# Integrality of Regularized Petersson Inner Product

( joint with M. Schwagenscheidt )

## 1 Introduction

Given $f, g \in S_k(\overset{SL_2(\mathbb{Z})}{\Gamma})$, Petersson defined (1932)

$$(f, g) := \int_{\Gamma \backslash \mathbb{H}} f(\tau)\, \overline{g(\tau)}\, v^k \, d\mu(\tau), \quad \tau = u + iv$$

* Pos. def. Herm. form
* Equiv. w.r.t. Hecke operators (1937)
* Generalized to non-hol, higher levels, char (rep), other groups, half int wt ....

Arithmetic Info:

* $f$ eigenform $\underset{\text{Manin}}{\overset{\text{Shimura}}{\rightsquigarrow}}$ periods $\omega^{\pm}$

  $$(f, f) \sim_{\overline{\mathbb{Q}}^\times} \omega^+ \omega^-$$

* Hida: $\lambda \mid \dfrac{(f,f)}{\text{can. period}}$

  $\Rightarrow \exists\, g$ (smaller level) s.t. $f \equiv g \mod \lambda$.

Regularization: if $f$ or $g$ has pole @ cusps

$$(f, g)^{\mathrm{reg}} := \mathrm{CT}_{s=0} \lim_{T \to \infty} \int_{\mathcal{F}_T} f(\tau)\, \overline{g(\tau)}\, v^{k-s}\, d\mu(\tau)$$


Petersson 1954, Harvey-Moore 1996, Borcherds 1997

## Arithmetic Info

* (Bruinier-Ono-Rhoades, Ono 2008)

$$f = q^{-1} + O(q^2) \in M^!_{12}, \quad g = \Delta = \sum_{n \geq 1} a(n) \cdot q^n \in S_{12}$$

$$\frac{(f,g)^{reg}}{(g,g)} \notin \mathbb{Q} \implies a(n) \neq 0 \quad \forall n \in \mathbb{N} \quad (\text{Lehmer's conj})$$

* (Bruinier-Ono 2010) $G \in S_2(\Gamma_0(p))$ newform w/ $\mathbb{Q}$-FC s.t.

$\varepsilon(G) = -1 \implies L(G,1) = 0$ | $\Delta > 0$ fund. disc s.t.

$g \in S^+_{3/2}(\Gamma_0(4p)) \xrightarrow[\text{lift}]{\text{Shimura}} G$ | $\chi_\Delta(p) := \left(\frac{\Delta}{p}\right) = 1$

$\mathbb{Q}$-FC

$$f_\Delta = q^{-\Delta} + O(q) \in M^{!,+}_{3/2}(\Gamma_0(4p)) \quad \text{w/ } \mathbb{Q}\text{-FC}$$

Then $\displaystyle \frac{(f_\Delta, g)^{reg}}{(g,g)} \in \mathbb{Q} \iff L'(G, \chi_\Delta, 1) = 0$

## 2. Unary Theta Functions

$(L, Q)$ even, integral lattice

$L \subset L^* = \mathrm{Hom}(L, \mathbb{Z})$, $\quad A_L := L^*/L$, $\quad Q : A_L \to \mathbb{Q}/\mathbb{Z}$

$\mathbb{C}[A_L] \circlearrowleft Mp_2(\mathbb{Z})$ via Weil rep $\rho_L$

$\cup$

$\{e_h : h \in A_L\}$ basis

Example: $N \in \mathbb{N}$, $P_N := (\mathbb{Z}, Nx^2)$, $A_{P_N} = \mathbb{Z}/2N\mathbb{Z}$

For $h \in \mathbb{Z}/2N\mathbb{Z}$, $\nu = 0, 1$, define

$$\Theta^{(\nu)}_{N,h}(\tau) := \sum_{n \in 2N\mathbb{Z} + h} n^\nu q^{n^2/4N} \quad \text{hol, wt } \tfrac{1}{2} + \nu, \ \Gamma(4N)$$

Rmk: $\Theta^{(\nu)}_{N,-h}(\tau) = (-1)^{\nu}\, \Theta^{(\nu)}_{N,h}(\tau)$.

