Euler systems for conjugate-symplectic motives

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The Birch and Swinnerton-Dyer conjecture

$E/\mathbf{Q}$ elliptic curve, $N = N_E$. BSD principle:

$$(|E(\mathbf{F}_p)|)_{p\nmid N} + \text{analysis} \quad \rightsquigarrow \quad r_E = \dim E(\mathbf{Q}) \otimes_{\mathbf{Z}} \mathbf{Q}.$$

Let $a_p := p + 1 - |E(\mathbf{F}_p)|$ and

$$L(E,s) := \prod_{p\nmid N} (1 - a_p p^{-s} + p^{1-2s})^{-1} \cdot L_N(E,s), \qquad \Re(s) > 3/2.$$

Hecke, Wiles et al.: $L(E,s) = L(\varphi_E, s)$ has *entire* continuation, f. equation $s \leftrightarrow 2-s$, for $\varphi_E(\tau) = \sum_{n\geq 1} a_n q^n \in S_2(N)$.

Conjecture (BSD)

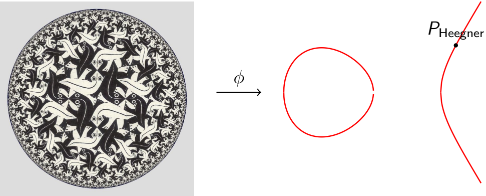
1. $\mathrm{ord}_{s=1} L(E,s) = r \implies r_E = r$.
2. $\kappa\colon E(\mathbf{Q}) \otimes \mathbf{Q}_p \to \mathrm{Sel}_p(E)$ is isomorphism $\forall p$ ($\iff \mathrm{Sha}(E)$ is finite), where

$$\mathrm{Sel}_p(E) = (\varprojlim_n \mathrm{Sel}_{p^n}(E)) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p \quad \overset{\text{Bloch–Kato}}{=} \quad H^1_f(\mathcal{G}_{\mathbf{Q}}, V_pE) \subset H^1(\mathcal{G}_{\mathbf{Q}}, V_pE)$$

The case $r = 1$: Heegner points

BSD is known for $r \leq 1$. If $r = 1$, for a suitable $K = K_D = \mathbf{Q}(\sqrt{-D})$:

- let $z_{K,1} :=$ CM-by-K point in $X_0(N)(\mathbf{C}) = \Gamma_0(N)\backslash\mathfrak{h}^*$.
  Theory of Complex Multiplication: $z_{K,1} \in X_0(N))(H_K)$, so
  $z_K := \mathrm{Tr}_{H_K/K}(z_{K,1}) - [H_K : K] \cdot \infty \quad \in \mathrm{Div}^0(X_0(N)_K) \to \mathrm{Jac}(X_0(N))(K)$.
- Modularity (Wiles et al., '93): there is $f \colon X_0(N) = X_{\mathrm{GL}_2, U_0(N)} \to E$.
- Heegner ('52): consider $z_K(f) := f_*(z_K) \in E(K)$.



- Gross–Zagier ('86): $\mathrm{ord}_{s=1} L(E_K, s) = 1 \iff z_K(f)$ non-torsion.
- **Kolyvagin** ('90): if $z_K(f)$ is non-torsion, then it spans $E(K)_{\mathbf{Q}}$ and $H^1_f(K, V_pE)$.

Heegner points/2

A modularity result can eliminate $K = K_D$ and yield $\mathrm{BSD}_{r=1}$ for $E/\mathbf{Q}$.

- Define $Z := \sum_D \mathrm{Tr}_{K_D/\mathbf{Q}}(z_{K_D})\ q^D \qquad \in \mathrm{Jac}(X_0(N))(\mathbf{Q}) \otimes \mathbf{Z}[\![q]\!]$
- Gross–Kohnen–Zagier's modularity: $Z \in \mathrm{Jac}(X_0(N))(\mathbf{Q}) \otimes M_{3/2}(4N)$.
- Kohnen, Shimura, Waldspurger: $S_2(N) \leftrightarrow S_{3/2}(4N)^+$.
- Let $\varphi \in S_{3/2}(4N)^+$ correspond to $\varphi_E \in S_2(N)$, and let

$$z(\varphi) := \Omega^{-1} \cdot \langle \varphi, Z \rangle_{\mathrm{Pet}} \qquad \in \mathrm{Jac}(X_0(N))(\mathbf{Q})_{\mathbf{Q}},$$

$$z(f, \varphi) := f_* z(\varphi) \qquad \in E(\mathbf{Q})_{\mathbf{Q}}.$$

This is a canonical point in $E(\mathbf{Q})_{\mathbf{Q}}$ and

$$\mathrm{ord}_{s=1} L(E, s) = 1 \overset{\mathrm{GKZ}}{\iff} z(f, \varphi) \neq 0 \Leftarrow\Rightarrow r_E = 1 \text{ and } |\mathrm{Sha}(E)| < \infty.$$

