Twisted Automorphic Descent and Gan-Gross-Prasad Conjecture

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Notation

- F is a number field and $\mathbb{A}$ is the ring of its adeles.
- G is a classical group defined over F:

$$G \in \{\mathrm{SO}(V_{\mathfrak{n}}), \mathrm{U}(V_{\mathfrak{n}}), \mathrm{Sp}_{2n}, \widetilde{\mathrm{Sp}}_{2n}\}.$$

- $\mathfrak{g} := Lie(G)$ is the Lie algebra of G.
- $\mathcal{N}(\mathfrak{g})$ is the nilpotent cone of $\mathfrak{g}(F)$ consisting of all nilpotent elements in $\mathfrak{g}$.
  The stable adjoint orbits in $\mathcal{N}(\mathfrak{g})$ are parameterized by the certain partitions of $\mathfrak{n}$ (or $2n$).

Unipotent subgroup and its character

Example

If $G = \mathrm{SO}(3,3)$, then the stable nilpotent orbits in $\mathcal{N}(\mathfrak{g})$ are parameterized by

$$\{[5,1],[3,3],[3,1,1,1],[2,2,1,1],[1^6]\}.$$

For each F-rational nilpotent orbit $\mathcal{O} \subset \mathcal{N}(\mathfrak{g})$, one can associate

a unipotent subgroup $N_{\mathcal{O}}$ and a non-degenerate character $\psi_{\mathcal{O}}$ of $N_{\mathcal{O}}(F)\backslash N_{\mathcal{O}}(\mathbb{A})$.

(Fix a non-trivial additive character $\psi\colon F\backslash\mathbb{A} \to \mathbb{C}^{\times}$.)

Fourier coefficients of automorphic forms

For an automorphic form φ of $G(\mathbb{A})$, **the Fourier coefficient** of φ associated to $\mathcal{O}$ is defined by

$$\mathcal{F}_{\mathcal{O}}(\varphi)(h) := \int_{N_{\mathcal{O}}(F)\backslash N_{\mathcal{O}}(\mathbb{A})} \varphi(nh)\psi_{\mathcal{O}}^{-1}(n)\,\mathrm{d}n.$$

Example

1. *If $\mathcal{O}$ is a regular nilpotent orbit (for the quasi-split groups), then $\mathcal{F}_{\mathcal{O}}(\varphi)$ is the Whittaker coefficient of φ.*
2. *Bessel-Fourier (Gelfand-Grev) coefficients for $\mathrm{SO}_{\mathfrak{n}}$ and $\mathrm{U}_{\mathfrak{n}}$:* $[2\ell+1, 1^{\mathfrak{n}-2\ell-1}]$
3. *Fourier-Jacobi coefficients for Sp_{2n}, $\widetilde{\mathrm{Sp}}_{2n}$ and $\mathrm{U}_{\mathfrak{n}}$:* $[2\ell, 1^{\mathfrak{n}-2\ell}]$.

Example: Ginzburg–Rallis model

Let $G = \mathrm{GL}_6$ and consider the unipotent subgroup associated to $[3,3]$:

$$[3^2] \to \begin{pmatrix} 0 & I_2 & 0 \\ 0 & 0 & I_2 \\ 0 & 0 & 0 \end{pmatrix}, \quad N_{\mathcal{O}} = \left\{ n = \begin{pmatrix} I_2 & A & B \\ 0 & I_2 & C \\ 0 & 0 & I_2 \end{pmatrix} : A, B, C \in M_{2\times 2} \right\}$$

and the non-degenerated character $\psi_{\mathcal{O}}(n) = \psi(\mathrm{tr}(A + C))$.

The corresponding Levi subgroup M of $N_{\mathcal{O}}$ is $\{\mathrm{diag}(a_1, a_2, a_3) : a_i \in \mathrm{GL}_2\}$.
The stabilizer $M_{\mathcal{O}}$ of M acting on $\psi_{\mathcal{O}}$ is $\{\mathrm{diag}(a, a, a)\} \cong \mathrm{GL}_2$.

