GEOMETRIC CONSTURCTION OF DISCRETE SERIES

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Discrete series representations are building blocks of all archimedean representation theory (depending on your viewpoint). But the very definition of discrete series is more or less that it is a building block... That these can be also explicitly parametrized and constructed is miraculous.

For the discrete series to even exist, we need $\operatorname{rank} G = \operatorname{rank} K$**, or equivalently** $Z_{\mathfrak{g}}(\mathfrak{c}) = \mathfrak{k}$ **(**$\mathfrak{c} = Z(\mathfrak{k})$**), i.e. there is a compact Cartan subgroup, and we assume this forever.**

1. Realization of holomorphic discrete series

This is probably exactly how people in arithmetic geometry think about discrete series. Namely you only consider the cases where the symmetric space G/K is Hermitian, so that there is nice notion of holomorphicity, and then you consider a holomorphic vector bundle over it coming from finite-dimensional representation of K ("weight" of a modular form). The space of smooth sections, L^2-sections, or sections under whatever norm condition, is then a realization of **holomorphic discrete series**. This is exactly how you would want to define classical modular forms.

1.1. $\mathrm{SU}(1,1)$ and more.

We know that holomorphic discrete series of $\mathrm{SU}(1,1)$ can be realized as some space of analytic functions on the upper half plane with certain transformation properties.

Another way of realizing this is found by HC, using the **complexification of** $\mathrm{SU}(1,1)$. Let $G = \mathrm{SU}(1,1) \subset \mathrm{SL}_2(\mathbb{C})$ and B be the lower standard Borel of $\mathrm{SL}_2(\mathbb{C})$. Then the elements of GB has a unique decomposition as $\overline{N}^{<1}TN$ where N is the unipotent radical of B, T is the diagonal and $\overline{N}^{<1} = \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \mid |z| < 1 \right\}$. Furthermore, $GB \subset \mathrm{SL}_2(\mathbb{C})$ is an open subset, and the complex structure is the same as the product complex structure one gets from the above decomposition (all three are complex analytic spaces).

Now the n-th discrete series is alternatively realized as

$$V_n := \{F : GB \to \mathbb{C} \text{ holomorphic} \mid |F|_{L^2} < \infty, F(xb) = \xi_n(b)^{-1}F(x)\}.$$

Here $\xi_n \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} = a^{-n}$, and G acts on V_n by $L(g)F(x) = F(g^{-1}x)$.

This kind of decomposition exists in general.

Theorem 1.1 (HC decomposition)**.** *Under the running assumption, $GB \subset G^{\mathbb{C}}$ is open, $P^+ \times K^{\mathbb{C}} \times P^- \to G^{\mathbb{C}}$ is one-to-one, holomorphic, regular with open image, and there is a bounded open subset $\Omega \subset P^+$ such that $GB = \Omega K^{\mathbb{C}} P^-$.*

Here the definitions are in accordance with the $\mathrm{SU}(1,1)$-case. For example, if $G = \mathrm{SU}(n,m)$, then the decomposition is

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix}.$$

Now we can mimic the definition for general G, except we need to define transformation factor ξ. For an analytically integral $\lambda \in (\mathfrak{h}^{\mathbb{C}})^*$, let ξ_λ be the corresponding holomorphic one dimensional representaiton of $T^{\mathbb{C}}$, and extend this trivially to B; this is the transformation factor. Let (L_λ, V_λ) be the representation constructed by this procedure.

Theorem 1.2. *If λ is dominant wrt Δ_K^+ then V_λ is a Hilbert space and L_λ is a unitary representation. It is a (nonzero) discrete series representation if furthermore $\langle \lambda + \delta, \alpha \rangle < 0$ for all positive noncompact roots α.*

λ here is rather the "Blattner parameter" because it is the (highest wt of the) lowest K-type (in contrast to HC parameter which is $\lambda + \delta$). You see that for $\mathrm{SU}(1,1)$ you only allow $n \geq 2$ because $\lambda = -ne_1$, $\delta = e_1$ and there is only one noncompact positive root, which is $2e_1$.

Proof sketch. You actually start from the purported lowest K-type and hope to generate all and nothing goes wrong in the process. More precisely, for the unitary representation of K associated to λ, consider the matrix coefficient map with respect to highest weight vector of norm one, extended to a map of GB using HC decomposition. Then this has right transformation property, so really the whole thing goes through if and only if this has finite L^2 norm, which is shown to be equivalent to the condition in the theorem via technical but not so difficult proof. $\square$

2. Realization of discrete series I: Unitary trick

This is due to Flensted-Jensen. This is what's in Knapp's book.

There are three ideas:

(1) **Analogue of Peter-Weyl theorem holds**: for π discrete series, $\widetilde{\pi} \otimes \pi$ occurs once in $L^2(G) \cong L^2((G \times G)/\Delta G)$. In this isomorphism $Z(\mathfrak{g}^{\mathbb{C}}) \cong D((G \times G)/\Delta G)$. So we can instead try to look for left-$(K \times K)$-finite functions on $(G \times G)/\Delta G$ that are eigenfunctions for all left-$(G \times G)$-invariant differential operators on $(G \times G)/\Delta G$.

