G)$ via Hecke operators, which acts on this sheaf via eigenvalues determined by the parameter ϕ_π.

Hecke Operators

- For any finite set I, let X^I be the product of I-copies of the diamond $X = Spd(\breve{\mathbb{Q}}_p)/Frob^{\mathbb{Z}}$.
- We then have the Hecke-Stack:

$$\begin{array}{ccc} & Hck & \\ \swarrow h^{\leftarrow} & & \searrow h^{\rightarrow} \times supp \\ Bun_G & & Bun_G \times X^I \end{array}$$

parametrzing triples $(\mathcal{E}_0, \mathcal{E}_1, j, F_i^{\sharp} i \in I)$, where $j : \mathcal{E}_0 \to \mathcal{E}_1$ is a modification of two G-bundles on X away from the closed points defined by characteristic 0 untilts $F_i^{\sharp}$ for $i \in I$ of F.

Hecke Operators

- Given $V \in Rep_{\overline{\mathbb{Q}}_\ell}({}^L G^I)$, Geometric Satake furnishes a sheaf $\mathcal{S}_V$ on Hck.
- We then define the Hecke operator:

$$T_V : D(Bun_G) \to D(Bun_G \times X^I)$$

$$\mathcal{F} \mapsto R(h^{\rightarrow} \times supp)_{\natural}(h^{\leftarrow *}(\mathcal{F}) \otimes^{\mathbb{L}} \mathcal{S}_V)$$

Drinfeld's Lemma

Theorem (Fargues-Scholze)

The Hecke operator T_V induces a functor:

$$T_V : D(Bun_G) \to D(Bun_G)^{BW^I_{\mathbb{Q}_p}}$$

where $D(Bun_G)^{BW^I_{\mathbb{Q}_p}}$ is the derived category of objects in $D(Bun_G)$ with a continuous action of $W^I_{\mathbb{Q}_p}$.

Excursion Operators

Definition

1. Let $(I, V, (\gamma_i)_{i \in I}, \beta, \alpha)$ be the datum of:
   - A finite set I.
   - A representation $V \in Rep_{\overline{\mathbb{Q}}_\ell}({}^L G^I)$.
   - A tuple of elements $(\gamma_i)_{i \in I} \in W_{\mathbb{Q}_p}^I$.
   - Maps of representations: $\overline{\mathbb{Q}}_\ell \xrightarrow{\alpha} \Delta^* V \xrightarrow{\beta} \overline{\mathbb{Q}}_\ell$.
2. Given such a datum, one defines the excursion operator on $D(Bun_G)$ to be the endomorphism of the identity functor:

$$id = T_{\overline{\mathbb{Q}}_\ell} \xrightarrow{\alpha} T_{\Delta^* V} = T_V \xrightarrow{(\gamma_i)_{i \in I}} T_V = T_{\Delta^* V} \xrightarrow{\beta} T_{\overline{\mathbb{Q}}_\ell} = id$$

Excursion Operators

Via looking at the action of this natural transformation on the triangulated sub-category:

$$D(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell) \simeq D(B\underline{G(\mathbb{Q}_p)}) \subset D(Bun_G)$$

we can apply V. Lafforgue's reconstruction theorem to deduce:

Theorem (Fargues-Scholze)

Given $\pi \in \Pi(G)$, there is a unique continuous semisimple map:

$$\phi_\pi^{FS} : W_{\mathbb{Q}_p} \rightarrow^L G(\overline{\mathbb{Q}}_\ell)$$

characterized by the property that the action on π by an excursion operator is the composite:

$$\overline{\mathbb{Q}}_\ell \xrightarrow{\alpha} \Delta^* V = V \xrightarrow{(\phi_\pi^{FS}(\gamma_i))_{i \in I}} V = \Delta^* V \xrightarrow{\beta} \overline{\mathbb{Q}}_\ell$$