- Ginzburg–Rallis model: $(\mathrm{GL}_6, \mathrm{GL}_2^{\triangle} \ltimes N_{[3^2]}, \psi_{\mathcal{O}})$.
- The associated period integral of automorphic forms is related to $L(\frac{1}{2}, \pi, \wedge^3)$.

Partition $[2\ell+1, 1^{\mathfrak{n}-2\ell-1}]$ of $\mathrm{SO}_{\mathfrak{n}}$

- $V_{\mathfrak{n}}$ is a quadratic space of dimension $\mathfrak{n}$, defined by

$$J_{\mathfrak{n}} = \begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & D & & & \\ & 1 & & & & \\ \iddots & & & & & \\ 1 & & & & & \end{pmatrix} \text{ and } D = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_{\mathfrak{r}} \end{pmatrix}.$$

- $G = \mathrm{SO}(V_{\mathfrak{n}})$ is a special orthogonal group.
- Take a partition of $\mathfrak{n}$:

$$p_{\underline{\ell}} = [2\ell+1, 1^{\mathfrak{n}-2\ell-1}] \rightsquigarrow \mathcal{O}_{p_{\underline{\ell}}} \text{ (a stable nilpotent orbit).}$$

- The associated nilpotent subgroup:

$$N_{p_{\underline{\ell}}} = \left\{ \begin{pmatrix} z & y & x \\ 0 & I_{\mathfrak{n}-2\ell} & y' \\ 0 & 0 & z^* \end{pmatrix} \in \mathrm{SO}(V_{\mathfrak{n}}) \colon z \in Z_\ell \right\},$$

where Z_ℓ is the standard maximal (upper-triangular) unipotent subgroup of GL_ℓ.

Rational orbits and Characters

The Levi subgroup normalizing $N_{\underline{p}_\ell}$ is

$$\underbrace{\mathrm{GL}_1 \times \cdots \times \mathrm{GL}_1}_{\ell} \times \mathrm{SO}(W),$$

where $W \subset V$ and $\dim(W) = \mathfrak{n} - 2\ell$.
Take an anisotropic vector $w_0 \in W$ and construct an F-rational nilpotent orbit $\mathcal{O}_{\ell, w_0}$ in $\mathcal{O}_{p_{\underline{\ell}}}$.

Definition $(N_{\underline{p}_\ell}, \psi_{\mathcal{O}_{\ell, w_0}})$

Define the character of $N_{\underline{p}_\ell}$ associated to $\mathcal{O}_{\ell, w_0}$

$$\psi_{\ell, w_0} \begin{pmatrix} z & y & x \\ & I_{\mathfrak{n}-2\ell} & y' \\ & & z^* \end{pmatrix} = \psi(\underbrace{z_{1,2} + \cdots + z_{\ell-1,\ell}}_{\textit{sub-diagonal of } z} + \underbrace{(w_0, y_\ell)}_{\textit{in } W}),$$

where y_ℓ is the bottom row of y and identified as a vector in W.

The stabilizer

This Levi subgroup M acts on ψ_{ℓ,w_0} via conjugation, where

$$M = \underbrace{\mathrm{GL}_1 \times \cdots \times \mathrm{GL}_1}_{\ell} \times \mathrm{SO}(W).$$

Denote by

$$G(\mathcal{O}_{\ell,w_0}) := \mathrm{SO}(W \cap w_0^{\perp}) \cong \mathrm{SO}_{\mathfrak{n}-2\ell-1}$$

the connected component of the stabilizer in $\mathrm{GL}_1^{\times \ell} \times \mathrm{SO}(W)$.