(2) **Solving a dual problem**: for simplicity we denote $G \times G = G^2$ etc. Then G^2 can be seen as $\begin{pmatrix} G & 0 \\ 0 & G \end{pmatrix}$, and ΔG can be seen as the fixed point subset under the involution ι given by $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mapsto \begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}$. An idea is to replace the triple $(K^2, G^2, \Delta G)$ with some

another triple where the corresponding involution is the Cartan involution Θ. This is somewhat analogous to Weil's unitary trick. In the Lie algebra level, if we denote the Cartan involution as θ, then $\theta\iota$ has ± 1 eigenspaces

$$\begin{pmatrix} X & \\ & \theta X \end{pmatrix} : +1$$

$$\begin{pmatrix} X & \\ & -\theta X \end{pmatrix} : -1$$

So if we were to apply unitary trick then the "dual" should be $\begin{pmatrix} X & \\ & \theta X \end{pmatrix} + i \begin{pmatrix} Y & \\ & -\theta Y \end{pmatrix}$. This is basically the same as

$$\begin{pmatrix} Z & \\ & \theta Z \end{pmatrix}, Z \in \mathfrak{g}^{\mathbb{C}}.$$

We call this $\mathfrak{g}_{\mathbb{C}}$. Similarly, the dual of $\mathfrak{k}^2$ becomes

$$\mathfrak{k}_{\mathbb{C}} := \{\begin{pmatrix} Z & \\ & \theta Z \end{pmatrix} \mid Z \in \mathfrak{k}^{\mathbb{C}}\}.$$

The dual of $\Delta\mathfrak{g}$ becomes

$$\mathfrak{u} := \{\begin{pmatrix} X & \\ & X \end{pmatrix} \mid X \in \mathfrak{k} \oplus i\mathfrak{p}\},$$

so the dual of the triple on the group level should be $(K_{\mathbb{C}}, G_{\mathbb{C}}, U)$, where U is the set of fixed points of the Cartan involution of $G_{\mathbb{C}}$. Here note that the complexification is still written in superscript so $K_{\mathbb{C}}$, $G_{\mathbb{C}}$ are some different real reductive groups different from complexification.

Now we have the following decompositions:

$$G^2 = (K \times 1)(\text{antidiag } \exp \mathfrak{p}(\mathfrak{g}))(\operatorname{diag} G),$$
$$G_{\mathbb{C}} = (\exp \mathfrak{p}(\mathfrak{k}_{\mathbb{C}}))(\text{antidiag } \exp \mathfrak{p}(\mathfrak{g}))U.$$

The **upshot** here, strategy something like Weil's unitary trick, is that

$$\left\{\text{left } K^2\text{-finite fn on } G^2/\operatorname{diag} G, \text{ eig. of } D(G^2/\operatorname{diag} G)\right\} \leftrightarrow \{\text{left } K_{\mathbb{C}}\text{-finite fn on } G_{\mathbb{C}}/U, \text{ eig. of } D(G_{\mathbb{C}}/U)\}.$$

For example, given a left K^2-finite function on $G^2/\operatorname{diag} G$, we find the corresponding $K_{\mathbb{C}}$-finite function on $G_{\mathbb{C}}/U$ by

- Restrict to $\operatorname{antidiag} \exp \mathfrak{p}(\mathfrak{g})$, extend trivially to U
- Left action by K^2 by fin-dim-rep, extend holomorphically to action by $(K^2)^{\mathbb{C}}$, and restrict to $K_{\mathbb{C}}$, because $(K^2)^{\mathbb{C}} = (K_{\mathbb{C}})^{\mathbb{C}}$!

(3) **Generalized Poisson kernel**: The problem now became something like finding function on G/K that are eigenfunctions on $D(G/K)$ for $K \subset G$ maximal compact. Over the upper half plane, this includes finding harmonic functions, so classically we would use the Poisson kernel

$$P(x+iy, t) = \frac{y}{(x-t)^2 + y^2}.$$

Under $G/K \cong \mathbb{H}$ for $G = \mathrm{SL}_2(\mathbb{R})$, one readily calculates that the Poisson kernel is the same as $\exp(-2\rho H(a^{-1}\overline{n}^{-1}\overline{n}'))$, where $\overline{n}aK \leftrightarrow x+iy$, $\overline{n}' = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}$, and $\exp H(g)$ is the A-component of the $G = NAK$ decomposition. More generally, for $\nu \in (a_{\mathfrak{p}}^*)^{\mathbb{C}}$ and $x \in G$,

$$g \mapsto \exp(-\nu H(g^{-1}x)),$$

is an eigenfunction of $D(G/K)$.