Hecke Operators and Local Shimura varieties

- Let (G, b, μ) be a basic local Shimura datum with E the reflex field of μ and $\rho \in \Pi(J_b)$.
- Consider the case where $I = \{*\}$ and $V = V_\mu$ is a highest weight rep of highest weight μ. The sheaf $\mathcal{S}_V$ is then supported on Hck_μ the subspace parametrizing modifications of type μ, $\mathcal{S}_V \simeq \overline{\mathbb{Q}}_\ell[d](\frac{d}{2})$, where $d = \langle 2\rho_G, \mu \rangle$.
- We look at the Hecke Correspondence:

$$\begin{array}{ccccc} & & Hck_\mu & & \\ & \swarrow^{h^{\leftarrow}} & & \searrow^{h^{\rightarrow} \times supp} & \\ Bun_G^b \xrightarrow{j} Bun_G & & & & Bun_G \times X \underset{j_1 \times id}{\longleftarrow} Bun_G^1 \times X \end{array}$$

- We consider the sheaf:

$$(j_1)^* T_\mu(j_!(\mathcal{F}_\rho)) \in D(G(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)^{BW_{\mathbb{Q}_p}}$$

Hecke Operators and Local Shimura varieties

- The simultaneous fibers of Hck_μ over $Bun_G^{\mathbf{1}}$ and Bun_G^b is given by:

$$[Gr^b_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}]$$

where $Gr^b_{G,\mu^{-1}} \subset Gr_{G,\mu^{-1}}$ is the open Newton strata in the Schubert cell of the B^+_{dR} -affine Grassmanian parametrizing modifications $\mathcal{E} \to \mathcal{E}_0$ of type μ, where $\mathcal{E} \simeq \mathcal{E}_b$, after pulling back to each geometric point.

- The Shtuka space sits as $G(\mathbb{Q}_p) \times J_b(\mathbb{Q}_p)$-torsor over this space:

$$Sht(G,b,\mu)_\infty \to [Gr^b_{G,\mu^{-1}}/\underline{G(\mathbb{Q}_p)}]$$

This gives rise to the key identification:

$$R\Gamma_c(G,b,\mu)[\rho][d](\frac{d}{2}) \simeq (j_1)^* T_\mu(j_{b!}(\mathcal{F}_\rho))$$

Compatibility for GL_n

- The Fargues-Scholze parameter of $\rho \in \Pi(J_b)$ is constructed from the action of excursion operators on the sheaf $j_!(\mathcal{F}_\rho)$. This is in turn constructed from the action of Hecke operators on $j_!(\mathcal{F}_\rho)$, which is related to the cohomology of local Shimura varieties and the classical local Langlands correpsondence.
- In particular, we can show the following:

Lemma (H,Koshikawa)

Let V be an irreducible representation of ${}^LG$, ϕ an irreducible representation of $W_{\mathbb{Q}_p}$, $b \in B(G)$, and $\rho \in \Pi(J_b)$.
If the cohomology sheaves of $T_V(j_{b!}(\mathcal{F}_\rho)) \in D(Bun_G)^{BW_{\mathbb{Q}_p}}$ admit a sub-quotient with action of ϕ then $r_V \circ \phi_\rho^{FS}$ also has such a sub-quotient.

Compatibility for GL_n

- If we let $(G, b, \mu) = (GL_n, b, (1, 0, \ldots, 0))$ and $\rho \in \Pi(D^*_{\frac{1}{n}})$ be a supercuspidal representation. Then, by Harris-Taylor, we know that:

$$R\Gamma_c(G, b, \mu)[\rho][n-1](\frac{n-1}{2}) \simeq j_1^* T_\mu j_{b!}(\mathcal{F}_\rho)$$

  admits a sub-quotient with $W_{\mathbb{Q}_p}$-action given by the semi-simplified parameter ϕ attatched to ρ
- Therefore, by the lemma $r_\mu \circ \phi_\rho^{FS} = \phi_\rho^{FS}$ also admits such a sub-quotient. Hence, we get that $\phi = \phi_\rho^{FS}$ and conclude compatibility for supercuspidal ρ.
- By compatibility of the Fargues-Scholze and Harris-Taylor correspondence with parabolic induction, we conclude compatibility in general.

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- Similar to the case of GL_n compatibility will be proven by describing the cohomology of certain local Shimura varieties attached to GSp_4/L for $L/\mathbb{Q}_p$ a finite extension.
- Consider $(G, b, \mu) := (Res_{L/\mathbb{Q}_p}(GSp_4), b, \mu)$, where $b \in B(G, \mu)$ is the unique basic element and μ is the Siegel cocharacter.
- Then $J_b = Res_{L/\mathbb{Q}_p} GU_2(D)$, where D is the quaternionic division algebra over L.
- Note that the rep of $\hat{G} \simeq GSp_4$ induced by μ is the standard embedding:

$$std : GSp_4 \hookrightarrow GL_4$$

  and that $\langle 2\rho_G, \mu \rangle = 3$.

