Towards a higher Siegel-Weil formula for unitary groups / function field.

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§1 Classical S-W formula. (unitary)

F'/F quadratic extn global fields

- (V, h) Herm. space dim n $/F'$

$G = U(V, h)$ $/F$

- $W = X \oplus Y$ 2n-dim'l Herm $/F'$

$\underline{X}$, $\underline{Y}$: max isotropic

$H = U(W)$ "$U(n,n)$".

$$G(\mathbb{A}) \times H(\mathbb{A}) \overset{\omega}{\circlearrowright} \mathcal{S}\big(\underbrace{(V \otimes_{F'} X)}(\mathbb{A})\big).$$

Weil repr.

$\Phi \in \mathcal{S}($ ——" —— $)$

Θ-function:

$$\Theta_\Phi(g,h) = \sum (\omega(g,h)\Phi)(x)$$
$$x \in (V \otimes_{F'} X)(F').$$

- $$\int_{[G]} \Theta_\Phi(g,h)\, dg$$
  $[G] = G(F)\backslash G(\mathbb{A})$

- $E(h, s, \Phi)$ — Eisenstein series on H induce from Siegel parab

  $h \in H(\mathbb{A})$.

  $P \subset H$, $P = \mathrm{Stab}(X)$, $\chi \cdot |\cdot|^{s+\frac{n}{2}}$

S-W formula:

$$\int_{[G]} \Theta_\Phi(g,h)\, dg = c \cdot E(h,0,\Phi).$$

counting "representations" of Herm matrix.

Kudla: Arithmetic version

Kudla-Rapoport.

arithmetic intersection number on unitary shimura variety $= E'(h,0,\Phi)$.

K-R conjecture on nonsingular Fourier coeff.

(inter of n

$\deg\left(\text{K-R divisors with "moment matrix } T\text{"}\right) = E'_T(h, 0, \Phi)$

T: $n \times n$ Herm.

Chao Li - Wei Zhang. proved this formula.

Function Fields :

Waldspurger formula	S-W formula
Gross-Zagier —	Arithmetic S-W.
(Function fields)	(Function fields)
Higher G-Z formula (Y.- Zhang).	Higher S-W

§2. Statement

$$\begin{array}{ccc} \sigma \circlearrowright X' & F' & \\ \downarrow \nu & | & \text{function fields } / k = \mathbb{F}_q \\ X & F & (q = \text{odd}) \end{array}$$

ν: étale double cover.

$[G] \rightsquigarrow \mathrm{Sht}^r_G$

$G = U(n)$

Hermitian vector bundle on X' :

$\mathcal{F}$ v.b. of rk n on X'

$h : \mathcal{F} \xrightarrow{\sim} \sigma^* \mathcal{F}^\vee$

$\mathcal{F}^\vee = \underline{\mathrm{Hom}}(\mathcal{F}, \omega_{X'})$

s.t. $\sigma^* h^\vee = h$.

$\rightsquigarrow$ Bun_G = moduli stack of $(\mathcal{F}, h)$.

Modifications

$$(\mathcal{F}_0, h_0) \overset{x'}{\dashrightarrow} (\mathcal{F}_1, h_1)$$

means: $x' \in X'$

$$\mathcal{F}_0 \big|_{X' \setminus \{x', \sigma x'\}} \xrightarrow[\sim]{\text{isometry}} \mathcal{F}_1 \big|_{X' \setminus \{x', \sigma x'\}}$$

at x', lowers by deg 1

at $\sigma x'$, raises by deg 1.

$$\begin{tikzcd} \mathcal{F}_0 & & \mathcal{F}_1 \\ & \mathcal{F}^\flat_{1/2} \arrow[ul, "\text{coker has length 1 at } x'"] \arrow[ur, "\underline{\sigma x'}"'] & \end{tikzcd}$$

A <u>Hermitian Shtuka</u> with legs $(x'_1, x'_2, \ldots, x'_r)$

$$(\mathcal{F}_0, h_0) \overset{x'_1}{\dashrightarrow} (\mathcal{F}_1, h_1) \overset{x'_2}{\dashrightarrow} (\mathcal{F}_2, h_2) \cdots \xrightarrow{x'_r} (\mathcal{F}_r, h_r)$$

$(\mathcal{F}_0, h_0)$ over $X' \times S$. $\qquad {}^\tau(\mathcal{F}_0, h_0)$

$$ {}^\tau(\mathcal{F}_0, h_0) = (id_{X'} \times Fr_S)^* (\mathcal{F}_0, h_0). $$

Sht_G^r = moduli stack of rk n Herm shtukas

$$\begin{tikzcd} \mathrm{Sht}_G^r \arrow[d, "\text{leg map}"] \\ (X')^r \end{tikzcd}$$

- leg map smooth of relative dim $= \underline{(n-1) \cdot r}$.
- Sht_G^r has dim $= nr$ smooth, loc. f.t., DM stack.

