UNDERSTANDING BEILINSON'S CONJECTURES

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1. Statements

Suppose you have a motive M over $\mathbb{Q}$, whatever that means. Then you can "localize" into different places of $\mathbb{Q}$, so that you have

$$\begin{tikzcd} & \mathbb{Q}\text{-MHS} \\ \mathbb{Q}\text{-MM} \arrow[ur, "\mathbb{R}"] \arrow[dr, "\ell"'] & \\ & \ell\text{-adic } G_{\mathbb{Q}}\text{-rep} \end{tikzcd}$$

and these realization functors are all conjectured to be fully faithful (corresponding to Hodge and Tate conjectures). Now given a $\mathbb{Q}$-motive or an ℓ-adic Galois representation, you have a definition of L-functions. Those relevant in this discussion are all conjectured to have nice L-functions. So the slogan for Beilinson's conjecture is that the datum of L-function should be also read off from archimedean realization.

Conjecture 1.1. *Let $M = h^i(X)(n)$, which is of weight $w = i - 2n$. Then it has a meromorphic continuation, and can only possibly have a pole at $\frac{w}{2} + 1$. There is a function equation with center of symmetry $\frac{1+w}{2}$. If w is odd, $L(m, s)$ is entire.*

The critical strip is $\frac{w}{2} < \mathrm{Re}(s) < 1 + \frac{w}{2}$. We call $1 + \frac{w}{2}$ near central point. It is expected to have no zero inside the critical strip except at the line of symmetry (Riemann hypothesis).

By the virtue of functional equation we can assume $w < 0$, if we were interested in $L(M, 0)$. Before stating Beilinson's conjecture we need some preparations.

- We have the regulator
$$r : H^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(n))_{\mathbb{Z}} \longrightarrow H^{i+1}_{\mathrm{AH}}(X_{\mathbb{R}}, \mathbb{R}(n)).$$
- We have a geometric-to-absolute spectral sequence
$$\mathrm{Ext}^p_{\mathrm{MHS}_{\mathbb{R}}}(\mathbf{1}, H^q_B(X(\mathbb{C}), \mathbb{R}(n))) \Rightarrow H^{p+q}_{\mathrm{AH}}(X, \mathbb{R}(n)).$$

Because we expect there is no $\mathrm{Ext}^{\geq 2}$ ($2 = \mathrm{tr.deg}_{\mathbb{Q}}\,\mathbb{Q} + 1$), we have

$$0 \longrightarrow \mathrm{Ext}^1_{\mathrm{MHS}_{\mathbb{R}}}(\mathbf{1}, H^i_B(X(\mathbb{C}), \mathbb{R}(n))) \longrightarrow H^{i+1}_{\mathrm{AH}}(X, \mathbb{R}(n)) \longrightarrow \mathrm{Hom}_{\mathrm{MHS}_{\mathbb{R}}}(\mathbf{1}, H^{i+1}_B(X(\mathbb{C}), \mathbb{R}(n))) \longrightarrow 0.$$

- There is also a precise formula for $\mathrm{Ext}^i_{\mathrm{MHS}_{\mathbb{R}}}(\mathbf{1}, N)$ for an $\mathbb{R}$-MHS N,

$$\mathrm{Ext}^i_{\mathrm{MHS}_{\mathbb{R}}}(\mathbf{1}, N) = \begin{cases} W_0N^+ \cap F^0N_{\mathbb{C}}, & i = 0 \\ W_0N^+ \backslash W_0N^{\mathrm{dR}}/F^0(W_0N^{\mathrm{dR}}) & i = 1 \\ 0 & i \geq 2 \end{cases},$$

where $N^+ = N^{\phi_\infty}$ an $N^{\mathrm{dR}} = (N_{\mathbb{C}})^{\phi_\infty \otimes c}$, where ϕ_∞ is the induced map on cohomology by the complex conjugation on the space $X(\mathbb{C}) \longrightarrow X(\mathbb{C})$ (and Tate twist), and c is the complex conjugation on the coefficients.