Automorphic Descent

For an automorphic form φ in π, the **Bessel-Fourier coefficients** (Gelfand-Graev) of φ are the Fourier coefficients associated to $(N_{\underline{p}_\ell}, \psi_{\ell,w_0})$, i.e.,

$$\mathcal{F}^{\psi_{\ell,w_0}}(\varphi)(h) := \int_{N_{\underline{p}_\ell}(F)\backslash N_{\underline{p}_\ell}(\mathbb{A})} \varphi(nh)\psi_{\ell,w_0}^{-1}(n)\,\mathrm{d}n.$$

Then $\mathcal{F}^{\psi_{\ell,w_0}}(\varphi)(h)$ is $G(\mathcal{O}_{\ell,w_0})(F)$-invariant.

Definition

Define **the ℓ-th automorphic descent** $\mathcal{F}^{\psi_{\ell,w_0}}(\pi)$ *of* π *to be the space generated by all* $\mathcal{F}^{\psi_{\ell,w_0}}(\varphi)$ *by varying* φ *in an automorphic representation* π *of* $G(\mathbb{A})$*, which is a* $G(\mathcal{O}_{\ell,w_0})(\mathbb{A})$*-module.*

Branching Problem

Remark

The nonvanishing of $\mathcal{F}^{\psi_{\ell,w_0}}(\varphi)$ *depends on the* $G(F)$*-adjoint orbit* $\mathcal{O}_{\ell,w_0}$.

Branching Problem: Study the automorphic descents in terms of Arthur's endoscopic classification of automorphic representations in the discrete spectrum?

$$\underbrace{H = G(\mathcal{O}_{\ell,w_0}) \subset G}_{\text{Branching}}$$

Tower property (Ginzburg-Rallis-Soudry)

$\pi \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{SO}_{\mathfrak{n}})$ $\qquad$ $\mathrm{SO}_{\mathfrak{n}-2\ell-1}$

$\mathcal{F}^{\psi_{\ell_0-1,w_0}}(\pi) \neq 0 \in \mathcal{A}(\mathrm{SO}_{\mathfrak{n}-2\ell_0+1})$

$\mathcal{F}^{\psi_{\ell_0,w_0}}(\pi) \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{SO}_{\mathfrak{n}-2\ell_0-1})$

$\mathcal{F}^{\psi_{\ell_0+1,w_0}}(\pi) \equiv 0$ on $\mathrm{SO}_{\mathfrak{n}-2\ell_0-3}$ $\forall w_0$

$\vdots$

0

ℓ_0 is called *the first occurrence index.*

Bessel Period

Let π and σ be irreducible automorphic representations of $G(\mathbb{A})$ and $G(\mathcal{O}_{\ell,w_0})(\mathbb{A})$, respectively, and one of them be cuspidal.

Definition (Bessel Period)

For $\varphi_\pi \in \pi$ and $\varphi_\sigma \in \sigma$, define the Bessel period to be

$$\mathcal{B}^{\mathcal{O}_{\ell,w_0}}(\varphi_\pi, \varphi_\sigma) := \int_{G(\mathcal{O}_{\ell,w_0})(F)\backslash G(\mathcal{O}_{\ell,w_0})(\mathbb{A})} \mathcal{F}^{\psi_{\ell,w_0}}(\varphi_\pi)(h)\varphi_\sigma(h)\,\mathrm{d}h.$$

Remark

In general, the Bessel periods can be regularized following by Jacquet-Lapid-Rogawski, Ichino-Yamana, Zydor, etc.

The spectrum of $\mathcal{F}^{\psi_{\ell_0,w_0}}(\pi)$

At the first occurrence index ℓ_0, one has

$$\mathcal{F}^{\psi_{\ell_0,w_0}}(\pi) = \sigma_1 \oplus \sigma_2 \oplus \cdots \oplus \sigma_r \oplus \cdots$$

which is a multiplicity free decomposition.