<u>Special cycles</u>

$\mathcal{E}$: v.b. on X' rk $= m$

$$ Z_{\mathcal{E}}^r = \left\{ \begin{tikzcd} & & \mathcal{E} \arrow[dll, "t_0"'] \arrow[dl, "t_1"] \arrow[d] \arrow[dr, "t_r"] \arrow[drr, "{}^\tau t_0"] & & \\ (\mathcal{F}_0, h_0) \arrow[r, dashed, "x_1'"] & \cdots \arrow[rr, dashed, "x_r'"] & & (\mathcal{F}_r, h_r) & \simeq {}^\tau(\mathcal{F}_0, h_0) \end{tikzcd} \right\} $$

$\mathcal{E}$ = line bundle. $\quad t_i \neq 0$

$(Z_{\mathcal{E}}^r)^*$ has dim $(n-1)r$

when $r=1$. this is a divisor.

analogue of K-R divisor on Sh.

(Note: $\mathrm{Sht}_G^r = \emptyset$ for r = odd).

$\mathcal{E} = L_1 \oplus L_2 \oplus \cdots \oplus \mathcal{L}_m$

$$Z_{\mathcal{E}}^r = \underbrace{Z_{\mathcal{L}_1}^r \times \cdots \times Z_{\mathcal{L}_m}^r}_{\mathrm{Sht}_G^r}$$

intersecting m cycles of codim r.

expected dim $= (n-m) \cdot r$.

$\mathrm{rk}\,\mathcal{E} = n$. expected dim $= 0$.

Nontrivial: define a 0-cycle $\in CH_0(\mathrm{Sht}_G^r)$ that is the "virtual fundamental cycle" of $Z_{\mathcal{E}}^r$.

$\mathcal{E}$ is not a direct sum of line bundle.

$\rightsquigarrow$ Can define the nonsingular part of the 0-cycle $[Z_{\mathcal{E}}^r]^{ns} \in CH_0(\mathrm{Sht}_G^r)$

$\mathcal{E}$: v.b. on X'. $\mathrm{rk} = n$.

$Z_{\mathcal{E}}^r$

$$\begin{tikzcd}
\mathcal{E} \arrow[d, "t_0"] \arrow[dddd, bend left=60, "a"] \\
(\mathcal{F}_0, h_0) \arrow[d, "s"] \\
\sigma^* \mathcal{F}_0^\vee \arrow[d, "\sigma^* t_0^\vee"] \\
\sigma^* \mathcal{E}^\vee
\end{tikzcd}$$

a induced Herm form on $\mathcal{E}$.

$a : \mathcal{E} \longrightarrow \sigma^* \mathcal{E}^\vee$ defined on $X'_{\mathbb{F}_q}$

discrete invariant.

$Z^r = \coprod Z_{\mathcal{E}}^r(a)$

$$Z_{\mathcal{E}} = \coprod_{\substack{a:\, \mathcal{E} \to \sigma^* \mathcal{E}^\vee \\ \sigma^* a^\vee = a}} Z_{\mathcal{E}}(a)$$

nonsingular part: a is injective.
(i.e. non-deg at generic pt of X')

$$[Z^r_{\mathcal{E}}(\underset{\substack{| \\ \text{nonsing}}}{a})] \in CH_0(Z^r_{\mathcal{E}}(a))$$

Fact: $Z^r_{\mathcal{E}}(a)$ is a proper scheme $/\mathbb{F}_q$

$$\deg [Z^r_{\mathcal{E}}(a)] \in \mathbb{Q}$$

$$E(h, s, \Phi) \qquad H = U(n,n).$$

Fourier expansion "at $\mathcal{E}$".

$$\mathcal{E} \in \underbrace{GL_n(F) \backslash GL_n(\mathbb{A}_{F'}) / GL_n(\hat{O}_{F'})}$$

$$\mathcal{E} \rightsquigarrow \underbrace{GL_n(\mathbb{A}_{F'})}.$$

$$\begin{bmatrix} \mathcal{E} & 0 \\ 0 & {}^t\mathcal{E}^{-1} \end{bmatrix} \in U(n,n)(\mathbb{A}).$$

Fourier exp. along $\begin{pmatrix} I_n & * \\ 0 & I_n \end{pmatrix}$ $n \times n$ Herm

Fourier coeff $\longleftrightarrow$ (rat'l) Hermitian forms on $\mathcal{E}$

$$\mathcal{E} \overset{a}{\dashrightarrow} \sigma^* \mathcal{E}^\vee \qquad \text{standard}$$

$\sigma^* a^\vee = a$.

$\boxed{E_{\mathcal{E}}(a,s)}$ = a^{th} coeff. of $E(h, s, \Phi_0)$ (expanded at $\mathcal{E}$)

rat'l function in q^s.

