Currents on Lubin-Tate space

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Overview

1. Motivating intersection problem
2. δ-forms on $\mathbb{R}^n$
3. δ-forms on non-archimedean spaces
4. δ-forms and formal intersection theory
5. Computation on ∞-level LT-space

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Lubin-Tate space

- Fix $h \geq 1$, set $\mathbb{F} := \bar{\mathbb{F}}_p$ and $\breve{\mathbb{Z}}_p := W(\mathbb{F})$
- $\mathbb{X}/\mathbb{F}$ a formal group of height h and dimension 1
- Lubin-Tate space is its formal deformation space

$$\mathcal{M}(\operatorname{Spec} R) = \left\{ (X, \rho) \,\middle|\, \begin{matrix} X/R \text{ of dimension 1 and height } h \\ \rho : X \otimes_R R/p \dashrightarrow \mathbb{X} \otimes_{\mathbb{F}} R/p \text{ of height 0} \end{matrix} \right\}$$

- $\mathcal{M} \cong \operatorname{Spf} \breve{\mathbb{Z}}_p[[t_1, \ldots, t_{h-1}]]$
- $\operatorname{End}(\mathbb{X}) \cong \mathcal{O}_D$, where $D/\mathbb{Q}_p$ is the CDA of invariant $1/h$
- So $\mathcal{O}_D^\times$ acts as
$$\gamma(X, \rho) = (X, \gamma \circ \rho)$$

Quadratic CM-cycles

Arise from analogous definition for quadratic $E/\mathbb{Q}_p$

- h even and $E/\mathbb{Q}_p$ unramified quadratic, fix

$$\iota : \mathcal{O}_E \longrightarrow \mathrm{End}(\mathbb{X})$$

- Lubin-Tate space for E and $h/2$

$$\mathcal{N}(\mathrm{Spec}\, R) = \left\{ (X, \iota, \rho) \,\middle|\, \begin{array}{l} (X, \rho) \text{ as above, } \iota : \mathcal{O}_E \to \mathrm{End}(X) \\ \text{s.th. } \rho \text{ is } \mathcal{O}_E\text{-linear} \end{array} \right\}$$

- Forgetful map is a closed immersion

$$\mathcal{N} \hookrightarrow \mathcal{M}$$

and $\mathcal{N} \cong \mathrm{Spf}\, \breve{\mathbb{Z}}_p[[s_1, \ldots, s_{h/2-1}]]$.

Intersection number

Let $C := Cent_D(E)$. For $\gamma \in \mathcal{O}_C^\times \backslash \mathcal{O}_D^\times / \mathcal{O}_C^\times$ regular semi-simple, set

$$\mathrm{Int}(\gamma) := \mathrm{len}_{\breve{\mathbb{Z}}_p} \mathcal{O}_{\mathcal{N} \cap \gamma \mathcal{N}}.$$

Analytic expression for $\mathrm{Int}(\gamma)$

Theorem (Q. Li '18, M. '20)

There is a constant c such that for all γ,

$$\mathrm{Int}(\gamma) = c \cdot \int_{GL_h(\mathbb{Z}_p)} |\operatorname{Res}(\gamma, g)|^{-1} dg.$$

$\operatorname{Res}(\gamma, g)$ is the resultant of the invariant polynomials of γ and g.

Idea (Q. Li): Consider analogous intersection problems for higher level

- $\mathcal{M}_n$ moduli of (X, ρ, α), where α Drinfeld level-p^n-structure

$$\alpha : (\mathbb{Z}_p/p^n)^{\oplus h} \longrightarrow X[p^n]$$

- Analogously define $\mathcal{N}_n$. Choice of $\mathcal{O}_E^{h/2} \cong \mathbb{Z}_p^h$ yields