Hansen-Kaletha-Weinstein

- Fix $\rho \in \Pi(J)$ a representation with discrete Gan-Tantono parameter ϕ.
- We define a complex

$$R\Gamma_c^\flat(G, b, \mu)[\rho] := R\mathcal{H}om_{J(\mathbb{Q}_p)}(R\Gamma_c(G, b, \mu), \rho)[-6](-3)$$

  of $G(\mathbb{Q}_p) \times W_L$-modules.
- We let $S_\phi := Cent(\phi, \hat{G})$.

Hansen-Kaletha-Weinstein

Theorem (Hansen-Kaletha-Weinstein)

The following is true.

- We have the following equality in $K_0(G(\mathbb{Q}_p))_{ell}$ the Grothendieck group of elliptic admissible $G(\mathbb{Q}_p)$-modules:

  $$[R\Gamma_c^\flat(G, b, \mu)[\rho])] = -\sum_{\pi \in \Pi_\phi(G)} Hom_{S_\phi}(\delta_{\pi,\rho}, std \circ \phi)\pi$$

  where $\delta_{\pi,\rho}$ is the algebraic representation of S_ϕ defined by the refined Local Langlands correspondence.
- If ϕ is a supercuspidal parameter the above is true in the Grothendieck group of all admissible representations.

The Refined LLC

- Note that we have $Z(\hat{G}) = GL_1$.
- The refined Local Langlands correspondence defines a bijection:

$$\Pi_\phi(GSp_4) \leftrightarrow \{\text{irred. reps } \tau \text{ of } S_\phi \text{ s.t } \tau|_{Z(\hat{G})} = \mathbf{1}\}$$

- And a bijection:

$$\Pi_\phi(GU_2(D)) \leftrightarrow \{\text{irred. reps } \tau \text{ of } S_\phi \text{ s.t } \tau|_{Z(\hat{G})} = id_{GL_1}\}$$

- These bijections are characterized by the endoscopic character identities proven by Chan-Gan, after fixing (B, ψ) a Whittaker datum.

The Refined LLC

- If ϕ is a stable discrete parameter then the L-packet are singletons and $S_\phi = Z(\hat{G}) = GL_1$.
- If ϕ is an endoscopic parameter $(std \circ \phi \simeq \phi_1 \oplus \phi_2)$. There is an identification

$$S_\phi = \{(a, b) \in GL_1 \times GL_1 | a^2 = b^2\} \subset GL_2 \times GL_2 \subset GL_4$$

  where $Z(\hat{G}) = GL_1$ embeds diagonally.
- We see that $\pi_0(S_\phi) \simeq \mathbb{Z}/2\mathbb{Z}$. The L-packet of $\Pi_\phi(GSp_4) = \{\pi^+, \pi^-\}$ is indexed by the reps τ_{π^+} and τ_{π^-} of S_ϕ defined by the trivial and non-trivial character of $\pi_0(S_\phi)$.

Applications

- The L-packet $\Pi_\phi(GU_2(D)) = \{\rho_1, \rho_2\}$ is indexed by the representations τ_{ρ_1} and τ_{ρ_2} corresponding to projection to the two GL_1 factors.

Definition

Given $\pi \in \Pi_\phi(GSp_4)$ and $\rho \in \Pi_\phi(GU_2(D))$, we set:

$$\delta_{\pi,\rho} := \tau_\pi^\vee \otimes \tau_\rho$$

where $\tau_\pi^\vee$ denotes the contragradient.

- With these identifications pinned down, we can now write the down the RHS of the previous theorem

$$-\sum_{\pi \in \Pi_\phi(G)} Hom_{S_\phi}(\delta_{\pi,\rho}, std \circ \phi)\pi$$

with $\delta_{\pi,\rho}$ as above.

Corollary

If $\rho \in \Pi(GU_2(D))$ has a discrete stable Gan-Tantono parameter ϕ, we have that

$$[R\Gamma_c^\flat(G, \mu, b)[\rho]] = -Hom_{id}(id, std \circ \phi)\pi = -4\pi$$

where $\Pi_\phi(GSp_4) = \{\pi\}$.