In particular, because $H^{i+1}_B(X(\mathbb{C}), \mathbb{R}(n))$ has weight $w+1$ and $H^i_B(X(\mathbb{C}), \mathbb{R}(n))$ has weight w, at most one of the two of the ends of the above ses survives. In particular,

$$H^{i+1}_{\mathrm{AH}}(X, \mathbb{R}(n)) = \begin{cases} H^i_B(X(\mathbb{C}), \mathbb{R}(n))^+ \backslash H^i_B(X(\mathbb{C}), \mathbb{R}(n))^{\mathrm{dR}}/F^0(H^i_B(X(\mathbb{C}), \mathbb{R}(n))^{\mathrm{dR}}) & w \leq -2 \\ H^{i+1}_B(X(\mathbb{C}), \mathbb{R}(n))^+ \cap F^0(H^{i+1}_B(X(\mathbb{C}), \mathbb{C}(n))) & w = -1 \\ 0 & w \geq 0 \end{cases}$$

- Now the Betti-to-de Rham comparison says

$$c_{B,\mathrm{dR}} : M_B \otimes \mathbb{C} \xrightarrow{\sim} M_{\mathrm{dR}} \otimes \mathbb{C},$$

where $\phi_\infty \otimes c \leftrightarrow 1 \otimes c$. Thus

$$(M_B \otimes \mathbb{R})^{\mathrm{dR}} \xrightarrow{\sim} M_{\mathrm{dR}} \otimes \mathbb{R}.$$

Also

$$(\oplus_{p \geq k} H^{p,q}) \leftrightarrow (F^k M_{\mathrm{dR}}) \otimes \mathbb{C}.$$

Also, $\phi_\infty \otimes c$ preserves each $H^{p,q}$ in $M_B \otimes \mathbb{C}$.

- So the above description on absolute Hodge cohomology simplifies, in terms of motivic terms, to

$$H^{i+1}_{\mathrm{AH}}(X, \mathbb{R}(n)) = \begin{cases} (h^i(X)(n)^+_B \otimes \mathbb{R}) \backslash h^i(X)(n)_{\mathrm{dR}} \otimes \mathbb{R}/(F^0h^i(X)(n)_{\mathrm{dR}}) \otimes \mathbb{R} & w \leq -2 \\ h^{i+1}(X)(n)^+_B \cap (F^0h^{i+1}(X)_{\mathrm{dR}}) \otimes \mathbb{C} & w = -1 \\ 0 & w \geq 0 \end{cases}$$

Note that by weight reason in the case of $w \leq -2$ the denominators of the double quotient has no intersection.

- The case where $\mathrm{Ext}^1_{\mathrm{MHS}_{\mathbb{R}}}(\mathbf{1}, H^i_B(X(\mathbb{C}), \mathbb{R}(N))) = 0$ is called ***critical***. Note that if $w = -1$ then it is automatically critical by weight reason. If so, the natural $\mathbb{Q}$-structures on each term assigns a well-defined number in $\mathbb{R}^\times/\mathbb{Q}^\times$ associated with $\mathrm{Ext}^1_{\mathrm{MHS}_{\mathbb{R}}}(\mathbf{1}, H^i_B(X(\mathbb{C}), \mathbb{R}(N)))$, which we call ***Deligne's period*** $c^+(M)$. We call the relevant period map

$$\alpha_M : h^i(X)(n)^+_B \otimes \mathbb{R} \longrightarrow (h^i(X)(n)_{\mathrm{dR}}/F^0h^i(X)(n)_{\mathrm{dR}}) \otimes \mathbb{R}.$$

The negative part of the motive then is realized as

$$M(-1)^+_B \otimes \mathbb{R} = M^-_B \otimes \mathbb{R}(-1) \xrightarrow{\sim} M_{\mathrm{dR}} \otimes \mathbb{R}/c_{B,\mathrm{dR}}(M^+_B \otimes \mathbb{R}),$$

so that we have a map

$$\beta_M : \ker(\alpha_{M(-1)}) \longrightarrow \mathrm{coker}(\alpha_M).$$

This is injective if $w = -2$.

Now we can formulate Beilinson's conjecture.