$$\mathcal{N}_n \hookrightarrow \mathcal{M}_n.$$

Analytic expression for $Int(\gamma)$

- $GL_h(\mathbb{Z}_p)$ acts on $\mathcal{M}_n$
- For $(\gamma, g) \in \mathcal{O}_D^\times \times GL_h(\mathbb{Z}_p)$, consider $\mathrm{len}_{\breve{\mathbb{Z}}_p} \mathcal{O}_{\mathcal{N}_n \cap (\gamma, g)\mathcal{N}_n}$
- Projection formula implies

$$\langle \mathcal{N}, \gamma\mathcal{N} \rangle = c \cdot \int_{GL_h(\mathbb{Z}_p)} \langle \mathcal{N}_n, (\gamma, g)\mathcal{N}_n \rangle dg.$$

- Key: $\langle \mathcal{N}_n, (\gamma, g)\mathcal{N}_n \rangle$ stabilizes for $n \to \infty$, limit equals $|\operatorname{Res}(\gamma, g)|^{-1}$

Today: Compute the limit in the generic fibers of $\mathcal{M}_n$ and $\mathcal{N}_n$

- Motivation/Hope is that methods from generic fiber can be applied to similar intersection problems of moduli spaces of p-divisible groups. There are usually no good formal models of these spaces with arbitrary level structure; LT-space is very special in this regard.

Overview

Smooth (p, q)-forms on $\mathbb{R}^n$

Definition (Lagerberg, Chambert-Loir – Ducros '12)

The sheaf of smooth (p, q)-forms $\mathcal{A}^{p,q}$ on $\mathbb{R}^n$ is

$$\mathcal{C}^\infty \otimes_{\mathbb{R}} \wedge^p(\mathbb{R}^n)^* \otimes_{\mathbb{R}} \wedge^q(\mathbb{R}^n)^* \quad (\cong \mathcal{A}^p \otimes_{\mathcal{C}^\infty} \mathcal{A}^q) .$$

- There are operators $\partial, \bar{\partial}$ and $\wedge$, relations as in complex analysis

$$\omega \wedge \omega' = (-1)^{(p+q)(p'+q')} \omega' \wedge \omega$$
$$\partial(\omega \wedge \omega') = \partial\omega \wedge \omega' + (-1)^{p+q} \omega \wedge \partial\omega'$$
$$\partial\bar{\partial} = -\bar{\partial}\partial.$$

- There is an integral operator $\int : \mathcal{A}^{n,n}_c(\mathbb{R}^n) \to \mathbb{R}$,

$$\int f \partial x_1 \wedge \bar{\partial} x_1 \wedge \ldots \wedge \partial x_n \wedge \bar{\partial} x_n := \int_{\mathbb{R}^n} f dx_1 \wedge \ldots \wedge dx_n.$$

- If $F : \mathbb{R}^n \to \mathbb{R}^n$ is affine linear, then

$$\int F^* \omega = |\det F| \int \omega.$$

Tropical cycles on $\mathbb{R}^n$

Definition (Fulton–Sturmfels '97)

A tropical cycle of codimension c on $\mathbb{R}^n$ is a pure c-codimensional, weighted rational polyhedral complex $(\mathcal{C}, \{m_\sigma\}_{\sigma \in \mathcal{C}^c})$ that is balanced.

- $\mathcal{C}$ is a finite set of rational simplices, stable under $\cap$ and taking faces
- $\mathcal{C}$ generated by its c-codimensional simplices
- $m_\sigma \in \mathbb{R}$ weights, σ maximal
- Let $\mathbb{L}_\sigma \subseteq \mathbb{Z}^n$ defined by affine linear space spanned by σ. Balanced means that for all $\tau \in \mathcal{C}^{c+1}$

$$\sum_{\tau \subset \sigma} m_\sigma n_{\sigma,\tau} = 0 \ \text{ in } \mathbb{Z}^n / \mathbb{L}_\tau .$$