Corollary

If $\rho = \rho_i \in \Pi(GU_2(D))$ has a discrete endoscopic Gan-Tantono parameter, we have that $[R\Gamma_c^\flat(G, \mu, b)[\rho]]$ is equal to

$$-Hom_{S_\phi}(\tau_{\rho_i}, std \circ \phi)\pi^+ + -Hom_{S_\phi}(\tau_{\rho_{3-i}}, std \circ \phi)\pi^-$$

Writing $std \circ \phi \simeq \phi_1 \oplus \phi_2$ this identifies with:

$$-Hom_{id}(id, \phi_i)\pi^+ + -Hom_{id}(id, \phi_{3-i})\pi^- = -2\pi^+ + -2\pi^-$$

where $\Pi_{\phi_\rho} = \{\pi^+, \pi^-\}$.

Compatibility in the non-supercuspidal case

- We want to use the previous corollaries to begin making progress to compatibility. We first note that for any $\rho \in \Pi(J)$ (resp. $\pi \in \Pi(G)$) an irreducible constituent of a parabolic induction, compatibility follows from compatibility of Fargues-Scholze with parabolic induction and compatibility for GL_n and its inner forms.
- Therefore, we can assume ρ (resp. π) is supercuspidal, which implies that their L-parameter is discrete and we can use the previous corollaries.

Compatibility in the non-supercuspidal case

- It follows from the commutation of excursion and Hecke operators that the following is true.

Lemma

For any $\rho \in \Pi(J_b)$, if $\pi \in \Pi(G)$ occurs as a sub-quotient of $R\Gamma_c^\flat(G, b, \mu)[\rho]$ or $R\Gamma_c(G, b, \mu)[\rho]$ then $\phi_\rho^{FS} = \phi_\pi^{FS}$.

- Now, assume ϕ is of Howe-Piatetski–Schapiro type, write $\Pi_\phi(G) = \{\pi_{sc}, \pi_{disc}\}$ and $\Pi_\phi(J) = \{\rho_{sc}, \rho_{disc}\}$. The previous corollary gives us an equality

$$[R\Gamma_c^\flat(G, b, \mu)[\rho_{disc}]_{sc}] = -2\pi_{sc}$$

in $K_0(G(\mathbb{Q}_p))$.

- Thus, by the Lemma, we get $\phi_{\pi_{sc}}^{ss} = \phi_{\rho_{disc}}^{ss} = \phi_{\rho_{disc}}^{FS} = \phi_\pi^{FS}$.

Compatibility in the supercuspidal case

- This allow us to reduce to checking compatibility for $\rho \in \Pi(J_b)$ with supercuspidal Gan-Tantono parameter ϕ.

Key Proposition

$$\bigoplus_{\rho' \in \Pi_\phi(J)} R\Gamma_c(G, b, \mu)[\rho']_{sc}$$

admits a non-zero W_L-stable sub-quotient with W_L-action given by $std \circ \phi \otimes |\cdot|^{-3/2}$.

- The previous Lemma and Corollaries tell us that the Fargues-Scholze parameter of all $\rho' \in \Pi_\phi(J_b)$ agree. Therefore, the same analysis as for GL_n allows us to conclude $std \circ \phi_\rho^{FS} = std \circ \phi$.

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We will now discuss the proof of the Key Proposition.

- As in Harris-Taylor, the key idea will be to relate the complex $\bigoplus_{\rho' \in \Pi_{\phi}(J)} R\Gamma_c(G, b, \mu)[\rho']_{sc}$ to the cohomology of a global Shimura variety.
- We let:
  - $F/\mathbb{Q}$ be a totally real field such that:
    - p is totally inert in F and $F_p \simeq L$.
    - q is an auxillary totally inert prime.
  - $\mathbf{G}$ a $\mathbb{Q}$-inner form of $Res_{F/\mathbb{Q}}GSp_4 =: \mathbf{G}^*$ such that:
    - $\mathbf{G}(\mathbb{R}) \simeq GSp_4(\mathbb{R}) \times GU_2(\mathbb{H})^{[F:\mathbb{Q}]-1}$
    - $\mathbf{G}_{\mathbb{Q}_p} \simeq Res_{L/\mathbb{Q}_p}GSp_4 = G$.
    - $\mathbf{G}_{F_v} \simeq GSp_4/F_v$ at all finite places v if $[F : \mathbb{Q}]$ is odd.
    - $\mathbf{G}_{F_v} \simeq GSp_4/F_v$ at all but the finite place q if $[F : \mathbb{Q}]$ is even.