Back to our setting, so we want to find good ν and h, a function on U, such that

$$\psi_{\mathbb{C}}(g_{\mathbb{C}}) := \int_U e^{-\nu H_{\mathbb{C}}(g_{\mathbb{C}}^{-1}u)} h(u)du,$$

transforms finite-dimensionally under left $K_{\mathbb{C}}$-action. As $K_{\mathbb{C}} \cap U = \operatorname{diag} K$, if we let $h(u)du$ to be the Haar measure on $\operatorname{diag} K$, i.e.

$$\psi_{\mathbb{C}}(g_{\mathbb{C}}) = \int_{\operatorname{diag} K} e^{-\nu H_{\mathbb{C}}(g_{\mathbb{C}}^{-1}u)} du,$$

then for $k_{\mathbb{C}} \in K_{\mathbb{C}}$,

$$\psi_{\mathbb{C}}(k_{\mathbb{C}} g_{\mathbb{C}}) = \int_{\operatorname{diag} K} e^{(\nu - 2\rho_{K_{\mathbb{C}}})H_{\mathbb{C}}(k_{\mathbb{C}}u)} e^{-\nu H_{\mathbb{C}}(g_{\mathbb{C}}^{-1}u)} du.$$

So the finite-dimensionality is achieved for a choice of ν if $\{\exp(\nu - 2\rho_{K_{\mathbb{C}}})H_{\mathbb{C}}(k_{\mathbb{C}}u) \mid k_{\mathbb{C}} \in K_{\mathbb{C}}\}$ is finite dimensional! This happens if ν is the highest restricted weight for an irreducible finite-dimensional representation of $K_{\mathbb{C}}$ having a nonzero $\operatorname{diag} K$-vector. This imposes some positivity conditions.

Integrality conditions are imposed when you consider transforming the function back to the original setting of $(K^2, G^2, \operatorname{diag} G)$. For example, for $G = \mathrm{SU}(1,1)$ again, an element of $K_{\mathbb{C}}$ can be written as $k_{\mathbb{C}} = \begin{pmatrix} e^{t+i\theta} & & & \\ & e^{-t-i\theta} & & \\ & & e^{-t+i\theta} & \\ & & & e^{t-i\theta} \end{pmatrix}$, and $\exp(\nu H_{\mathbb{C}}(k_{\mathbb{C}}))$ is of form e^{2rt} for some complex number r. Now that this function can be holomorphicall extended to $(K_{\mathbb{C}})^{\mathbb{C}}$ means sth like $r \in \mathbb{Z}$!

There are some other subtleties but this is the gist of the argument.

That this is everything is also extremely difficult to prove, but this is not the point of this.

3. Realization of discrete series II: Dirac operator and spin structure on G/K

To motivate the use of Dirac operators, we review realization of holomorphic discrete series using Dolbeault cohomology (Narasimhan–Okamoto [?]). Motivation for [?] is pretty obvious: BWB. I.e. you already know for complex groups htwtreps are all obtained as some cohomology group over a complex variety. For reference we state

Theorem 3.1 (Borel–Weil–Bott). *Let G be a semisimple complex algebraic group, and let λ be an integral weight of T. Let L_λ be the line bundle over G/B twisting the principal B-bundle $G \to G/B$ by λ by which we mean the character of B pulled back from $B/U = T$. Then, either $H^*(G/B, L_\lambda) = 0$, or $H^*(G/B, L_\lambda)$ is concentrated in one degree and the only nonvanishing cohomology, endowed with the natural G-action, is the contragredient of the htwtrep of G w/ htwt $w * \lambda$, where $w \in W$ is the unique $w \in W$ where $w * \lambda$ is dominant (and the nonvanishing cohomological degree is $\ell(w)$).*

Here note the dominant chamber is contained in exactly one of the "shifted Weil chambers" so there can be at most one such w.

So we imagine something like this can happen also for real group reps. Indeed for $\mathrm{SL}_2(\mathbb{R})$ this certainly is the case.

3.1. **Narasimhan–Okamoto.** Notation: $\mathfrak{g}, \mathfrak{g}_{\mathbb{C}}$

Suppose G/K has hermitian symmetric structure. Choose an ordering of roots of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ compatible with the complex structure on G/K. For τ_Λ an irreducible unitary rep of K of htwt Λ, let E_Λ be the holomorphic vector bundle on G/K associated to the contragredient representation. Let $H_2^{0,q}(E_\Lambda)$ be the Hilbert space of square-interable harmonic forms of type $(0,q)$ with coefficient in E_Λ. The G-rep $\pi_\Lambda^q = H_2^{0,q}(E_\Lambda)$ is unitary and it decomposes into finite number of irreps. One is then trying to say discrete series is realized by these constructions.

- First one shows π_Λ^q decomposes into finite sum of discrete series.
- One gives a character of alternating sum of π_Λ^q.
- One shows that π_Λ^q is concentrated in one degree.