Beilinson–Bloch–Kato Conjecture on algebraic cycles

Let K number field, $X_{/K}$ proper smooth variety of *odd* dimension $n-1$. Consider

- $\mathrm{Ch}^{n/2}(X) := \mathbf{Q}[Y \subset^{(n/2)} X]_{/\sim}, \quad \supset \quad \mathrm{Ch}^{n/2}(X)^0 := \mathrm{Ker}\,(\mathrm{cl});$
- $V = V_pX := H^{n-1}_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_p(\tfrac{n}{2})) \circlearrowleft \mathcal{G}_K,$
  $\rightsquigarrow L(V,s) := \prod_{v\nmid p} \det(1 - \mathrm{Fr}_v \cdot q_v^{-s} | V^{I_v})^{-1} L_{(p)}(V,s).$

Conjecture (BBK). – $L(V,s)$ *has entire continuation, f.eq.* $s \leftrightarrow -s$, *and*

1. $\mathrm{ord}_{s=0} L(V,s) = r \implies \dim \mathrm{Ch}^{n/2}(X)^0 = r$;
2. $\mathrm{AJ}_p \colon \mathrm{Ch}^{n/2}(X)^0_{\mathbf{Q}_p} \to H^1_f(K, V_pX)$ *is an isomorphism.*

Variant: if V irreducible, geometric p-adic $\mathcal{G}_K$-representation of weight -1, let

$$H^1_f(K,V)^{\mathrm{mot}} := \sum_{(X, f\colon V_pX \twoheadrightarrow V)} f_* \mathrm{AJ_p}(\mathrm{Ch}^{n/2}(X)^0_{\mathbf{Q}_p}) \quad \subset \quad H^1_f(K,V).$$

Conjecture (BBK'). – $\mathrm{ord}_{s=0} L(V,s) = \dim H^1_f(K,V)^{\mathrm{mot}} = \dim H^1_f(K,V).$

Main result

Among n-dimensional $\mathcal{G}_K$-representations as above:

(1) 'Easier': K is TR field, symplectic $\mathcal{G}_K$-pairing $V \times V \to \mathbf{Q}_p(1)$ (e.g. V_pX), Hodge–Tate weights are distinct (e.g. $V \doteq \mathrm{Sym}^{n-1} V_pE$).

(2) 'Easiest': K is CM field, *conjugate-symplectic* pairing $V \times V^c \to \mathbf{Q}_p(1)$, (e.g. $V = \mathrm{Ind}_F^K V^{(1)}$, $F = K^+$), HT weights are distinct and consecutive.

Conjecture: $V = V_\Pi$ for cuspidal automorphic $\Pi \circlearrowleft \mathbf{GL}_n(\mathbf{A}_K)$. If so, $L(V, s)$ ✓.

Theorem A**. – *Let V be irreducible, automorphic, (2). Assume (ES), (Mod), for simplicity: $F \neq \mathbf{Q}$, n even. Then for an explicit*

$$\Theta \in H^1_f(K, V)^{\mathrm{mot}},$$

$$\mathrm{ord}_{s=0} L(V, s) = 1 \overset{?}{\iff} \Theta \neq 0 \implies \dim H^1_f(K, V)^{\mathrm{mot}} = \dim H^1_f(K, V) = 1.$$

Li–Liu ('21): *if AJ_p is injective* + some assumptions, then $\Rightarrow$ holds.

Previous results: low rank V [many!]; $V^{(2)} = V_n \otimes V_{n+1}$ [LTXZZ]; $V^{(1)}$ symplectic [Cornut]; $V^{(2)}$ c-symplectic + symplectic, any regular HT type [Graham–Shah].

Construction of Θ

As $V = V_\Pi$ c-symplectic $\Longrightarrow \Pi = \mathrm{BC}(\sigma)$ for $\sigma \circlearrowleft \mathbf{G}(\mathbf{A}_F)$, $\mathbf{G} = \mathbf{U}(\frac{n}{2}, \frac{n}{2})_{/F}$;
if $\varepsilon(V) = -1$ (at least) $\Longrightarrow \Pi = \mathrm{BC}(\pi)$ for $\pi \circlearrowleft \mathbf{H}(\mathbf{A}_F)$, $\mathbf{H} = \mathbf{U}(W)_{/F}$,
$\mathrm{sig}(W_\infty) = \{(n-1, 1); (n, 0), \ldots, (n, 0)\}$ [GRS, Mok, KMSW].
In fact V appears in the cohomology of the Shimura variety $X = (X_{\mathbf{H},U})_U / K$:

$$X_{\mathbf{H},U}(\mathbf{C}) = \mathbf{H}(F)\backslash \mathbf{H}(\mathbf{A}_F^\infty) \times \mathbf{B}_\mathbf{C}^{(n-1)}, \quad V_p X_{\mathbf{H},U} = \bigoplus_{\pi' \in \mathcal{A}(\mathbf{H})^\heartsuit} \pi'^{\vee,\infty,U} \otimes V_{\pi'}$$ [Kisin-Shin-Zhu].