Basic Uniformization

- We let $(\mathbf{G}, X)$ be the Shimura datum of abelian type, where $X_{\mathbb{C}}$ under the isomorphism $j : \overline{\mathbb{Q}}_p \simeq \mathbb{C}$ agrees with the conjugacy class of cocharacters defined by μ.
- Let $\mathbb{A}$ (resp. $\mathbb{A}_f$) denote the (resp. finite) adeles of $\mathbb{Q}$.
- For any compact open $K \subset \mathbf{G}(\mathbb{A}^f)$, we let $\mathcal{S}(\mathbf{G}, X)_K$ be the rigid analytic global Shimura variety over $\mathbb{C}_p$ of level K.
- We consider the infinite level Shimura variety:

$$\mathcal{S}(\mathbf{G}, X)_{K^p} = \lim_{K_p \to \{1\}} \mathcal{S}(\mathbf{G}, X)_{K^p K_p}$$

This is representable by a perfectoid space after completing the structure sheaf.

Basic Uniformization

- By results of Hansen, there exists a canonical $G(\mathbb{Q}_p)$-equivariant Hodge-Tate period map:

$$\pi_{HT} : \mathcal{S}(\mathbf{G}, X)_{K^p} \to \mathcal{F}\ell_{G,\mu^{-1}}$$

where $\mathcal{F}\ell_{G,\mu^{-1}} := (G_{\mathbb{C}_p}/P_{\mu^{-1}})^{ad}$.

- For $b \in B(G, \mu)$, we define the Newton strata $\mathcal{S}(\mathbf{G}, X)^b_{K^p}$ by pulling back the Newton strata $\mathcal{F}\ell^b_{G,\mu^{-1}}$ along π_{HT}.

Basic Uniformization

- We let $\mathbf{G}'$ denote another $\mathbb{Q}$-inner form of $\mathbf{G}^*$ satisfying:
  - $\mathbf{G}'(\mathbb{R}) \simeq GU_2(\mathbb{H})^{[F:\mathbb{Q}]}$
  - $\mathbf{G}'_{\mathbb{Q}_p} \simeq J_b$.
  - $\mathbf{G}'(\mathbb{A}_f^p) \simeq \mathbf{G}(\mathbb{A}_f^p)$
- Under the assumptions on L, by results of Shen there exists a $\mathbf{G}(\mathbb{A}_f)$-equivariant isomorphism of diamonds over $\mathbb{C}_p$:

$$\lim_{K^p} \mathcal{S}(\mathbf{G}, X)^b_{K^p} \simeq \underline{\mathbf{G}'(\mathbb{Q})} \backslash \underline{\mathbf{G}'(\mathbb{A}^f)} \times_{Spd(\mathbb{C}_p)} Sht(G, b, \mu)_\infty / \underline{J_b(\mathbb{Q}_p)}$$

for the basic element $b \in B(G, \mu)$. Moreover,

$$\pi_{HT} : \lim_{K^p} \mathcal{S}(\mathbf{G}, X)^b_{K^p} \to \mathcal{F}\ell^b_{G,\mu^{-1}} \simeq Sht(G, b, \mu)_\infty / \underline{J_b(\mathbb{Q}_p)}$$

agrees with projection to the second factor.

Basic Uniformization

This allows us to deduce:

Corollary

There exists a natural $G(\mathbb{Q}_p) \times W_L$ and Hecke equivariant excision map:

$$\Theta : R\Gamma_c(G, b, \mu) \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}^f)/K^p, \mathcal{L}_\xi) \to$$

$$R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)$$

where (G, b, μ) is the local Shimura datum from before and $\mathcal{L}_\xi$ is the $\overline{\mathbb{Q}}_\ell$-local system defined by a representation of $\mathbf{G}$ of some highest weight ξ.