Some steps:

(1) Preliminaries: let $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ where $\mathfrak{p}_+$ is identified with antiholomorphic tangent vectors at eK of G/K. We take P_n to be the set of positive noncompact roots (so that $\mathfrak{p}_+ = \sum_{\alpha \in P_n} \mathbb{C}X_\alpha$) and P_k be the set of positive compact roots.

Let $\mathcal{F}$ be the set of all integral forms on $\mathfrak{h}_{\mathbb{C}}$ (i.e. λ's where $2\langle \lambda, \alpha\rangle/\langle \alpha, \alpha\rangle$ is integer for every root α). Let $\mathcal{F}'$ be the subset where $\langle \lambda + \rho, \alpha\rangle \neq 0$ for all root α, and $\mathcal{F}_0'$ be the subset where $\langle \lambda+\rho, \alpha\rangle > 0$ for all compact positive root α.

E_Λ is a hermitian vb on G/K corresponding to the contragredient of V_Λ a K-rep. Let $(v_1, \cdots, v_r)$ be a ONB of V_Λ. Let $z_1, \cdots, z_n$ be ONB of $\mathfrak{g}_{\mathbb{C}}$ where from 1 to m its in $\mathfrak{p}_+$, from $m+1$ to $n-m$ it's in $\mathfrak{k}_{\mathbb{C}}$, from $n-m+1$ to n it's in $\mathfrak{p}_-$ and furthermore from $m+k+1$ to $m+k+l$ it's in $\mathfrak{h}^{\mathbb{C}}$.

Let $C^q(G, V_\Lambda^*) = \wedge^q \mathfrak{p}- \otimes C^\infty(G) \otimes V_\Lambda^*$ and $L_2^q(G, V_\Lambda^*)$ accordingly. We have an embedding $C^{0,q}(E_\Lambda) \hookrightarrow C^q(G, V_\Lambda^*)$ onto the 0-weight isotypic component (K acts trivially). Similarly $L_2^{0,q}(E_\Lambda)$ maps isometrically onto $L_2^q(G, V_\Lambda^*)(0)$. Let $\mathrm{Ad}_\pm^q$ be the K-rep on $\Lambda^q \mathfrak{p}_\pm$.

(2) Now you've connected Lie algebra world and geometry using this embedding η. The Laplacian, from geometry world, in Lie algebra language acts as an operator

$$\frac{1}{2}(\langle \Lambda + 2\rho, \Lambda\rangle - 1 \otimes \nu(C) \otimes 1)$$

where ν is the obvious action of $U(\mathfrak{g}_{\mathbb{C}})$ on $C^\infty(G)$, and C is the Casimir. This is called the **Okamoto-Ozeki formula**, similar to the Kuga's formula; philosophy is the same, namely Casimir and Laplacian are the same up to linear change.

(3) Discrete series is an infinitesimal equivalence class of irreducible unitary G-reps of a subrepresentation (closed invariant subspace) of $L^2(G)$. For a unitary rep π of G, let π_d, the discrete part, be the smallest closed invariant subspace which contains every irreducible closed invariant subspace of π.

DS is parametrized by $\mathcal{F}_0'$ as follows: for each $\lambda \in \mathcal{F}'$ let $\Theta_{\lambda+\rho}$ be the unique invariant eigendistribution corresponding to $\lambda + \rho$, which satisfies

$$\Delta(\exp H)\Theta_{\lambda+\rho}(\exp H) = \sum_{s \in W_G} \epsilon(s) e^{s(\lambda+\rho)(H)},$$

where $H \in \mathfrak{h}$ and $\Delta(\exp H) = \prod_{\alpha \in P}(e^{\alpha(H)/2} - e^{-\alpha(H)/2})$, just as in Weyl character formula. Then for each $\lambda \in \mathcal{F}_0'$, there is a unique DS $\omega_{\lambda+\rho}$ such that $\Theta_{\omega_{\lambda+\rho}} = (-1)^m \epsilon(\lambda + \rho)\Theta_{\lambda+\rho}$ where $m = -\frac{1}{2}\dim G/K$ and $\epsilon(\lambda+\rho) = \mathrm{sgn}(\prod_{\alpha\in P}\langle \lambda+\rho, \alpha\rangle)$.

(4) Let $\widetilde{\pi}_\Lambda^q$ be the representation $L_2^{0,q}(E_\Lambda)$. Then, the discrete part of $\widetilde{\pi}_\Lambda^q$ is infinitesimally equivalent to a sum

$$\bigoplus_{\omega \text{ DS}} m_{\tau_\Lambda^q}(\omega)\omega^*,$$

where m means the multiplicity of K-type, and $\tau_\Lambda^q = \mathrm{Ad}_+^q \otimes \tau_\Lambda$. We know the sum is finite because a K-type occurs in only finitely many DS (Harish-Chandra).

(5) Let $L_2(G)_\omega$, for DS ω, be the smallest closed subspace of $L_2(G)$ containing all matrix coefficients of ω. Let $C_\omega(G) = L_2(G)_\omega \cap C(G)$. Then for any ω' DS, $f \in C_{\omega'}(G)$, $\Theta_\omega(f) = 0$ if $\omega' \neq \omega^*$, and is $d(\omega)^{-1} f(1)$ if $\omega' = \omega^*$, where $d(\omega)$ is the formal degree.