Conjecture 1.2 (Beilinson's conjecture)**.** *We divide into cases.*

- *Suppose $w < -2$.*
  (1) *Then, the regulator map gives an isomorphism*
$$r_{\mathbb{R}} : H^{i+1}_{\mathcal{M}}(X, \mathbb{Q}(n))_{\mathbb{Z}} \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow H^{i+1}_{\mathrm{AH}}(X_{\mathbb{R}}, \mathbb{R}(n)).$$
  (2) *The LHS of $r_{\mathbb{R}}$ obviously has a $\mathbb{Q}$-structure, and the RHS of $r_{\mathbb{R}}$, from our earlier discussion, has a $\mathbb{Q}$-structure because it is a double quotient where each term has a $\mathbb{Q}$-structure. Thus* $\det r$ *makes sense as a well-define element of $\mathbb{R}^\times/\mathbb{Q}^\times$, and*
$$0 \neq L(M, 0) \equiv \det r (\mathrm{mod}\, \mathbb{Q}^\times).$$
- *Suppose $w = -2$.*
  (1) *Then, $s = 0$ is the near-critical point, and may be a pole of $L(M, s)$. The order of the pole should be*
$$\mathrm{ord}_{s=0} L(M, s) = -\dim_{\mathbb{Q}} N^{n-1},$$
  where $N^{n-1}(X) = \mathrm{CH}^{n-1}(X)/\mathrm{CH}^{n-1}(X)_0$ (Tate's conjecture).
  (2) *The cycle class map $cl : N^{n-1}(X) \longrightarrow h^{2n-2}(X)(n-1)_B$ lies actually in $\ker(\alpha_{h^{2n-2}(X)(n-1)})$, so we can send via β to $\mathrm{coker}(\alpha_{h^{2n-2}(X)(n)}) = H^{2n-1}_{\mathrm{AH}}(X, \mathbb{R}(n))$. Then this with the regulator induce an isomorphism*
$$(r \oplus cl) \otimes \mathbb{R} : H^{2n-1}_{\mathcal{M}}(X, \mathbb{Q}(n))_{\mathbb{Z}} \otimes \mathbb{R} \oplus N^{n-1}(X) \otimes \mathbb{R} \longrightarrow H^{2n-1}_{\mathrm{AH}}(X, \mathbb{R}(n)),$$
  and its determinant is the leading term of $L(M, s)$ at $s = 0$.
- *Suppose $w = -1$.*
  (1) *There is a nondegenerate height pairing*
$$h : \mathrm{CH}^n(X)_0 \otimes \mathrm{CH}^{\dim X + 1 - n}(X)_0 \longrightarrow \mathbb{R}.$$
  (2) $\mathrm{ord}_{s=0} L(M, s) = \dim_{\mathbb{Q}} \mathrm{CH}^n(X)_0$.
  (3) *The leading term of $L(M, s)$ at $s = 0$ is up to $\mathbb{Q}^\times$ the same as $c^+(M) \det(h)$.*

Some remarks.

- Any cycle in the part $\ker(AJ : \mathrm{CH}^i(X)_{\text{alg-0}} \longrightarrow J^i(X_{\mathbb{C}}))$ should lie in the kernel of the height pairing (believable), so Belinson-Bloch is saying that the ***Abel-Jacobi map is injective up to torsion, on the algebraically trivial Chow groups***. That the AJ map is injective up to torsion on the whole $\mathrm{CH}^i(X)_0$ is also expected, but Beilinson says "there is no definite reason for this conjecture."

2. Filtration on Chow group

So why not higher order invariants, above Abel-Jacobi? Turns out that there is such thing; namely there is a filtration on the Chow group
$$\mathrm{CH}^*(X) \supset F^1 \mathrm{CH}^*(X) \supset \cdots,$$
where $F^1 \mathrm{CH}^*(X) = \mathrm{CH}^*(X)_0$ and stuff. How? It is just the existence of regulator (to ***absolute cohomology***) and the geometric-to-absolute spectral sequence that degenerates. Namely

- so you start with a cycle class map
$$cl : \mathrm{CH}^j(X) \longrightarrow H^{2j}_{abs}(X, A(j)),$$
for whatever cohomology theory you use (either ℓ-adic cohomology or absolute Hodge cohomology).
- There is a degenerating geometric-to-absolute ss. and you can pullback the filtration coming from ss to the Chow group.

But WTF why is there no higher Exts in MHS category?