(0) For $\underline{x} \in W^{n/2}$ with $T(\underline{x}) = (x_i, x_j)_{ij} > 0$, let $W(\underline{x}) := \mathrm{Span}(x_i)^\perp \subset^{(n/2)} W$,
let $\mathbf{H}(\underline{x}) := \begin{pmatrix} \mathbf{U}(W(\underline{x})) & \\ & 1 \end{pmatrix} \subset \mathbf{H}$ and $U(\underline{x}) := U \cap \mathbf{H}(\underline{x}))$
$\rightsquigarrow$ sub-unitary Shimura variety $Z(\underline{x})_U := [X_{\mathbf{H}(\underline{x}), U(\underline{x})}] \in \mathrm{Ch}^{n/2}(X_{\mathbf{H},U})$.

(1) Kudla: form

$$\Theta(\phi) := \sum_{\underline{x} \in U \backslash W^{n/2} \otimes \mathbf{A}^\infty :\ T(\underline{x}) \geq 0} \phi(\underline{x}) \cdot Z(\underline{x})\, \mathbf{q}_\tau^{T(\underline{x})}, \quad \tau \in \mathfrak{h}_\mathbf{G}, \phi \in \mathcal{S}(W_{\mathbf{A}^\infty}^{n/2}).$$

Conjecture (Mod): $\Theta(\phi) \in \mathrm{Ch}^{n/2}(X) \otimes S_\mathbf{G}$.
Assume this. *Known: orthogonal Shimura varieties [Zhang, Bruinier–W.Raum]; if $n = 2$ [Liu]; 'formally' [Liu]; for some $[K : \mathbf{Q}] = 2$ [J. Xia].*

Construction of Θ / continuation

(1) Kudla: $\Theta(\phi) := \sum_{T(\underline{x})\geq 0} \phi(x) \cdot Z(\underline{x})\, \mathbf{q}^{T(\underline{x})} \overset{\text{(Mod)}}{\in} \mathrm{Ch}^{n/2}(X_U) \otimes S_{\mathbf{G}}$.

(2) Liu: for $\varphi \in \sigma^\vee$ antiholomorphic, consider the *arithmetic theta lift*

$$\Theta(\varphi, \phi) := \Omega^{-1} \cdot \langle \varphi, \Theta(\phi) \rangle_{\mathrm{Pet}} \quad \in \mathrm{Ch}^{n/2}(X_U)^0_V \quad \xrightarrow{\mathrm{AJ}_p} H^1_f(K, \bigoplus_{V_{\pi'}=V} \pi'^{\vee,\infty,U} \otimes V).$$

(3) for $f \in \pi^{\infty,U}$, consider

$$\Theta(f, \varphi, \phi) := f_* \mathrm{AJ}_p \Theta(\varphi, \phi) \qquad \in H^1_f(K, V)^{\mathrm{mot}}.$$

This defines
$\Theta \in \mathrm{Hom}_{(\mathbf{H}\times\mathbf{G})(\mathbf{A}_F^\infty)}(\pi^\infty \otimes \sigma^{\vee,\infty} \otimes \mathcal{S}_{\mathrm{Weil}}(W^{n/2}_{\mathbf{A}^\infty_K}), \mathbf{Q}_p) \otimes H^1_f(K, V)^{\mathrm{mot}}$.
The space $\theta(\pi, \sigma) := \nearrow$ is 1D. Our cycle is $\Theta := \Theta(f^1, \varphi^1, \phi^1)$.

Theta dichotomy (correcting half-lies) [Harris-Kudla-Sweet, Gan-Ichino]:
given Π thus σ, there is a unique pair $(W, \pi \circlearrowleft \mathbf{H}_W(\mathbf{A}_F))$ such that
$\dim \theta(\pi, \sigma) = 1 > 0$. We pick this pair.

Euler systems according to Jetchev–Nekovář–Skinner

We'll construct an *Euler system* based on Θ, i.e. companion classes $(\Theta_m)_{m\in R}$ satisfying compatibility conditions related to the Euler product of $L(V,s)$.