Conjecture (Generic Summand Conjecture (Jiang-Zhang))

Assume that $\pi \in \mathcal{A}_{\mathrm{cusp}}(G)$ is of a generic Arthur parameter.
If $\mathcal{F}^{\mathcal{O}_{\ell_0,w_0}}(\pi)$ is nonzero for some w_0 at the first occurrence index ℓ_0,
then there exists σ in $\mathcal{A}_{\mathrm{cusp}}(G(\mathcal{O}_{\ell_0,w_0}))$ such that σ is of a generic Arthur parameter and $\mathcal{B}^{\mathcal{O}_{\ell_0,w_0}}(\varphi_\pi, \varphi_\sigma)$ is nonzero.

Nonvanishing Bessel-Fourier coefficients

Application: Non-vanishing twist of central value of L-function.

Theorem (Jiang-Zhang)

For a generic $\pi \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{U}_{2,2})$, *there exists a Hecke character* χ *of* U_1 *such that*

$$L(\frac{1}{2}, \pi \times \chi) \neq 0.$$

Idea:

1. Ginzburg-Rallis-Soudry's Exchange Root Lemma: $\mathcal{F}^{\psi_{3,w_0}}(\pi) \neq 0$ for some w_0;
2. $\mathcal{B}^{\mathcal{O}_{3,w_0}}(\varphi_\pi, \chi) \neq 0$ for some χ;
3. $L(\frac{1}{2}, \pi \times \chi) \neq 0$ by one direction of global Gan-Gross-Prasad Conjecture.

Construction of automorphic representations

Goal: Given an Arthur parameter ϕ of $G(\mathbb{A})$, construct the spaces of automorphic representations in the Vogan packet $\Pi_\phi[G]$.
Construct a concrete module of $\pi \in \mathcal{A}_{\text{cusp}}(G)$:

$$
\begin{array}{ccc}
\Pi_\phi[G] & \longrightarrow & \mathcal{E}_{\tau\otimes\sigma_0} \\
\uparrow\vdots & \overset{\sigma_0}{\nearrow} & \downarrow \mathcal{F}^{\psi_{\ell,w_0}} \\
\pi & & \mathcal{F}^{\psi_{\ell,w_0}}(\mathcal{E}_{\tau\otimes\sigma_0}) \cong \pi
\end{array}
$$

We need to choose σ_0 such that $\mathcal{B}^{\mathcal{O}_{\ell,w_0}}(\varphi_\pi, \varphi_{\sigma_0}) \neq 0$.

Remark

1. *Ginzburg-Rallis-Soudry: Automorphic Descent, where σ_0 is on the trivial group.*
2. *Cai-Friedberg-Ginzburg-Kaplan: Doubling Construction.*
3. *Ginzburg-Soudry: Double Descent.*

Group H

- V': a quadratic space of dimension $2a+m$ such that
  1. m and $\mathfrak{n}$ have the different parity;
  2. the Witt index $\mathrm{Witt}(V') \geq a$.
- $H_{2a+m} = \mathrm{SO}(V')$.
- P_a: the parabolic subgroup H_{2a+m} of Levi subgroup $\mathrm{GL}_a \times H_m$;
- $\tau \in \mathcal{A}(\mathrm{GL}_a(\mathbb{A}))$ and $\tau = \tau_1 \boxplus \tau_2 \boxplus \cdots \boxplus \tau_r$, where $\tau_i \in \mathcal{A}_{\mathrm{cusp}}(\mathrm{GL}_{a_i})$, $\sum_{i=1}^r a_i = a$, and $\tau_i \not\cong \tau_j$ if $i \neq j$.
- σ: a cuspidal automorphic representation of $H_m(\mathbb{A})$.