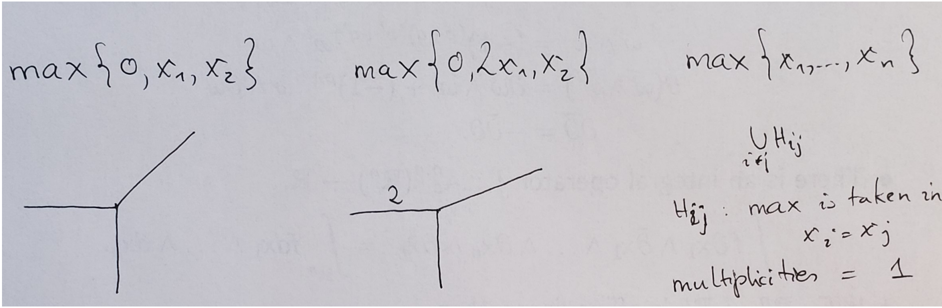
Here, $n_{\sigma,\tau} \in \mathbb{L}_\sigma / \mathbb{L}_\tau$ is a generator pointing from τ into σ.

Hypersurfaces

Any $\phi : \mathbb{R}^n \to \mathbb{R}$ that is continuous and p.w. linear with finite rational polyhedral complex of definition $\mathcal{D}$ defines a tropical hypersurface $\partial\bar{\partial}\phi := (\mathcal{D}^1, m_\tau)$ as follows:

$$m_\tau = \langle \nabla \phi_\sigma, \tilde{n}_{\sigma,\tau} \rangle - \langle \nabla \phi_{\sigma'}, \tilde{n}_{\sigma,\tau} \rangle$$

Here, $\tau = \sigma \cap \sigma'$ and $\tilde{n}_{\sigma,\tau}$ lifts $n_{\sigma,\tau}$.



Tropical Intersection Product

Let $TZ^c(\mathbb{R}^n)$ be the group of tropical cycles of codimension c, up to refinement.

Theorem (Fulton–Sturmfels '97)

There is an intersection product

$$TZ^c(\mathbb{R}^n) \times TZ^{c'}(\mathbb{R}^n) \longrightarrow TZ^{c+c'}(\mathbb{R}^n)$$

given by the fan displacement rule. It makes $TZ^{\cdot}$ into a commutative ring.

- Given $\mathcal{C}$ and $\mathcal{C}'$, refine them s.th. $\mathcal{C} \cup \mathcal{C}'$ is a polyhedral complex, choose v small generic
- Set $\mathcal{C} \cdot \mathcal{C}'$ as $(\mathcal{C} \cup \mathcal{C}')^{\geq c+c'}$ with multiplicities

$$m_\tau = \sum_{(\sigma,\sigma') \in \mathcal{C}^c \times \mathcal{C}'^{c'},\ \sigma \cap \sigma' = \tau,\ \sigma \cap (\sigma' + v) \neq \emptyset} m_\sigma m_{\sigma'} [\mathbb{Z}^n : \mathbb{L}_\sigma + \mathbb{L}_{\sigma'}].$$

- E.g. $(\partial\bar{\partial} \max\{x_1, \ldots, x_n\})^{n-1} = \{x_1 = \cdots = x_n\}$ with multiplicity 1

δ-forms on $\mathbb{R}^n$

- Sheaf of currents $\mathcal{D}^{\cdot,\cdot}(U) := \mathrm{Hom}_{cont}(\mathcal{A}_c^{n-\cdot,n-\cdot}(U), \mathbb{R})$
- Contains both $\mathcal{A}^{\cdot,\cdot}$ and $TZ^{\cdot}$ via

$$\omega \longmapsto [\omega] = \left[\eta \mapsto \int \omega \wedge \eta\right]$$

$$(\mathcal{C}, \{m_\sigma\}) \longmapsto \delta_{\mathcal{C}} = \left[\eta \mapsto \sum_{\sigma \in \mathcal{C}^c} \int_\sigma \eta|_\sigma\right]$$

- For $\omega \in \mathcal{A}^{\cdot,\cdot}$ and $\mathcal{C} \in TZ^{\cdot}$, may combine to $\omega \wedge \delta_{\mathcal{C}}$

Definition (Gubler–Künnemann '16)

The sheaf of δ-forms $\mathcal{B}^{\cdot,\cdot}$ is the subsheaf of $\mathcal{D}^{\cdot,\cdot}$ generated by all $\omega \wedge \delta_{\mathcal{C}}$. It is stable under $\partial, \bar\partial$ and there is a $\wedge$-product, extending $\wedge$ on $\mathcal{A}^{\cdot,\cdot}$ and the tropical intersection product on $TZ^{\cdot}$.