Conjecture 2.1 (Version 1 of Beilinson's motivic filtration conjecture)**.** *For X/k, there is a descending filtration $F^* \mathrm{CH}^*(X)_{\mathbb{Q}}$, where $F^0 \mathrm{CH}^j(X)_{\mathbb{Q}} = \mathrm{CH}^j(X)_{\mathbb{Q}}$, $F^1 \mathrm{CH}^j(X)_{\mathbb{Q}} = \mathrm{CH}^j(X)_{\mathrm{hom},\mathbb{Q}}$, and*
- *filtration is multiplicative,*
- *filtration is respected by pushforward/pullbacks,*
- $\mathrm{gr}^{\nu}_F \mathrm{CH}^j(X)_{\mathbb{Q}}$ *depends only on the motive modulo homological equivalence, $h^{2j-\nu}(X)$,*
- $F^{\nu} \mathrm{CH}^j(X)_{\mathbb{Q}} = 0$ *for $\nu \gg 0$.*

In fact one expects that $F^{j+1} \mathrm{CH}^j(X)_{\mathbb{Q}} = 0$.

Conjecture 2.2 (Version 2)**.** *There is an abelian category $\mathcal{MM}_k$ such that it contains $\mathcal{M}_k$ (category of Grothendieck motives up to homological equivalence) as a full subcategory, and*
$$\mathrm{gr}^{\nu}_F \mathrm{CH}^j(X)_{\mathbb{Q}} = \mathrm{Ext}^{\nu}_{\mathcal{MM}_k}(\mathbf{1}, h^{2j-\nu}(X)(j)).$$

But this is when k has large trascendence degree. So if k is a number field, still everything is valid and even on the motivic level we expect
$$0 \longrightarrow \mathrm{Ext}^1_{\mathcal{MM}_k}(1, h^{i-1}(X)(j)) \longrightarrow H^i_{\mathcal{M}}(X, \mathbb{Q}(j)) \longrightarrow \mathrm{Hom}_{\mathcal{M}_k}(1, h^i(X)(j)) \longrightarrow 0.$$

3. Coniveau filtration

- Deligne cohomology $H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i)) := \mathbb{H}^{2i}(X, \mathbb{Z}_{\mathcal{D}}(i))$ where
$$\mathbb{Z}_{\mathcal{D}}(i) := 0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \Omega^1 \longrightarrow \cdots \longrightarrow \Omega^{i-1} \longrightarrow 0.$$
So
$$0 \longrightarrow \tau_{\leq i-1}\Omega^*_X[-1] \longrightarrow \mathbb{Z}_{\mathcal{D}}(i) \longrightarrow \mathbb{Z} \longrightarrow 0,$$
so that
$$0 \longrightarrow J^{2i-1}(X) \longrightarrow H^{2i}_{\mathcal{D}}(X, \mathbb{Z}(i)) \longrightarrow \mathrm{Hdg}^{2i}(X, \mathbb{Z}) \longrightarrow 0.$$
- (geometric) Coniveau of $\alpha \in H^*_B(X, \mathbb{Q})$ is the smallest number c such that there is a closed algebraic subset $Y \subset X$ of codimension c such that $\alpha|_{X-Y} \in H^*_B(X - Y, \mathbb{Q})$ is zero.

  Theorem 3.1 (Deligne)**.** *If $\alpha \in H^*_B(X, \mathbb{Q})$ is zero as a class in $H^*_B(X - Y, \mathbb{Q})$ where Y is pure codimension c, then $\alpha = j_*\beta$, where*
  - *$j : \widetilde{Y} \longrightarrow X$ is a resolution of singularities of Y,*
  - *$\beta \in H^{*-2c}_B(\widetilde{Y}, \mathbb{Q})$.*

  This follows from theory of MHS.
- A wt k pure HS L is said to have Hodge coniveau c if L's Hodge decomposition takes form
$$L = L^{k-c,c} \oplus \cdots \oplus L^{c,k-c},$$
with $L^{k-c,c} \neq 0$. This is relevant because coniveau $\geq c$-part of $H^k_B(X, \mathbb{Q})$, denoted $H^k_C(X, \mathbb{Q})_c$, is a Hodge substructure of Hodge coniveau $\geq c$.

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Conjecture 3.1 (Generalized Hodge conjecture, Grothendieck). *If $L \subset H_B^k(X, \mathbb{Q})$ is a Hodge substructure of Hodge coniveau $\geq c$, then $L \subset H_B^k(X, \mathbb{Q})_c$.*

- A variety X is of geometric coniveau $\geq c$ if $H_B^*(X, \mathbb{Q}) = H_B^*(X, \mathbb{Q})_{\text{alg}} + H_B^*(X, \mathbb{Q})_c$. Under GHC, this is equivalent to $H^{p,q}(X) = 0$ for $p > c, p \neq q$.