(6) Now the key part is the following. For a K-finite function φ, let $T_\varphi^q = \int_G \varphi(g) T_g^q dg$ be the operator on $L_2^q(G, V_\Lambda^*)$ where T_g^q is the left-action of g on $L_2(G)$. Consider $\widetilde{K}_\varphi^q$ which involves first projecting down to discrete-part, then to weight 0 part (K-invariant part) and then composing with T_φ^q. This is an operator of finite rank, and is an integral transform with $\mathrm{End}(\wedge^q \mathfrak{p}_- \otimes V_\Lambda^*)$-valued C^∞ kernel function K_φ^q given by

$$K_\varphi^q(x,y) = \int_K \varphi(xky^{-1}) \, \mathrm{Ad}_-^q(k) \otimes \tau_\Lambda^*(k) dk.$$

Using this,

$$\sum_{q=0}^m (-1)^q \operatorname{tr} \widetilde{K}_\varphi^q = \sum_{q=0}^m (-1)^q \int_G dx \int_K \operatorname{tr} \mathrm{Ad}_-^q(k) \varphi(xkx^{-1}) \overline{\chi_\Lambda}(k) dk.$$

Because $\sum_{q=0}^m (-1)^q \operatorname{tr} \mathrm{Ad}_-^q(k) = \det(1 - \mathrm{Ad}_-^1(k))$, the above alternating sum becomes

$$\int_G dx \int_K \det(1 - \mathrm{Ad}_-^1(k)) \varphi(xkx^{-1}) \overline{\chi}_\Lambda(k) dk.$$

If you apply Weyl's integral formula and Weyl character formula this simplifies into $(-1)^{q_\Lambda} \Theta_{\omega_{\Lambda+\rho}^*}(\varphi)$, where q_Λ is the number of noncompact positive roots α such that $\langle \Lambda + \rho, \alpha \rangle > 0$. Now the original character formula we want follows from Hodge-theoretic style consequence that cohomology is harmonic form.. using Okamoto-Ozeki...

(7) Now using very soft analysis one proves that the vanishing when $q \neq q_\Lambda$ and $|\langle \Lambda + \rho, \alpha \rangle| > c_\Lambda^q$ for all $\alpha \in P_n$, where
   - $Q_\Lambda = \{\alpha \in P_n \mid \langle \Lambda + \rho, \alpha \rangle > 0\}$
   - $\Gamma_q = \{\sum_{\alpha \in Q} \alpha \mid Q \subset P_n, |Q| = q\}$
   - $\gamma_\Lambda = \sum_{\alpha \in Q_\Lambda} \alpha$
   - $c_\Lambda^q = \frac{1}{2} \max(\langle 2w\gamma_\Lambda - \rho - \gamma, \rho - \gamma \rangle + \langle \rho, \rho \rangle \mid w \in W_G, \gamma \in \Gamma_q)$.

3.2. **Dirac cohomology.** Notation: $\mathfrak{g}_0, \mathfrak{g}$

Basic idea is that things appearing in real group reps (Laplacian, $\mathfrak{g}/\mathfrak{k}$) are squares. Suppose $\mathfrak{g}_0$ has a nondegenerate symmetric bilinear form B extending Killing form on the semisimple part, and suppose B is positive definite on $\mathfrak{p}_0$ and negative definite on $\mathfrak{k}_0$. Then natural actions on $\mathfrak{p}_0$ are form-preserving, i.e. acts as an element of $\mathfrak{so}(\mathfrak{p})$. Then this admits a double cover by a spin group...! This is the basic idea behind everything.

In particular, adjoint $\mathfrak{k}$-action, which is a Lie algebra morphism $\mathfrak{k} \to \mathfrak{so}(\mathfrak{p})$, embeds further into $\mathrm{Cl}(\mathfrak{p})$, the Clifford algebra of $\mathfrak{p}$ with respect to B. Then...

- can consider $\widetilde{K}$, the double cover of K by pulling back using $\mathrm{Spin}(\mathfrak{p}_0) \to \mathrm{SO}(\mathfrak{p}_0)$

- also $\mathcal{A} = U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})$, associative $\mathbb{Z}/2\mathbb{Z}$-graded superalgebra (grading comes from Clifford side),
- then can consider $(\mathcal{A}, \widetilde{K})$-modules, which is sort of extension of $(\mathfrak{g}, K)$-modules. Indeed $\widetilde{K}$-action and $\mathcal{A}$-action can be compared because $\mathfrak{k}$-action is connected to both $\mathrm{Cl}(\mathfrak{p})$-action and $U(\mathfrak{g})$-action.