Eisenstein series

Let $\phi_{\tau\otimes\sigma}$ be a section in $\mathrm{Ind}_{P_a(\mathbb{A})}^{H_{2a+m}(\mathbb{A})}\tau|\det|^s \otimes \sigma$ and form an Eisenstein series

$$E(\phi_{\tau\otimes\sigma}, s)(h) = \sum_{\gamma\in P_a(F)\backslash H_{2a+m}(F)} \phi_{\tau\otimes\sigma}(\gamma h).$$

Lemma

Let τ be as above and σ be of generic Arthur parameter. Then $E(\phi_{\tau\otimes\sigma}, s)$

1. *has a pole of order r at $s = \frac{1}{2}$ if and only if $L(s, \tau_i, \rho)$ has a pole at $s = 1$ for all i and $L(\frac{1}{2}, \tau \times \sigma) \neq 0$;*
2. *has a pole at $s = 1$ if and only if $L(s, \tau \otimes \sigma)$ has a pole at $s = 1$.*

- $\rho = \wedge^2$ if H_{2a+m} is an even orthogonal group;
- $\rho = \mathrm{sym}^2$ if H_{2a+m} is an odd orthogonal group.

Global zeta integral

We consider two cases:

1. $G = H_{2a+m}(\mathcal{O}_{\ell,w_0}) \subset H_{2a+m}$ where $\ell = a + \frac{m-\mathfrak{n}-1}{2}$;
2. $H_{2a+m} = G(\mathcal{O}_{\ell,w_0}) \subset G$ where $\ell = \frac{\mathfrak{n}-m-1}{2} - a$.

Assumption: π is cuspidal.

Definition

Define the **global zeta integral**

$$\mathcal{Z}(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0}) = \begin{cases} \mathcal{B}^{\mathcal{O}_{\ell,w_0}}(E(\phi_{\tau\otimes\sigma}, s), \varphi_\pi), & \text{if } G \subset H_{2a+m} \\ \mathcal{B}^{\mathcal{O}_{\ell,w_0}}(\varphi_\pi, E(\phi_{\tau\otimes\sigma}, s)), & \text{if } H_{2a+m} \subset G. \end{cases}$$

Key properties:

1. $\mathcal{Z}(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0})$ is eulerian;
2. If $\mathcal{B}^{\mathcal{O}_{\ell,w_0}}(\varphi_\pi, \varphi_\sigma) = 0$ ($\mathcal{B}^{\mathcal{O}_{\ell,w_0}}(\varphi_\sigma, \varphi_\pi) = 0$), then $\mathcal{Z}(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\ell,w_0}) = 0$.

Unramified calculation

Lemma (Jiang-Soudry-Zhang)

With the notation given as above, one has

$$\mathcal{Z}(s, \phi_{\tau\otimes\sigma}, \varphi_\pi, \psi_{\mathcal{O}_{\ell,w_0}}) = \prod_{\nu\in S} \mathcal{Z}_\nu(s,\cdot)\frac{L^S(s+\frac{1}{2}, \tau\times\pi)}{L^S(s+1,\tau\times\sigma)L^S(2s+1,\tau,\rho)}.$$

Theorem (One direction of GGP Conjecture for tempered Arthur Parameters)

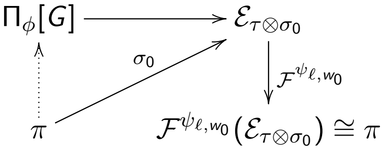
Let π and σ be cuspidal automorphic representations of $G(\mathbb{A})$ and $G(\mathcal{O}_{\ell,w_0},\mathbb{A})$, respectively. Assume that π and σ are of generic Arthur parameters.
If $\mathcal{B}^{\mathcal{O}_{\ell,w_0}}(\varphi_\pi,\varphi_\sigma)\neq 0$, then $L(\frac{1}{2},\pi\times\sigma)\neq 0$.

Some known cases of Gan-Gross-Prasad conjecture

Global Gan-Gross-Prasad conjecture:
Waldspurge, Ginzburg-Rallis-Soudry, Ginzburg-Rallis-Jiang, Wei Zhang, Ichino-Yamana, Gan-Ichino, Furusawa-Morimoto, Morimoto, Beuzart-Plessis-Chaudouard-Zydor, Beuzart-Plessis-Liu-W. Zhang-Zhu, etc.