  A variety is ***strongly of geometric coniveau $\geq c$*** if there is a decomposition of $\Delta_X \in H_B^{2n}(X \times X, \mathbb{Q})$ as
  $$\Delta_X = Z_1 + Z_2,$$
  where Z_1 is a decomposable cycle (i.e. rational linear combination of product cycles $W_i \times V_i$) and Z_2 is supported in $Y \times X$ for some closed algebraic subset $Y \subset X$ of codimension $\geq c$.

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Theorem 3.2. *Strongly of geometric coniveau $\geq c$ implies geometric coniveau $\geq c$. Under Hodge conjecture, the converse holds.*

Proof. If SGC $\geq c$, then for any $\alpha \in H_B^*(X, \mathbb{Q})$,
$$\alpha = [Z_1]^*\alpha + [Z_2]^*\alpha.$$
Now $[Z_2]^*\alpha$ vanishes away of Y, so it is of geometric coniveau $\geq c$. Also $[Z_1]^*\alpha$ is algebraic (if you act by $W \times V$ then the result is just a multiple of V).

For the converse, by HC we have
$$H_B^*(X, \mathbb{Q}) = H_B^*(X, \mathbb{Q})_{\text{alg}} + H_B^*(X, \mathbb{Q})_c = H_B^*(X, \mathbb{Q})_{\text{alg}} \oplus H_B^*(X, \mathbb{Q})_c^{\perp\text{alg}}.$$
This is because $H_B^*(X, \mathbb{Q})_{\text{alg}}$ is stable under the Lefschetz decomposition by HC. So Δ_X is a sum of class in $H_B^*(X, \mathbb{Q})_{\text{alg}} \otimes H_B^*(X, \mathbb{Q})_{\text{alg}}$ (decomposable) and $H_B^*(X, \mathbb{Q})_c \otimes H_B^*(X, \mathbb{Q})$ ..? And this second factor is Hodge class, and we know this comes from some desingularization so over that desingularization it comes from Hodge class which is also by HC algebraic. □

- Following Mumford's counterexample, Roitman proved that

Theorem 3.3. *If $X/\mathbb{C}$ smooth projective and $Y \subset X$ dimension $\leq r$ and $\mathrm{CH}_0(Y) \longrightarrow \mathrm{CH}_0(X)$ surjective, then $H^0(X, \Omega_X^k) = 0$ for any $k > r$.*

This is reproved by Bloch–Srinivas using the "decomposition of the diagonal" principle.

Theorem 3.4. *Under the above situation there is a decomposition $\Delta_X = Z_1 + Z_2$ in $\mathrm{CH}(X \times X)_{\mathbb{Q}}$ where Z_1 is a cycle supported in $X \times Y$ and Z_2 is a cycle supported in $D \times X$ for some proper $D \subset X$.*

which is furher generalized to

Theorem 3.5. *If $X/\mathbb{C}$ smooth projective, and if*
$$cl : \mathrm{CH}_i(X, \mathbb{Q}) \longrightarrow H^{2n-2i}(X, \mathbb{Q})$$
is injective for $i < c$ then there is a decomposition
$$\Delta_X = Z_1 + Z_2$$
in $\mathrm{CH}^n(X \times X)_{\mathbb{Q}}$ where Z_1 is decomposable and Z_2 is a cycle supported in $D \times X$ for some closed proper algebraic subset $D \subset X$ of codimension $\geq c$. Under this assumption, X is of geometric coniveau $\geq c$.

Conjecture 3.2. *The converse holds. Ie if X is of geometric coniveau $\geq c$, then the cycle class maps are injective. Equivalently,* $\mathrm{CH}_i(X)_{\mathrm{hom},\mathbb{Q}} = 0$ *for $i < c$.*

A weaker variant, asserting SGC $\geq c$ implies the conclusion, holds for a general complete intersection in a variety with trivial Chow groups (Voisin).

- Bloch-Beilinson conjecture on filtration on Chow groups

  implies

  Voevodsky's smash nilpotence conjecture (correspondence for $Z \in \mathrm{CH}^n(X \times X)_{\mathrm{hom},\mathbb{Q}}$ is nilpotent)

  implies

  The above conjecture on coniveau.

Definition 3.1. *Coniveau filtration is*

$$N^p H^i(X) = \sum_{\operatorname{codim} S \geq p} \ker(H^i(X) \longrightarrow H^i(X - S)).$$

Niveau filtration uses instead homology.