Extension given by $(\omega \wedge \delta_{\mathcal{C}}) \wedge (\omega' \wedge \delta_{\mathcal{C}'}) := (\omega \wedge \omega') \wedge \delta_{\mathcal{C}.\mathcal{C}'}$.

Overview

Tropicalization of varieties

- K non-archimedean field, $|\cdot| : K \to \mathbb{R}_{\geq 0}$ non-trivial, $v := -\log|\cdot|$
- Continuous surjective map $v : \mathbb{G}_{m,K}^{an,n} \longrightarrow \mathbb{R}^n$

Theorem (Bieri–Groves '84, ...)

Let $X \hookrightarrow \mathbb{G}_{m,K}^n$ be a closed subvariety, pure of dimension d. Then $v(X^{an}) \subset \mathbb{R}^n$ is naturally a d-dimensional tropical cycle.

- For X as above, let $\mathcal{C}$ be complex of definition
- For $\sigma \in \mathcal{C}_d$, choose affine linear $q : \mathbb{G}_m^n \longrightarrow \mathbb{G}_m^d$ such that $q|_\sigma$ is injective. Set $X^{an}(\sigma) := X^{an} \cap (v \circ q)^{-1}(\sigma)$ and

$$m_\sigma := [\mathbb{Z}^d : q(\mathbb{L}_\sigma)]^{-1} \cdot \deg(X^{an}(\sigma) \to \mathbb{G}_m^{an,d}(\sigma)).$$

δ-forms on varieties

X/K a variety

- Tropical Chart is a tuple (U, f, V), where $U \subset X$ is open, $f : U \to \mathbb{G}_m^n$ a map and $V \subseteq X^{an}$ of the form $(v \circ f^{an})^{-1}(\Omega)$ for some open $\Omega \subset \mathbb{R}^n$
- Morally, for $\omega, \omega' \in \mathcal{B}^{\cdot,\cdot}(\Omega)$, the pull backs to U^{an} agree if

$$\omega|_{(v \circ f^{an})(V)} = \omega'|_{(v \circ f^{an})(V)}$$

Definition (Gubler–Künnemann '16)

1) A presented δ-form on $W \subset X^{an}$ is given by tropical charts $\{(U_i, f_i, V_i)\}$ such that $W = \cup V_i$ together with δ-forms ω_i on $\mathbb{R}^{n_i}$ that agree on overlaps.
2) A δ-form is a presented δ-form up to local agreement.

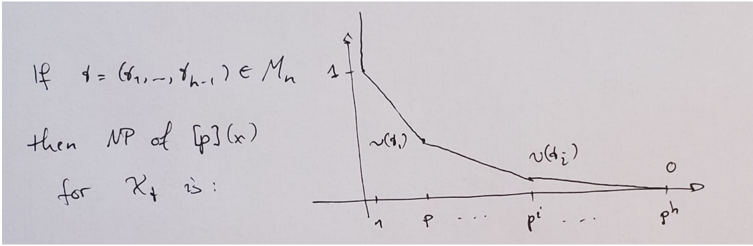
- δ-forms form a sheaf $\mathcal{B}^{\cdot,\cdot}$ on X^{an}
- Operators $\wedge, \partial, \bar{\partial}, \int$ are computed in charts