Theorem* (JNS). – *Let* $\rho\colon \mathcal{G}_K \to \mathrm{Aut}_{\mathbf{Q}_p}(V)$ *be abs. irreducible, c-symplectic.*
Assume (ES): 1. there is $\gamma_1 \in \mathcal{G}_K$ *fixing* $K[1](\mathcal{O}_F^{\times,1/p^\infty})$*, such that* $\dim V^{\gamma_1} = 1$*.*
2. there is $\gamma_2 \in \mathcal{G}_K$ *fixing* $K[1](\mathcal{O}_K^{\times,1/p^\infty})$*, such that* $V^{\gamma_2} = 0$*.*
Let $S := \{$**split** *primes* $v = w\overline{w} \nmid p$ *of* F *such that* $V \circlearrowleft \mathcal{G}_{K_w}$ *is unramified*$\}$*.*
Let $R := \{$*squarefree products of primes in* $S\}$*.*
For $m \in R$*, let* $K[m] \subset K^{\mathrm{ab}}$ *with* $\mathrm{Gal}(K[m]/K) = \frac{\mathrm{Cl}(\mathcal{O}_F + m\mathcal{O}_K)}{\mathrm{Cl}(\mathcal{O}_F)} =: C[m]$*.*
Assume given a collection $z_m \in H^1_f(K[m], V_{\mathbf{Z}_p})$*,* $m \in R$*, satisfying:*

$$\mathrm{Tr}_{K[mv]/K[m]}\, z_{mv} = P_w(\mathrm{Fr}_w)\, z_m, \qquad P_w(T) := \det(1 - \mathrm{Fr}_w T | V)$$

for all $m \in R$*,* $v \in S$*,* $v \nmid m$*.*
Then

$$z := \mathrm{Tr}_{K[1]/K}\, z_1 \neq 0 \qquad \Longrightarrow \qquad H^1_f(K,V) = \mathbf{Q}_p \cdot z.$$

* statement may not be entirely correct, any errors mine.

Remark: there is enhanced notion of ES when V is P.-ordinary, with classes "z_{mp^s}",
$\Longrightarrow$ finer Iwasawa-theoretic consequences.

The Euler system of theta cycles

Theorem B. – *Let V be as in Theorem A. There is an Euler system* $(\Theta_m)_{m\in R}$ *with* $\mathrm{Tr}_{K[1]/K}\Theta_1 = \Theta$. *If V is P.-ordinary, there is a p-enhanced Euler system.*

Let's explain the construction. Recall that

$$\Theta := \Theta(f^1, \varphi^1, \phi^1) = f^1_* \mathrm{AJ}_p(\langle \varphi^1, \Theta(\phi^1)\rangle), \qquad \Theta(\phi^1) = \sum_{T(\underline{x})\geq 0} \phi^1(\underline{x}) \cdot Z(\underline{x})\, \mathbf{q}^{T(\underline{x})}.$$

First, $\Theta_1 \overset{\mathrm{Tr}_{K[1]/K}}{\longmapsto} \Theta$ will be a "connected component of the cycle Θ".
Let $\mathbf{T} = \mathbf{U}(1)$; for $C \subset \mathbf{T}(\mathbf{A}^\infty_F)$, we have $X_{\mathbf{T},C} \cong \mathrm{Spec}\, K_C$ (class field theory).
Fix a basepoint $y_C \in X_{\mathbf{T},C}(K_C) = \bullet \circ \bullet \ \bullet$. By Shimura and Deligne,
$\det\colon X_{\mathbf{H}',U'} \to X_{\mathbf{T},\det(U')}$ induces a bijection on π_0. Let $C(\underline{x}) := \det U(\underline{x})$, and let

$$Z(\underline{x})^C := \begin{cases} [\det^{-1}_{\mathbf{H}(\mathbf{x})}(y_C)] \subset [X_{\mathbf{H}(\underline{x}),U(\underline{x}),K_C}] & \text{if } C(\underline{x}) \overset{\subseteq}{=} C \\ |C(\underline{x})/C|^{-1} \cdot [\det^{-1}_{\mathbf{H}(\mathbf{x})}(y_{C(\underline{x})})] & \text{if } C(\underline{x}) \supsetneq C \end{cases} \quad \overset{\mathrm{Tr}_{K_C/K}}{\longmapsto} Z(\underline{x}).$$