Local Gan-Gross-Prasad conjecture:
Waldspurge, Moeglin-Waldspurge, Beuzart-Plessis, Gan-Ichino, Atobe-Gan, H. He, Kobayashi-Speh, Max Gurevich, Kei Yuen Chan, Hang Xue, Zhilun Luo, etc.

Twist Automorphic Descent



Theorem (Jiang-Zhang (even case))

Let τ and σ be as above. Assume that m is even.

- *If $\ell > \frac{m}{2} - 1$ (then $G = \mathrm{SO}_{\mathfrak{n}}$ with $\mathfrak{n} < 2a + 1$), then $\mathcal{F}^{\psi_{\ell,w_0}}(\mathcal{E}_{\tau\otimes\sigma}) = 0$.*
- *If $\ell = \frac{m}{2} - 1$, then $\mathcal{F}^{\psi_{\ell,w_0}}(\mathcal{E}_{\tau\otimes\sigma}) = \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_r \oplus \cdots$, where π_i is of the Arthur parameter ψ_τ.*
- *If $\mathcal{F}^{\psi_{\ell,w_0}}(\mathcal{E}_{\tau\otimes\sigma}) \neq 0$ at $\ell = \frac{m}{2} - 1$, then $\mathcal{F}^{\psi_{\ell,w_0}}(\mathcal{E}_{\tau\otimes\sigma}) = \pi$ for the unique member π in the Vogan packet $\Pi_{\psi_\tau}[\mathrm{SO}_{2a+1}]$.*

Remarks

In the above theorem,

- σ is of a generic Arthur parameter.
- ψ_τ is the Arthur parameter associated to τ.
- $\mathcal{E}_{\tau\otimes\sigma_0}$ is the residue of the Eisenstein series $E(\phi_{\tau\otimes\sigma}, s)$ at $s = \frac{1}{2}$.

Remark

- *Under the assumption of Generic Summand conjecture, any π of a generic Arthur parameter can be constructed via this way by choosing a suitable σ.*
- *If $G = \mathrm{SO}_{2a}$, the descent $\mathcal{F}^{\psi_{\ell,w_0}}(\mathcal{E}_{\tau\otimes\sigma_0})$ might not be irreducible.*
- *Jiang, Baiying Liu, Zhang: the Fourier-Jacobi cases.*

Non-tempered Gan-Gross-Prasad Conjecture

Assumption: $\pi \in \mathcal{A}_{\mathrm{cusp}}(G)$ is of **generic** Arthur parameter of form

$$\psi_\pi^A = (\eta_1, 1) \boxplus \cdots \boxplus (\eta_s, 1).$$

Write the global Arthur parameter of $\sigma \in \mathcal{A}_{\mathrm{cusp}}(G(\mathcal{O}_{\ell,w_0}))$ by

$$\psi_\sigma^A = (\zeta_1, 2b_1 + 1) \boxplus \cdots \boxplus (\zeta_l, 2b_l + 1) \boxplus (\xi_1, 2a_1) \boxplus \cdots \boxplus (\xi_k, 2a_k).$$

Proposition (Structure of ψ_σ^A (Jiang-Zhang 2019))

If the Bessel period $\mathcal{B}^{\mathcal{O}_{\ell,w_0}}(\varphi_\pi, \varphi_\sigma)$ is non-zero, then the Arthur-parameter ψ_σ^A of σ must be of the form:

$$\psi_\sigma^A = (\zeta_1, 1) \boxplus \cdots \boxplus (\zeta_l, 1) \boxplus (\xi_1, 2) \boxplus \cdots \boxplus (\xi_k, 2),$$

and $\{\xi_1, \xi_2, \ldots, \xi_k\}$ is a subset of $\{\eta_1, \eta_2, \ldots, \eta_s\}$.