Miracle: $(\mathfrak{g}, K)$-**modules and** $(\mathcal{A}, \widetilde{K})$-**modules are the same thing!!!!** In this regard there is a hidden extra "square-root symmetry". The equivalence is given by $M \mapsto M \otimes S$, for a $(\mathfrak{g}, K)$-module, where S is a **spinor representation** of $\mathrm{Cl}(\mathfrak{p})$. This is an irreducible $\mathbb{Z}/2\mathbb{Z}$-graded $\mathrm{Cl}(\mathfrak{p})$-module which is unique when $\dim \mathfrak{p}$ is odd and there are two if $\dim \mathfrak{p}$ is even.

In this context the **Dirac operator** is $D \in \mathscr{A}^K$, given by

$$D = \sum Y_i \otimes Z_i \in U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}),$$

where Y_i is a basis of $\mathfrak{p}$ and Z_i is a dual basis.

As expected, Dirac operator is something like the square root of Casimir. For γ an irreducible fdrep of $\widetilde{K}$ of htwt τ, D^2 acts on the γ-isotypic component of $X \otimes S$ by the scalar $-\|\Lambda\|^2 + \|\tau + \rho_{\mathfrak{k}}\|^2$, where Λ is the infinitesimal character of X. If X is unitary, then D is symmetric wrt the (induced) hermitian form, so D^2 acts positively; thus we immediately get the restriction of possible K-types that could appear:

$$\|\tau + \rho_{\mathfrak{k}}\|^2 \geq \|\Lambda\|^2$$

(Parasarathy-Dirac inequality)

Dirac operator gives an interesting invariant for each $(\mathfrak{g}, K)$-module X, the **Dirac cohomology** (of Vogan):

$$H_V^D(X) = \ker D / (\ker D \cap \operatorname{im} D)$$

This inherits $\widetilde{K}$-action because $D \in \mathcal{A}^K$. If X is unitary admissible, then $H_V^D(X) = \ker D$, so it is some sum of K-types. If it is nonvanishing, then it is extremely powerful invariant:

Theorem 3.2. *(1) (Huang-Pandzic) If a $\widetilde{K}$-type of htwt τ contributes to nonzero Dirac cohmology $H_V^D(X)$, then Λ and $\tau + \rho_{\mathfrak{k}}$ are in the same Weyl orbit!!!*
(2) Unipotent reps, highest weight modules, discrete series, Vogan-Zuckerman modules all have nonvanishing Dirac cohomology.

3.3. **Parthasarathy, *Dirac operator and the discrete series.*** Notation: $\mathfrak{g}, \mathfrak{g}_{\mathbb{C}}$

Now the idea is structurally similar to Narasimhan–Okamoto but the reason why it worked is because 1. there is Hodge-type theorems where harmonic forms = cohomology, and 2. Kuga-type lemma could apply to something that has geometric meaning b/c basically $\Omega^{0,q} = \wedge^q \mathfrak{p}^-$. Here still one wants to use E_V, the vb corresponding to (contragredient of) K-rep V, but here what we rather use is the bundle corresponding to $L^\pm \otimes V$ where $L^\pm$ are half-spin representations of $\mathrm{Spin}(\mathfrak{p})$ corresponding to a choice of a G-invariant spin structure on G/K. Letting $C^\pm(E_V)$ be the space of smooth sections of $E_{L^\pm \otimes V}$, the Dirac operator sends $D : C^\pm(E_V) \to C^\mp(E_V)$. If we define $H_2^\pm(E_V)$ be the space of square-integrable sections in $\ker D$, then $H_2^\pm$ are the stages of realization. Some steps:

(1) A spin structure on a Riemannian manifold of dimension n means a principal $\mathrm{Spin}(n)$-bundle $\widetilde{F}$ on M such that the principal $\mathrm{SO}(n)$-bundle $\widetilde{F} \times_{\mathrm{Spin}(n)} \mathrm{SO}(n)$ is equivlent to the principal $\mathrm{SO}(n)$-bundle F of orthogonal frames of M. In the case when $M = G/K$, F

is given by $G \times_K \mathrm{SO}(\mathfrak{p})$. Thus, the spin structure on G/K means $K \to \mathrm{Spin}(\mathfrak{p})$ lifting $K \to \mathrm{SO}(\mathfrak{p})$ (up to going into covering space).

In our case $\operatorname{rank} G = \operatorname{rank} K$ means $\mathfrak{p}$ has even dimension, so the spin rep $\sigma : \mathrm{Spin}(\mathfrak{p}) \to \mathrm{Aut}(L)$ splits into $L = L^+ \oplus L^-$ each of which is of dimension 2^{m-1}, where $m = \dim \mathfrak{p}/2$; these are called half spin representations. We let χ be the K-rep on L and $\chi^\pm$ on $L^\pm$.