Example: ① $N=2,\ \nu=1$ : $\Theta^{(1)}_{2,0} \equiv 0 \equiv \Theta^{(1)}_{2,2}$

$$\Theta^{(1)}_{2,1}(\tau) = -\Theta^{(1)}_{2,3}(\tau) = q^{1/8} - 3q^{9/8} + 5q^{25/8} + \cdots = \eta(\tau)^3$$

$$\eta(\tau) := q^{1/24} \prod_{n\geq 1} (1-q^n).$$

② $N=6,\ \nu=0$ : $\Theta^{(0)}_{6,1}(\tau) - \Theta^{(0)}_{6,5}(\tau) = q^{1/24} - q^{25/24} - q^{49/24} + q^{121/24} + \cdots$

$= \eta(\tau) \in S_{1/2}(\chi)$, $\chi : Mp_2(\mathbb{Z}) \to \mathbb{C}^\times$

Rmk: Serre-Stark: $M_{1/2}$ spanned the $\Theta^{(0)}_{N,h}$'s.

Def: $\Theta^{(\nu)}_N(\tau) := \sum_{h(2N)} \Theta^{(\nu)}_{N,h}(\tau)\, e_h \in M_{\frac{1}{2}+\nu}(\rho_N)$ ← Weil rep for $P_N$

Rmk: More generally, given $L$ of signature $(p,q)$, and $W \subset L \otimes_{\mathbb{Q}} \mathbb{R}$ neg def of dim $= q$, define $\Theta_L(\tau, W)$ of wt $\frac{p+q}{2}$ w.r.t. $\rho_L$ by

$$\Theta_L(\tau, W) := v^{q/2} \sum_{\lambda \in L^*} \exp\left(2\pi i \left( Q(\lambda_{W^\perp})\tau + Q(\lambda_W)\bar{\tau} \right)\right) e_\lambda$$

We omit $W$ if $q=0$.

Given $f, g \in S_k(\rho_L)$, denote $\langle f, g\rangle := \sum_h f_h \overline{g_h}$

$f = \sum f_h e_h$, $g = \sum g_h e_h$

and the function $\langle f, g\rangle v^k$ is $\Gamma$-inv

$$\rightsquigarrow (f, g) := \int_{\Gamma\backslash\mathbb{H}} \langle f, g\rangle v^k \, d\mu(\tau)$$

Rmk: $\nu=0$ : $(\Theta_N, \Theta_N)^{reg} = \frac{\pi(N+1)}{3\sqrt{N}}$

$\nu=1$ : $(\Theta_N, \Theta_N) = \frac{\sqrt{N}(N-1)}{6}$

Rmk: $M^!_k(\rho)$ has basis w/ $\mathbb{Z}$-FC, where $\rho$ is any Weil rep. (McGraw 2003)

## 3. Result and Pf ideas

Known: (Zagier (Borcherds), Zwegers, Bringmann-Folsom-Ono, Bruinier-Schwagenscheidt)

If $f \in M^!_{\frac{1}{2}+\nu}(\rho_N)$ w/ $\mathbb{Z}$-FC, then

$$\frac{(f, \Theta_N)^{reg}}{(\Theta_N, \Theta_N)^{reg}} \in \mathbb{Q}$$

Thm (L-S, 2021) In the notation above

$$\frac{(f, \Theta_N)^{reg}}{(\Theta_N, \Theta_N)^{reg}} \text{ is in } \frac{1}{12N(N+1)}\mathbb{Z} \text{ if } \nu = 0$$
$$\text{in } \frac{1}{4N(N-1)}\mathbb{Z} \text{ if } \nu = 1$$

Pf idea: $\nu = 0$

$$(f, g)^{reg} = \int^{reg}_{\Gamma\backslash\mathbb{H}} \langle f, g \rangle v^k \, d\mu(\tau)$$

$$\langle f, g \rangle = \sum_h f_h \cdot \overline{g_h} = \sum_h f_h \cdot \eta^{-r} \cdot \eta^r \cdot \overline{g_h}$$

$$= \langle \underbrace{f \cdot \eta^{-r}}_{\in M^!}, \underbrace{\overline{\eta^r g}}_{\text{For } g = \Theta_N \text{, this is a } \Theta\text{-fun. for an indef latt}((1,r))} \rangle.$$

$\rightsquigarrow (f, \Theta_N)^{reg}$ = special value of theta lift [for $(1,r)$ latt]

$\overset{\text{Borcherds}}{\doteq}$ fin sum of rat. #'s.