$\neq 0$ only when $a : \mathcal{E} \longrightarrow \sigma^* \mathcal{E}^\vee$.

Theorem (Feng - Y. - Zhang). $r \geq 0$.

$\mathcal{E}$: rk n on X'

a : $\mathcal{E} \longrightarrow \sigma^* \mathcal{E}^\vee$ nonsing.

normalizing factors added.

$$\deg \left[Z^r_{\mathcal{E}}(a) \right] = \frac{1}{(\log q)^r} \widetilde{E}^{(r)}_{\mathcal{E}}(a, 0).$$

Rk. $r = 0$. S-W.

$r =$ odd. both sides are 0.

LHS : $Sht^r_G = \emptyset$.

RHS. $\widetilde{E}(h, s, \Phi) = \widetilde{E}(, -s,)$

Idea of proof.

Elementary rep theory :

$W_d = (\mathbb{Z}/2)^d \rtimes S_d \hookleftarrow W_i \times W_{d-i}$

sgn

$$\overline{\mathrm{sgn}}_d : \quad W_d \longrightarrow S_d \longrightarrow \{\pm 1\}$$

$$\chi_d : \quad W_d \xrightarrow[\substack{\text{adding} \\ \mathbb{Z}/2\text{-coordinates.}}]{} \mathbb{Z}/2 \simeq \{\pm 1\}$$

$$R_d^{\mathrm{Int}}(t) = \bigoplus_{i=0}^{d} \mathrm{Ind}_{W_i \times W_{d-i}}^{W_d} (\chi_i \boxtimes 1)\, t^i$$

graded rep. of W_d.

$$R_d^{\mathrm{Eis}}(t) = \bigoplus_{0 \le j \le i \le d} (-1)^j \, \mathrm{Ind}_{S_{i-j} \times W_j \times W_{d-i}}^{W_d} (1 \boxtimes \overline{\mathrm{sgn}}_j \boxtimes 1)\, t^i$$

graded virtual rep of W_d.

<u>Exercise</u> $\quad R_d^{\mathrm{Int}}(t) = R_d^{\mathrm{Eis}}(t)$

$$\deg[Z_\varepsilon^r(a)] = E_\varepsilon^{(r)}(a)$$

$\mathrm{Herm}_{2d}(X'/X) =$ moduli stack of (Q, h)

Q: torsion coh. sh on X' of length $2d$.

h: $Q \xrightarrow{\sim} \sigma^* Q^\vee$

$\sigma^* h^\vee = h \qquad \underline{\mathrm{Ext}}^1(Q, \omega_{X'})$.

étale locally. it looks like

$$[\mathfrak{o}_{2d} / O_{2d}]$$

Springer theory on $\mathrm{Herm}_{2d}(X'/X)$

$$\begin{array}{ccc} \mathrm{Rep}(W_d) & \longrightarrow & \mathrm{Perv}(\mathrm{Herm}(X'/X)) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \rho & \longmapsto & \mathcal{F}_\rho. \end{array}$$

middle extn from any open dense subset.

$$R_d^{\mathrm{Int}}(t) = R_d^{\mathrm{Eis}}(t).$$

Herm-Springer th.

$$K_d^{\mathrm{Int}}(t) = K_d^{\mathrm{Eis}}(t)$$

(as virtual perv. sh on $\mathrm{Herm}_{2d}(X'/X)$).

$(\underline{\underline{\mathcal{E}, a}})$. $\rightsquigarrow$ $\mathrm{coker}(a)$ is a torsion sh. with Herm str.

$a: \mathcal{E} \to \sigma^* \mathcal{E}^\vee$. $\in \mathrm{Herm}_{2d}(X'/X)$.

both sides of higher S-W formula only depend on $Q = \underline{\underline{\mathrm{coker}(a)}}$.

$$\boxed{K_d^{\mathrm{Int}}(t)} = K_d^{\mathrm{Eis}}(t)$$

Lefschetz TF

$\mathrm{Tr}(\mathrm{Frob}_Q \circ C^r, -)$

$\mathrm{Tr}(\mathrm{Frob}_Q, -)$

$t = q^{-s}$

$\deg[\mathrm{Sht}^r_{\mathcal{M},\mathcal{E}}(a)]$

Hitchin -Shtukas

$\|$ change order of intersection

$\deg[\mathcal{Z}^r_{\mathcal{E}}(a)]$

$\boxed{\text{function in } q^{-s}}$

$\|$ Cho-Yamauchi

$\boxed{\mathrm{Den}(\mathcal{E}, a, q^{-2s})}$

$\|$

$\tilde{E}_{\mathcal{E}}(a,s)$.

$\tilde{E}^{(r)}(a,0)$

$\| \ \deg[\mathcal{Z}^r_{\mathcal{E}}(a)]$

$\tilde{E}^{(r)}(a,0) = \sum_{i=-d}^{d} i^r \cdot$