Tropicalization of Lubin-Tate space

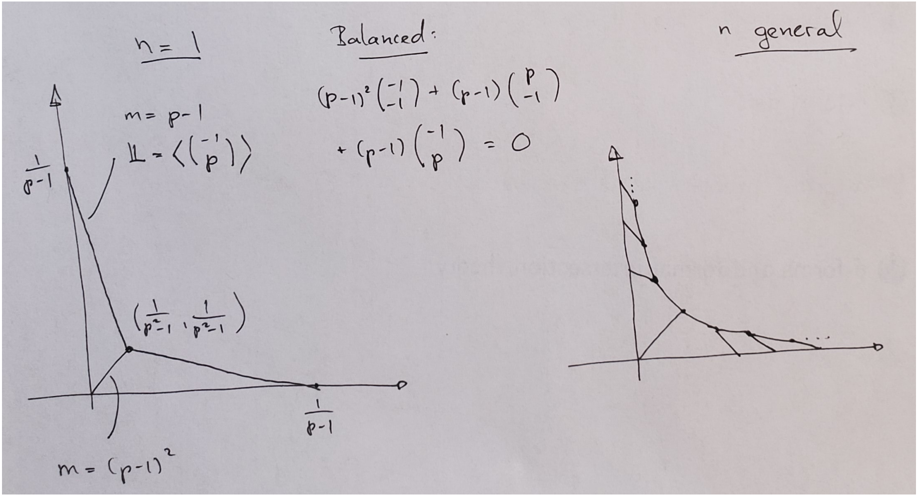
- Over $\mathcal{M}$, fix coordinate on universal formal group, $\mathcal{X} \cong \operatorname{Spf} \mathcal{O}_{\mathcal{M}}[[x]]$
- Level structure $\alpha : \mathcal{M}_n \longrightarrow \mathcal{X}^h$ then induces a map

$$M_n := \mathcal{M}_n^{an} \longrightarrow \mathbb{G}_{m,K}^{an,h}$$

- Then $\big(v \circ (\alpha_1, \ldots, \alpha_h)\big)(M_n) \subset (0, \infty)^n$ is naturally a tropical cycle
- Support determined by possible Newton polygons for $[p]$
- May choose $\mathcal{M} \cong \operatorname{Spf} \breve{\mathbb{Z}}_p[[t_1, \ldots, t_{h-1}]]$ such that



Example $h = 2$



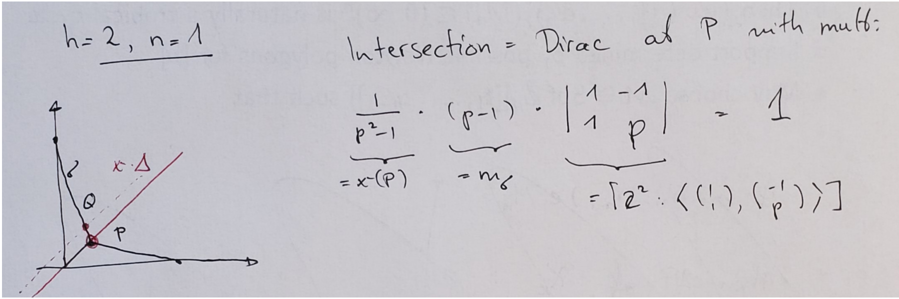
h = 1
Balanced:
n general
m = p-1
$\mathbb{1} = \langle \binom{-1}{p} \rangle$
$(p-1)^2\binom{-1}{-1} + (p-1)\binom{p}{-1} + (p-1)\binom{-1}{p} = 0$
$\frac{1}{p-1}$
$\left(\frac{1}{p^2-1}, \frac{1}{p^2-1}\right)$
$\frac{1}{p-1}$
$m = (p-1)^2$