Let $\Theta(\phi)^C := \sum_{T(\underline{x})\geq 0} \phi(\underline{x}) \cdot Z(\underline{x})^C\, \mathbf{q}^{T(\underline{x})}$, still modular if (Mod) holds.
Define

$$H^1_f(K[1], V_{\mathbf{Z}_p}) \ni \qquad \Theta_1 := f_{1,*}\mathrm{AJ}_p(\langle \varphi^1, \Theta(\phi^1)^{C[1]}\rangle) \qquad \overset{\mathrm{Tr}_{K[1]/K}}{\longmapsto} \Theta.$$

The Euler system of theta cycles / 2: construction of Θ_v

For $\chi \in \widehat{C[1]}$, $\kappa \in H^1_f(K[1], V)$, let $\kappa(\chi) := \sum_{\tau \in C[1]} \chi^{-1}(\tau)\kappa^{\tau} \in H^1_f(K, V(\chi))$.

Now, for $v \in S$, we look for $\Theta_v \in H^1_f(K[v], V_{\mathbf{Z}_p})$ satisfying

$$\mathrm{Tr}_{K[v]/K[1]}\Theta_v =: \ \Theta_v^{K[1]} \overset{\heartsuit}{=} P_w(\mathrm{Fr}_w)\,\Theta_1 \quad \overset{\forall \chi \in \widehat{C[1]}}{\Longleftrightarrow}$$

$$\Theta_v^{K[1]}(\chi) \overset{\heartsuit}{=} \ P_w(\mathrm{Fr}_w)\,\Theta_1(\chi) \overset{\mathrm{rec}}{=} L(V_w, \chi_w, 0)^{-1} \cdot \Theta_1(\chi).$$

For $\chi = 1$, want: $\Theta_v^{K[1]}(\mathbf{1}) := \Theta(f^1, \varphi^1, \phi^v) \overset{\heartsuit}{=} L(V_w, 0)^{-1} \cdot \Theta(f^1, \varphi^1, \phi^1)$.
Only change ϕ at v. By multiplicity 1 (cf. [YZZ, LSZ]), can replace Θ with any

$$0 \neq \theta_v \colon \pi_v \otimes \sigma_v^{\vee} \otimes \mathcal{S}_{\mathrm{Weil}}(W_v^{n/2}) \to \mathbf{C}$$

that is equivariant for $\mathbf{H}(F_v) \times \mathbf{G}(H_v) \cong \mathbf{GL}_n(F_v) \times \mathbf{GL}_n(F_v)$. Note: $\sigma_v \cong \pi_v$.
Fact: there is $\mathcal{F} \colon \mathcal{S}_{\mathrm{Weil}}(W_v^{n/2}) \to \mathcal{S}_{\mathrm{lin}}(M_n(F_v))$. So take $\theta_v = \zeta_{\mathrm{GJ}} \circ \mathcal{F}^{-1}$ for

$$\zeta_{\mathrm{GJ}}(f', \varphi', \phi') = \frac{1}{L(\pi_v, \frac{1}{2})} \int_{\mathbf{GL}_n(F_v)} (g.f', \varphi')\phi'(g)dg = \begin{cases} 1 & \text{if unramified} \\ L(\pi_v, \frac{1}{2})^{-1} & \phi' = 1_{\mathbf{GL}_n(\mathcal{O}_{F_v})} \end{cases}.$$

The Euler system of theta cycles / 3: conclusion

Summing up: for $\chi = \mathbf{1}$, let $\Theta_v^{K[1]}(\chi) = \Theta(f^1, \varphi^1, \phi^v)$ with $\phi_v^v := \mathcal{F}^{-1}(\mathbf{1}_{\mathbf{GL}_n(\mathcal{O}_{F_v})})$.

In fact, this ✓ $\forall \chi \in \widehat{C[1]} \rightsquigarrow \Theta_v^{K[1]} \in H^1_f(K[1], V_{\mathbf{Z}_p})$.

Is there $\Theta_v = \mathrm{Tr}^{-1}_{K[v]/K[1]}(\Theta_v^{K[1]}) \in H^1_f(K[v], V_{\mathbf{Z}_p})$?

Easy: just do as above, take $\Theta_v := f^1_* \mathrm{AJ}_p(\Theta(\varphi, \phi^v)^{C[v]})$.

Not so easy: we may lose integrality of Θ_v! (Recall $Z(\underline{x})^{C[v]}$ may have denominators.)
So we have to show that for every $\underline{x} \in \mathrm{Spt}(\phi^v)$:

- either $Z(\underline{x})$ has "Galois-level" $\preceq C[v]$ (i.e. $\det U(\underline{x}) \subset C[v]$),
- or $\phi^v(\underline{x}) \in |C[1]/C[v]| \cdot \mathbf{Z}$, killing denominators.

It works by explicit computation.

Thanks!