F.E. along 0 dim'l cusp assoc. to an isotropic $\ell \in L$

$\square$

## 4. Consequence

Construct mock modular forms w/ $\mathbb{Z}$-FC.

<u>Def</u>: $f^+ = \sum_{m \in \mathbb{Q}} a(m) q^m$ is a mock mod form of wt $k$

if $\exists$ real analytic $f^*$ s.t.

(1) $f^+ + f^*$ is modular of wt $k$

(2) $v^{k-2}\, \overline{L_\tau(f^+ + f^*)} = v^{k-2}\, \overline{L_\tau f^*} =: \underset{\text{called the shadow of } f^+}{g}$ is hol.

<u>Example</u>: (1) $E_2(\tau) = 1 - 24 \sum_{m \geq 1} \sigma_1(m) q^m$, mmf wt 2

shadow $\doteq 1$.

(2) $\sum_{n \geq 0} \underset{\text{Hurwitz class \#'s}}{H(n)}\, q^n$ is mmf wt $3/2$ w/ shadow $\doteq \underset{\text{wt } 1/2}{\theta}$.

<u>Thm</u> (L-S, 2021) For $N \in \mathbb{N}$, $\nu = 0, 1$, $\exists$ $\Theta_N^{(\nu),+}(\tau)$ w/

shadow $\frac{1}{\sqrt{N}} \Theta_N^{(1)}(\tau)$ s.t. $48N\, \Theta_N^{(1),+}(\tau)$ has $\mathbb{Z}$-FC.

$\frac{\sqrt{N}}{\pi} \theta_N^{(0)}(\tau)$ $\qquad$ $72N\, \Theta_N^{(0),+}(\tau)$

<u>Pf</u>: Stokes' Thm $\underline{(f, \Theta_N^{(\nu)})^{reg}} = CT\left( \langle f, \Theta_N^{(\nu),+} \rangle \right)$

which also characterizes $\Theta_N^{(\nu),+}$ by Serre duality (see e.g. Borcherds). $\square$

<u>Some interesting identities:</u>

$$\sum_{n \geq 0} H(n) \cdot q^n = \frac{1}{24\eta(4\tau)} \sum_{\mathfrak{a} \subset \mathbb{Z}[\sqrt{6}]} \phi_6(\mathfrak{a})\, q^{Nm(\mathfrak{a})/6}$$

$$= \frac{1}{24\,\eta(4\tau)^3} \sum_{\mathfrak{a} \subset \mathbb{Z}[\sqrt{2}]} \phi_2(\mathfrak{a})\, q^{Nm(\mathfrak{a})/2}$$

$$\phi_6(\mathfrak{a}) := \begin{cases} \left(\dfrac{12}{tr(\lambda)/2}\right) Tr\left(\dfrac{|\lambda|}{2-\sqrt{6}}\right), & \text{if } \mathfrak{a} = (\lambda) \text{ w/ } 49-20\sqrt{6} < \lambda/\lambda' \leq 1, \\ 0, & o/w. \end{cases}$$

$$\phi_2(\mathfrak{a}) := \begin{cases} -\left(\dfrac{-4}{Tr(|\lambda|/2)}\right) Tr\left(\dfrac{\lambda^2\sqrt{2}}{3-\sqrt{2}}\right), & \text{if } \mathfrak{a} = (\lambda) \text{ w/ } 17-12\sqrt{2} < \lambda/\lambda' \leq 1 \\ 0, & o/w. \end{cases}$$