Boyer's Trick

Proposition

Let $R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)_{sc}$ denote the part of the cohomology where $G(\mathbb{Q}_p)$ acts via a supercuspidal representation. Then Θ induces an isomorphism:

$$\Theta : R\Gamma_c(G, b, \mu)_{sc} \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}^f)/K^p, \mathcal{L}_\xi) \simeq$$

$$R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)_{sc}$$

of $G(\mathbb{Q}_p) \times W_L$-representations.

Globalization

Let $\rho \in \Pi(J)$ be a representation with supercuspidal Gan-Tantono parameter ϕ. Using the simple trace formula, we choose a globalization

$$\Pi' \in \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^p, \mathcal{L}_\xi)$$

of ρ to a cusp form of $\mathbf{G}'(\mathbb{A}_F)$ of fixed level K^p satisfying the following:

- Π'_∞ is cohomological of regular weight ξ.
- Π' is unramified away from finite set of finite places S.
- Π' is an unramified twist of Steinberg at some finite set of finite places $\{q\} \subset S_{st}$.

Strong Multiplicity One

By combing the stable trace formula and the simple twisted trace formula of Kottwitz-Shelstad, we can deduce the following:

Proposition (H.)

Assume that $|S_{st}| \geq 3$. Let π be a cuspidal automorphic representation of $\mathbf{G}'$, $\mathbf{G}$, or $\mathbf{G}^* = GSp_4/F$ such that:

- $\pi^{S\cup\{\infty\}} \simeq \Pi'^{S\cup\{\infty\}}$
- π_v is an unramified twist of Steinberg at all $v \in S_{st}$
- π_∞ is cohomological of regular weight ξ

then we have an equality: $\phi_{\pi_p} = \phi$.

- If we let $\mathfrak{m} \subset \mathbb{T}^S$ be the maximal ideal in the commutative Hecke algebra defined by Π', we can localize the uniformization isomorphism at $\mathfrak{m}$

$$(R\Gamma_c(G, b, \mu)_{sc} \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^p, \mathcal{L}_\xi))_{\mathfrak{m}} \simeq$$

$$R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^p}, \mathcal{L}_\xi)_{\mathfrak{m},sc}$$

- We write $K^p = K^{S_{st}\cup\{p\}}K^p_{S_{st}}$ for $K^{S_{st}\cup\{p\}} \subset \mathbf{G}(\mathbb{A}_f^{S_{st}\cup\{p\}})$. Taking colimits on both sides as $K^p_{S_{st}} \to \{1\}$, we obtain an isomorphism

$$(R\Gamma_c(G, b, \mu)_{sc} \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^{S_{st}\cup\{p\}}, \mathcal{L}_\xi))^{st}_{\mathfrak{m}} \simeq$$

$$R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^{S_{st}\cup\{p\}}}, \mathcal{L}_\xi)^{st}_{\mathfrak{m},sc}$$

  on the summands where $\mathbf{G}$ and $\mathbf{G}'$ act via unramified twist of Steinberg at all $v \in S_{st}$.

Applying strong multiplicity one to both sides of the uniformization isomorphism, We have:

- All automorphic representations of $\mathbf{G}$ occurring in

$$R\Gamma_c(\mathcal{S}(\mathbf{G},X)_{K^{S_{st}\cup\{p\}}},\mathcal{L}_\xi)^{st}_{\mathfrak{m},sc}$$

  have local constituent at $\mathbf{G}_{\mathbb{Q}_p} \simeq G$ with L-parameter equal to ϕ. Since ϕ is supercuspidal, we have:

$$R\Gamma_c(\mathcal{S}(\mathbf{G},X)_{K^{S_{st}\cup\{p\}}},\mathcal{L}_\xi)^{st}_{\mathfrak{m},sc} = R\Gamma_c(\mathcal{S}(\mathbf{G},X)_{K^{S_{st}\cup\{p\}}},\mathcal{L}_\xi)^{st}_{\mathfrak{m}}$$

- Moreover, all automorphic representations of $\mathbf{G}'$ occurring in

$$(R\Gamma_c(G,b,\mu)_{sc} \otimes^{\mathbb{L}}_{\mathcal{H}(J_b)} \mathcal{A}(\mathbf{G}'(\mathbb{Q})\backslash\mathbf{G}'(\mathbb{A}_f)/K^{S_{st}\cup\{p\}},\mathcal{L}_\xi))^{st}_{\mathfrak{m}}$$

  have local constiuent at $\mathbf{G}'_{\mathbb{Q}_p} \simeq J_b$ with L-parameter ϕ.