Integral of a specific form

Proposition

Set $\phi := \min\{v(\alpha_1), \ldots, v(\alpha_h)\}$, which is a p.w. linear function on M_n. Then $\partial\bar{\partial}\phi$ is a $(1,1)$-δ-form and $\omega := \phi(\partial\bar{\partial}\phi)^{h-1}$ has compact support with

$$\int_{M_n} \omega = 1.$$



Overview

Analytic intersection numbers

X separated analytic space over K of dimension d

- Sheaf of currents on X is

$$\mathcal{D}^{\cdot,\cdot}(W) := \mathrm{Hom}_{cont}(\mathcal{B}_c^{d-\cdot,d-\cdot}(W), \mathbb{R})$$

- $Z \subset X$ Zariski closed, of codimension c defines a (c, c)-current

$$\delta_Z : \eta \longmapsto \int_Z \eta|_Z$$

Definition

A Green current for Z is an element $h \in \mathcal{D}^{c-1,c-1}(X)$ such that

$$\omega(Z, h) := \partial\bar{\partial}h + \delta_Z \in \mathcal{B}^{c,c}(X).$$

- For (Z_1, h_1) and (Z_2, h_2) such that $c_1 + c_2 = d + 1$ and $Z_1 \cap Z_2 = \emptyset$, we define

$$\langle (Z_1, h_1), (Z_2, h_2) \rangle := \int_{Z_2} h_1|_{Z_2} + \int_X \omega(Z_1, h_1) \wedge h_2.$$

Formal schemes define Currents

- Assume $\mathcal{O}_K$ noetherian, $\pi \in \mathcal{O}_K$ uniformizer, $v(\pi) = 1$
- $\mathfrak{X}/\operatorname{Spf}\mathcal{O}_K$ of dimension $(d+1)$, formally of finite type, flat, separated

Definition

For $\mathcal{Z} \subset \mathfrak{X}$ closed formal subscheme, $\mathcal{Z} = V(\mathcal{I})$, set

$$\psi_{\mathcal{Z}} : |\mathfrak{X}^{an}| \longrightarrow \mathbb{R} \cup \{\infty\}, \; x \longmapsto \min\{v(f(x)) \mid f \in \mathcal{I}_{\bar{x}}\}.$$

Theorem (M.)

Assume $\mathcal{Z}$ is of codimension c and a complete intersection or a local complete intersection and $\mathfrak{X}^{an}$ projective. Then $h_{\mathcal{Z}} := \psi_{\mathcal{Z}} \cdot (\partial\bar{\partial}\psi_{\mathcal{Z}})^{c-1}$ is a Green current for $\mathcal{Z}^{an}$.

Intersection numbers

Proposition (M.)

$\mathfrak{X}/\operatorname{Spf}\mathcal{O}_K$ as above, $\mathcal{Z}_1, \mathcal{Z}_2 \subset \mathfrak{X}$ local complete intersections of pure codimensions $c_1 + c_2 = d + 1$. Assume $\mathcal{Z}_1$ flat over $\mathcal{O}_K$ and $\mathcal{Z}_1 \cap \mathcal{Z}_2$ artinian. Then

$$\operatorname{len}_{\mathcal{O}_K} \mathcal{O}_{\mathcal{Z}_1 \cap \mathcal{Z}_2} = \int_{\mathcal{Z}_1^{an}} h_{\mathcal{Z}_2}|_{\mathcal{Z}_1^{an}}.$$

Lubin-Tate situation is a complete intersection:

- $h_n \in \mathcal{D}^{h/2-1,h/2-1}(M_n)$ Green current defined by $\mathcal{N}_n \subset \mathcal{M}_n$
- Then

$$\langle \mathcal{N}_n, (\gamma, g)\mathcal{N}_n \rangle = \int_{N_n} (\gamma, g)^* h_n|_{N_n}.$$