A-parameter structure

Assumption: $\pi \in \mathcal{A}_{\mathrm{cusp}}(G = H(\mathcal{O}_{\ell,w_0}))$ is of **generic** Arthur parameter of form

$$\psi_\pi^A = (\zeta_1, 1) \boxplus \cdots \boxplus (\zeta_s, 1)$$

Write the global Arthur parameter of $\sigma \in \mathcal{A}_{\mathrm{cusp}}(H)$ by

$$\psi_\sigma^A = (\eta_1, 2a_1 + 1) \boxplus \cdots \boxplus (\eta_k, 2a_k + 1) \boxplus (\delta_1, 2b_1) \boxplus \cdots \boxplus (\delta_l, 2b_l).$$

Proposition (Structure of ψ_σ^A (Jiang-Zhang 2019))

If the Bessel period $\mathcal{B}^{\mathcal{O}_{\ell,w_0}}(\varphi_\sigma, \varphi_\pi)$ is non-zero, then the global Arthur parameter ψ_σ^A of σ must be of the form:

$$\psi_\sigma^A = (\eta_1, 1) \boxplus \cdots \boxplus (\eta_k, 1) \boxplus (\delta_1, 2) \boxplus \cdots \boxplus (\delta_l, 2)$$

and $\{\delta_1, \delta_2, \ldots, \delta_l\}$ is a subset of $\{\zeta_1, \zeta_2, \ldots, \zeta_s\}$.

One direction of non-tempered Gan-Gross-Prasad conjecture

Assumption: π or σ is of generic Arthur-parameter.

Theorem (Non-tempered GGP (Jiang-Zhang 2020))

Assume that $\prod_{1\leq i\leq \ell, 1\leq j\leq k} L(\frac{1}{2}, \zeta_i \times \xi_j) \neq 0$.
If the Bessel period $\mathcal{B}^{\mathcal{O}_{\ell,w_0}}(\varphi_\pi, \varphi_\sigma)$ *is non-zero, then*

1. $(\psi_\pi^A, \psi_\sigma^A)$ *is relevant;*
2. *the local multiplicities over all places are nonzero;*
3. $$L(s, \psi_\pi^L, \psi_\sigma^L) = \frac{L(s+\frac{1}{2}, (\psi_\pi^L)^\vee \times \psi_\sigma^L) L(s+\frac{1}{2}, \psi_\pi^L \times (\psi_\sigma^L)^\vee)}{L(s+1, \psi_\pi^L \times (\psi_\pi^L)^\vee) L(s+1, \psi_\sigma^L \times (\psi_\sigma^L)^\vee)}$$

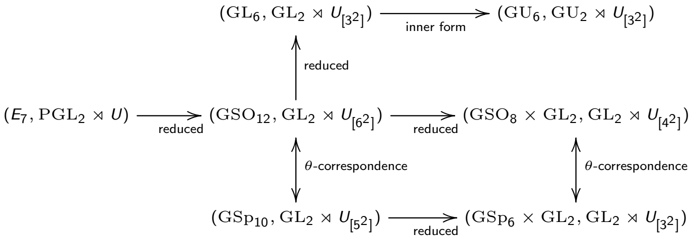
   is nonzero at $s = 0$.

Spherical Varieties $(G, M_{\mathcal{O}} \rtimes N_{\mathcal{O}}, \psi_{\mathcal{O}})$

What about the other models arisen from Fourier coefficients $(N_{\mathcal{O}}, \psi_{\mathcal{O}})$?