(2) For a K-rep τ, $\chi \otimes \tau$ gives a vb $E_{L\otimes V}$. Now L is a module for $\mathrm{Cl}(\mathfrak{p}_{\mathbb{C}})$ and since $\mathfrak{p}_{\mathbb{C}} \subset \mathrm{Cl}(\mathfrak{p}_{\mathbb{C}})$, one has a natural bilinear pairing $\mathfrak{p}_{\mathbb{C}} \otimes L \to L$ given by the spin rep of Clifford algebra. Now this is a $\mathrm{Spin}(\mathfrak{p})$-module homomorphism, where $\mathfrak{p}_{\mathbb{C}}$ is regarded as a $\mathrm{Spin}(\mathfrak{p})$-module also. In this pairing we know $\mathfrak{p}_{\mathbb{C}} \otimes L^\pm$ maps into $L^\mp$. So this gives $E_{\mathfrak{p}_{\mathbb{C}}\otimes L\otimes V} \to E_{L\otimes V}$ and $E_{\mathfrak{p}_{\mathbb{C}}\otimes L^\pm\otimes V} \to E_{L^\mp\otimes V}$.

On the other hand, there is a connection for each vb, and because the cotangent bundle is identified with $E_{\mathfrak{p}_{\mathbb{C}}}$, so connection is thought as

$$\nabla : C(E_{L\otimes V}) \to C(E_{\mathfrak{p}_{\mathbb{C}}\otimes L\otimes V})$$

and similarly for $L^\pm$. Now the Dirac operator is given by

$$D : C(E_{L\otimes V}) \to C(E_{\mathfrak{p}_{\mathbb{C}}\otimes L\otimes V}) \to C(E_{L\otimes V})$$

composing connection and the Clifford spin action.

(3) Now we define $\square = D^2$. It has Kuga-like formula. We define $L_2^\pm(E_V)$, $H_2^\pm(E_V)$ (the latter defined as kernel of Dirac operator). Let $\pi^\pm$ be the G-rep $H_2^\pm(E_V)$. Then for φ a K-finite function, we define $\widetilde{\pi}^\pm_\varphi = \int_G \varphi(g)\widetilde{\pi}^\pm(g)dg$, and then $\operatorname{tr} \widetilde{\pi}^+_\varphi - \operatorname{tr} \widetilde{\pi}^- = (-1)^{p_\lambda}\Theta_{w(\lambda+\rho)}(\varphi)$. The rest is similar.

3.4. **Huang-Pandzic.** This treats Renard's lecture note in more detail. That a nonzero $\widetilde{K}$-type in the Dirac cohomology predics the infinitesimal character is called **Vogan's conjecture**. How is it proved?

- Vogan showed that this would follow from two smaller conjectures. First is that, for any $z \in Z(\mathfrak{g})$, there is unique $\zeta(z) \in Z(\mathfrak{k}_\Delta)$ and some $a, b \in U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})$ such that $z \otimes 1 = \zeta(z) + Da + bD$. Here $\mathfrak{k}_\Delta$ is the diagonal embedding $\mathfrak{k} \to U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})$. Second is that, the map $\zeta : Z(\mathfrak{g}) \to Z(\mathfrak{k}_\Delta)$ is a homomorphism of algbras, and under Harish-Chandra isomorphisms this is $S(\mathfrak{h})^W \to S(\mathfrak{t})^{W_K}$, restriction of polynomials on $\mathfrak{h}^*$ to $\mathfrak{t}^*$. This is because, if $x \in (X \otimes S)(\gamma)$ represents a nonzero Dirac cohomology class (i.e. $Dx = 0$ but not in the image of D) then $z \otimes 1$ acts by $\Lambda(z)$ on x, where Λ is the infinitesimal charcater of X. Because $(z\otimes 1 - \zeta(z))x = Dax + bDx = Dax$ and because $\zeta(z)$ also acts as a scalar $(\gamma+\rho_{\mathfrak{k}})(\zeta(z))$, as x is not in the image of D, $(z\otimes 1-\zeta(z))x = 0$. So $\Lambda(z) = (\gamma+\rho_{\mathfrak{k}})(\zeta(z))$, and this means Λ is the extension of $\gamma + \rho_{\mathfrak{k}}$ to $\mathfrak{h}$ given as 0 on $\mathfrak{a}$.
- Let's try to prove the first. We are using the $\mathbb{Z}/2\mathbb{Z}$-grading. Let $d(a) = Da - \epsilon_a aD$ for homogeneous $a \in U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})$, which defines an operator supercommuting with D. This is K-equivariant, and defines a differential on $\mathcal{A}^K$, and changes odd/evenness.

  Huang-Pandzic proved the first by showing that $\ker d = Z(\mathfrak{k}_\Delta) \oplus \operatorname{im} d$. This would immediately imply the firs assertion because, for any $z \in Z(\mathfrak{g})$, $z\otimes 1$ is central in $\mathcal{A}^K$, so it commutes with D, and therefore $d(z\otimes 1) = Dz\otimes 1 - (z\otimes 1)D = 0$. Thus $z\otimes 1 = \zeta(z) + da$ for some $a \in \mathcal{A}^K$ and $\zeta(z) \in Z(\mathfrak{k}_\Delta)$, and $da = Da + aD$, so thsi gives the first assertion with in fact $a = b$.