Galois Representations in the Cohomology of Shimura varieties

- This reduces us to checking that

$$R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^{S_{st}\cup\{p\}}}, \mathcal{L}_\xi)^{st}_{\mathfrak{m},sc}$$

  has W_L-action given (up to multiplicity) by $std \circ \phi \otimes |\cdot|^{-3/2}$
- Using recent work of Kisin-Shin-Zhu, Kret and Shin show that the RHS is concentrated in middle degree (=3). Moreover, they compute the trace of Frobenius on the Shimura variety over $\overline{\mathbb{Q}}$ in terms of the Satake parameters of a weak transfer τ of Π' to GSp_4.

Galois Representations in the Cohomology of Shimura varieties

Letting τ be such a (globally generic) weak transfer, we can identify the Galois action (up to multiplicity) with:

Theorem (Sorensen)

There exists, a unique (after fixing an isomorphism $i : \mathbb{C} \xrightarrow{\simeq} \overline{\mathbb{Q}}_\ell$) irreducible continuous representation $\rho_\tau : Gal(\overline{F}/F) \to GSp_4(\overline{\mathbb{Q}}_\ell)$ characterized by the property that, for each finite place $v \nmid \ell$ of F, we have:

$$WD(\rho_{\tau,i}|_{W_{F_v}})^{F-s.s} \simeq \phi_{\tau_v} \otimes |\cdot|^{-3/2}$$

where ϕ_{τ_v} is the semi-simplified Gan-Takeda parameter.

- Therefore $R\Gamma_c(\mathcal{S}(\mathbf{G}, X)_{K^{S_{st} \cup \{p\}}}, \mathcal{L}_\xi)^{st}_{\mathfrak{m},sc}$ has W_L-action by $std \circ \phi \otimes |\cdot|^{-3/2}$, and the Proposition follows.

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The Kottwitz Conjecture for GSp_4

- To show compatibility, it was enough to provide an indirect description of $R\Gamma_c(G, b, \mu)[\rho]$. Amazingly, by using compatibility, and the machinery of Fargues-Scholze, we can say much more about this complex. Namely, we can show:

Theorem (H)

Let $L/\mathbb{Q}_p$ be a finite unramified extension with $p > 2$ and ϕ a supercuspidal parameter. If ϕ is stable and $\Pi_\phi(G) = \{\pi\}$ and $\Pi_\phi(J) = \{\rho\}$. We have isomorphisms:

$$R\Gamma_c(G, b, \mu)[\pi] \simeq \rho \boxtimes (std \circ \phi)^\vee \otimes |\cdot|^{-3/2}[-3]$$

$$R\Gamma_c(G, b, \mu)[\rho] \simeq \pi \boxtimes (std \circ \phi) \otimes |\cdot|^{-3/2}[-3]$$

The Kottwitz Conjecture for GSp_4

Theorem (cont.)

If ϕ is endoscopic with $std \circ \phi \simeq \phi_1 \oplus \phi_2$ and $\Pi_\phi(G) = \{\pi^+, \pi^-\}$ and $\Pi_\phi(J) = \{\rho_1, \rho_2\}$. We have isomorphisms:

$$R\Gamma_c(G,b,\mu)[\pi] \simeq \begin{array}{c} \rho_1 \boxtimes \phi_1^\vee \otimes |\cdot|^{-3/2} \oplus \rho_2 \boxtimes \phi_2^\vee \otimes |\cdot|^{-3/2}[-3] \\ \rho_1 \boxtimes \phi_2^\vee \otimes |\cdot|^{-3/2} \oplus \rho_2 \boxtimes \phi_1^\vee \otimes |\cdot|^{-3/2}[-3] \end{array}$$

and

$$R\Gamma_c(G,b,\mu)[\rho] \simeq \begin{array}{c} \pi^+ \boxtimes \phi_1 \otimes |\cdot|^{-3/2} \oplus \pi^- \boxtimes \phi_2 \otimes |\cdot|^{-3/2}[-3] \\ \pi^+ \boxtimes \phi_2 \otimes |\cdot|^{-3/2} \oplus \pi^- \boxtimes \phi_1 \otimes |\cdot|^{-3/2}[-3] \end{array}$$

Both possibilities occur and knowing the precise form in one case determines the precise form of the cohomology in all other cases.