	G	$M_{\mathcal{O}} \rtimes N_{\mathcal{O}}$	ρ_X
1	$\mathrm{GL}_4 \times \mathrm{GL}_2$	$\mathrm{GL}_2 \times \mathrm{GL}_2$	$(\wedge^2 \otimes std_2) \oplus std_4 \oplus std_4^\vee$
2	$\mathrm{GU}_4 \times \mathrm{GU}_2$	$(\mathrm{GU}_2 \times \mathrm{GU}_2)^\circ$	$(\wedge^2 \otimes std_2) \oplus std_4 \oplus std_4^\vee$
3	$\mathrm{GSp}_6 \times \mathrm{GSp}_4$	$(\mathrm{GSp}_4 \times \mathrm{GSp}_2)^0$	$\mathrm{Spin}_7 \otimes \mathrm{Spin}_5$
4	GL_6	$\mathrm{GL}_2 \ltimes U_{[3^2]}$	$\wedge^3$
5	GU_6	$\mathrm{GU}_2 \ltimes U_{[3^2]}$	$\wedge^3$
6	GSp_{10}	$\mathrm{GL}_2 \ltimes U_{[5^2]}$	Spin_{11}
7	$\mathrm{GSp}_6 \times \mathrm{GL}_2$	$\mathrm{GL}_2 \ltimes U_{[3^2]}$	$\mathrm{Spin}_7 \otimes std_2$
8	$\mathrm{GSO}_8 \times \mathrm{GL}_2$	$\mathrm{GL}_2 \ltimes U_{[4^2]}$	$\mathrm{HSpin}_8 \otimes std_2$
9	GSO_{12}	$\mathrm{GL}_2 \ltimes U_{[6^2]}$	HSpin_{12}
10	E_7	$\mathrm{PGL}_2 \ltimes U$	ω_7

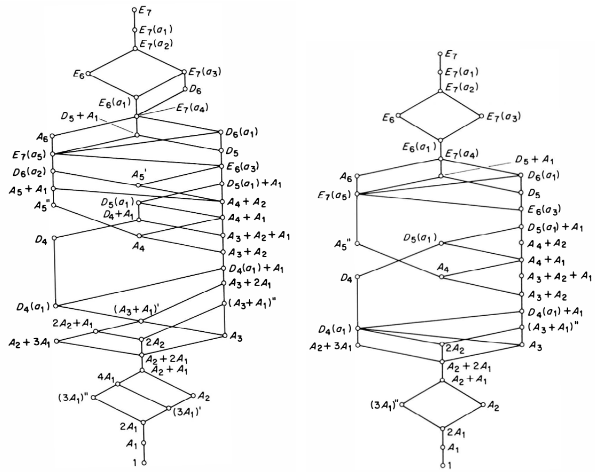
Relations among 10 models

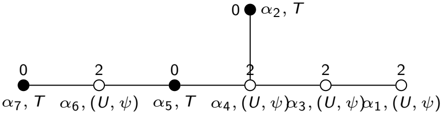


Remark

The θ-correspondences for degenerate Whittaker models are given by Gomez and Zhu.

E_7 Nilpotent orbit





Ichino-Ikeda type formula

Conjecture (Wan-Zhang)

Let G and H be in the above table, π be an irreducible cuspidal automorphic representation of generic A-parameter. Then

$$
\begin{aligned}
&|\int_{Z(\mathbb{A})H(F)\backslash H(\mathbb{A})} \phi(h)\,\mathrm{d}h|^2 \\
=&\frac{1}{|S_\phi|} \cdot \frac{C_{H/Z_{G,H}}}{\Delta_{H/Z_{G,H}}(1)} \cdot \lim_{s\to 1} \frac{\Delta_G(s)}{L(1,\pi,Ad)} \cdot L(\frac{1}{2},\pi,\rho_X) \cdot \Pi_{v\in S} I^{\sharp}_{H_v}(\phi_v).
\end{aligned}
$$

Remark.

1. Sakellaridis-Venkatesh Conjecture
2. Refined Gan-Gross-Prasad Conjecture: Ichino-Ikeda, N. Harris, Y. Liu, H. Xue, ect.

Thank You!