Overview

Green current at ∞ level

- $M_\infty := \varprojlim M_n$ as adic space. Then
$$\mathcal{B}_c(M_\infty) = \varinjlim \mathcal{B}_c(M_n) \quad \text{and} \quad \mathcal{D}(M_\infty) = \varprojlim \mathcal{D}(M_n).$$
- Integral not intrinsic, set $\int_{M_\infty} := \left([M_n : M]^{-1} \int_{M_n}\right) \in \mathcal{D}^{0,0}(M_\infty)$.
- Similarly $\int_{N_\infty}$, defining $\delta_{N_\infty} \in \mathcal{D}^{h/2,h/2}(M_\infty)$

$$\delta_{N_\infty}(\eta) \overset{n \gg 0}{=} [N_n : N]^{-1} \int_{N_n} \eta = -[N_n : N]^{-1} \int_{M_n} h_n \wedge \partial\bar{\partial}\eta$$
$$= -[M_n : M] \cdot [N_n : N]^{-1} \int_{M_\infty} h_n \wedge \partial\bar{\partial}\eta$$

Proposition

The limit $h_\infty := \lim_{n\to\infty} [M_n : M] \cdot [N_n : N]^{-1} h_n$ exists as a δ-form on $M_\infty \setminus N_\infty$ and defines Green current for N_∞.

Our aim is to compute $\int_{N_\infty} (\gamma, g)^* h_\infty$.

Scholze–Weinstein description

- $\tilde{\mathbb{X}} := \varprojlim \mathbb{X}$ universal cover, lifts over Spf $\breve{\mathbb{Z}}_p$:

$$\tilde{\mathbb{X}}(R) := \varprojlim X(R) \xrightarrow{\cong} \tilde{\mathbb{X}}(R/p).$$

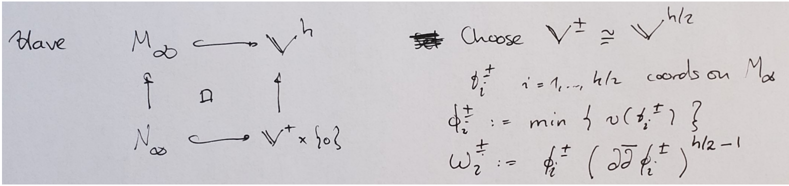
- $\mathbb{V} := \tilde{\mathbb{X}}_\eta^{ad}$ its generic fiber, isomorphic to $\{|t| < 1\} \subseteq \mathrm{Spa}\, \breve{\mathbb{Q}}_p\langle t^{1/p^\infty}\rangle$.

Theorem (Scholze–Weinstein '13)

There is a closed immersion $M_\infty \hookrightarrow \mathbb{V}^h$ given by

$$(X, \rho, (\alpha_n)_{n\geq 1}) \longmapsto (\rho(\alpha_n \bmod p^\epsilon))_n \in \tilde{\mathbb{X}}(\mathcal{O}_C/p^\varepsilon)^h.$$

- $\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathcal{O}_E$ acts on $\mathbb{V}^h$, decomposing it $\mathbb{V} \cong \mathbb{V}^+ \times \mathbb{V}^-$



Main Result

Theorem (M.)

For $(\gamma, g) \in \mathcal{O}_D^\times \times GL_h(\mathbb{Z}_p)$, γ regular semi-simple,

$$\int_{N_\infty} (\gamma, g)^* h_\infty^-|_{N_\infty} = \frac{[M_1 : M]}{[N_1 : N]^2} |\operatorname{Res}(\gamma, g)|^{-1}.$$

- First verify $h_\infty = [M_1 : M] \cdot [N_1 : N]^{-1} \omega_\infty^-$
- Key: $(\gamma, g)^* \phi_\infty^- = \min \left\{ v\left((g^+\gamma^- + g^-\gamma^+) \cdot (t_1^+, \ldots, t_{h/2}^+)\right) \right\}$ implies

  $$\int_{N_\infty} \omega_\infty^- = |\det(g^+\gamma^- + g^-\gamma^+)|^{-1} \int_{N_\infty} \omega_\infty^+.$$

  This uses tropical methods.
- Determinant is $|Res(\gamma, g)|^{-1}$, Integral is $[N_1 : N]^{-1}$ by our previous computation