The Spectral Action

- We will explain the proof in the stable case for simplicity. This uses the spectral action.
- The cocharacter μ defines a vector bundle C_μ on the stack of Langlands parameters $[\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}]$, whose evaluation at a $\overline{\mathbb{Q}}_\ell$-point corresponding to $\phi : W_{\mathbb{Q}_p} \to \hat{G}(\overline{\mathbb{Q}}_\ell)$ is $r_\mu \circ \phi$.
- The spectral action $C_\mu * j_!(\mathcal{F}_\rho)$ is precisely $T_\mu j_!(\mathcal{F}_\rho)$.
- If ρ has supercuspidal Gan-Tantono ($=$ Fargues-Scholze) parameter then $\phi = \phi_\rho^{FS}$ defines a connected component of $[\mathcal{Z}^1(W_{\mathbb{Q}_p}, \hat{G})_{\overline{\mathbb{Q}}_\ell}/\hat{G}]$ which (up to unramified twists) is given by:

$$[\overline{\mathbb{Q}}_\ell/S_\phi]$$

  and the action of C_μ factors over restriction to this connected component.

The Spectral Action

- Vector bundles on $[\overline{\mathbb{Q}}_\ell/S_\phi]$ are the same thing as $W \in Rep_{\overline{\mathbb{Q}}_\ell}(S_\phi)$. Given such W, the spectral action gives us functors:

$$Act_W : \bigoplus_{b \in B(G)_{basic}} D(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell) \to \bigoplus_{b \in B(G)_{basic}} D(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$$

- In particular, if we consider $r_\mu \circ \phi|_{S_\phi} = id_{GL_1} \boxtimes \phi$ as a rep of $S_\phi \times W_L$. We get:

$$R\Gamma_c(G, b, \mu)[\rho] \simeq C_V * j_!(\mathcal{F}_\rho)[-3](-3/2) =$$

$$Act_{id_{GL_1}}(\rho) \boxtimes \phi \otimes |\cdot|^{-3/2}[-3]$$

The Spectral Action

- Recent work of Hansen tells us that, since ϕ_{ρ}^{FS} is supercuspidal, $R\Gamma_c(G, b, \mu)[\rho]$ is concentrated in middle degree 3.
- Therefore, by Hansen-Kaletha-Weinstein, we deduce that $Act_{id_{GL_1}}(\rho) \simeq \pi$ and the result follows.
- The spectral action is monoidal so in particular

$$Act_{id_{GL_1}^{-1}}(\pi) = Act_{id_{GL_1}^{-1}} \circ Act_{id_{GL_1}}(\rho) =$$

$$Act_{\mathbf{1}}(\rho) = \rho$$

so we get the same result for the π-isotypic part!

The non-minuscule Case

- Using the monoidal property, we can show that $Act_{id^n_{GL_1}}(\rho)$ (resp. $Act_{id^n_{GL_1}}(\pi)$) is equal to π (resp. ρ) if n is odd and is ρ (resp. π) if n is even.
- This gives us a complete description of $R\Gamma_c(G, b, \mu)[\rho]$ (resp. $R\Gamma_c(G, b, \mu)[\pi]$) for μ any cocharacter, which in turn implies Fargues' conjecture for GSp_4.
- Using compatibility of Fargues-Scholze with isogenies of groups, we can deduce compatibility for Sp_4 and $SU_2(D)$, arguing again using the spectral action, we can verify the Kottwitz's conjecture in the Grothendieck group.

Future Work

- Alexander Bertoloni-Meli and Kieh-Hieu Nguyen recently verified Kottwitz's conjecture in the Grothendieck group for odd unitary similitude groups. In joint work in progress, we verify compatibility for these groups and obtain more precise descriptions of the cohomology even in the non-minuscule case!
- Recently, Koshikawa reproved Caraini-Scholze's vanishing results, using their semi-perversity results and compatibility for GL_n and its inner forms. While the relevant semi-perversity statement is not known currently, our result allows the remaining part of the argument to go through for GSp_4!

Thanks

I would like to thank:

- Eric Urban and Chao Li for organizing the seminar!
- All of you for participating!