  To prove this we consider the filtration on $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})$, which is induced from the filtration on $U(\mathfrak{g})$. It is K-invariant, so it induces a filtration on $\mathcal{A}^K$. Obviously $D \in$

$F_1\mathcal{A}^K$, the differential raises the filtration degree by 1. We denote the corresponding graded differential by $\overline{d}$, because $d^2a = [D,a]$ so it really defines a differential. The graded differential on $\operatorname{gr}^* U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}) \otimes \mathrm{Sym}(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p}) \cong \mathrm{Sym}(\mathfrak{k}) \otimes \mathrm{Sym}(\mathfrak{p}) \otimes \wedge\mathfrak{p}$ is given by $(-2d)\operatorname{id} \otimes d_{\mathfrak{p}}$ where $d_{\mathfrak{p}} : \mathrm{Sym}(\mathfrak{p}) \otimes \wedge\mathfrak{p} \to \mathrm{Sym}(\mathfrak{p}) \otimes \wedge\mathfrak{p}$ is the Koszul differential. The proof is just straightforward calculation.

Now that we're given with the fact, we can use a very well-known fact about Koszul differentials: if V is a vector space with a Koszul differential $d_V : S(V)\otimes\wedge(V) \to S(V)\otimes\wedge(V)$, then $\ker d_V = \mathbb{C}1 \otimes 1 \oplus \operatorname{im} d_V$. This is because of the following: by definition for $v \in V$, $d_V(v \otimes 1) = 0$, $d_V(1 \otimes v) = v \otimes 1$, and if you consider another map h by $h(v\otimes 1) = 1\otimes v$, $h(1\otimes v) = 0$, then $hd_V + d_V h = \deg$, the degree operator (multiplication by the degree), which implies that if $\deg a \neq 0$, $d_V a = 0$ implies $a = \frac{1}{\deg a} d_V h(a)$; and $\mathbb{C}1$ is in the kernel while is not in the image. Now using this we see that d on the associated graded gives $\ker d = S(\mathfrak{k}) \otimes 1 \otimes 1 \oplus \operatorname{im} d$. As this commutes with K, on the K-invariant associated graded the kernel is $S(\mathfrak{k})^K \otimes 1 \otimes 1 \oplus \operatorname{im} d$. Now we can do induction on the degree on actual $\mathcal{A}$ using the associated graded version.

- Second is not so hard..

Now it talks about cubic Dirac operator. Let R be a closed subgroup of a compact semisimple Lie group. Let $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$. Then one defines a cubic Dirac operator $D \in U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{s})$ similarly. this is cubic because $\mathrm{Cl}(\mathfrak{s})$-part is something like $B(X, [Y, Z])$. This is the ordinary Dirac operator when $(\mathfrak{g}, \mathfrak{r})$ is symmetric, and the cubic term is neccessary modification to have a nice square. Most part of Vogan's conjecture carries over to this case. Also Kostant showed that

$$H^*(G/R, \mathbb{C}) \cong \mathrm{Tor}_*^{Z(\mathfrak{g})}(\mathbb{C}, Z(\mathfrak{r})),$$

where $Z(\mathfrak{r})$ is $Z(\mathfrak{g})$-module via the ζ map.

Now the cubic Dirac operator is deifned by using complexified Lie algebras so we can extend this to any real Lie groups.

3.5. **Atiyah–Schmid.** For the vanishing appearing in Parthasaranthy you need some nice regularity condition (slightly worse than regularity you want for discrete series in general). Atiyah–Schmid used instead L^2-sections of twisted spinor bundles. They proved a general L^2-index theorem (as Fredholm..) for Dirac operators. Then L^2-Plancherel theorem + general abstract nonsense from functional analysis shows that if index is nonzero (which is computed by L^2-index theorem), then ker gives you a nonzero rep and anything appearing inside is discrete series. This general abstract nonsense sort of circumvents the use of Dirac-Parthasaranthy inequality (at least it seemed like to me) because you can just talk about K-types appearing in nonzero Dirac cohomology which determines what the discrete series is by Vogan's conjecture.

Or maybe not..

4. Realization of discrete series III: L^2-Dolbeault cohomology of G/T

Notation: $\mathfrak{g}_0, \mathfrak{g}$

This is called the Langlands conjecture (or the multiplicity is the Langlands conjecture.. idk). Namely, that a discrete series should be realize as an L^2-Dolbeault cohomology of a line bundle over G/T where T is compact Cartan.

This is somehow more effective.. This can go a bit beyond, realizing certain limit of discrete series. And also it is a nice picture that it talks with its complex analogue which is BWB.

4.1. **Schmid's two papers.** Namely, *On a conjecture of Langlands* and *L^2-cohomology and the discrete series.*

(1) Let $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$ be Borel such that $\mathfrak{n}$ corresponds to antiholomorphic vectors of $D := G/T$. Given an element $\lambda \in \mathfrak{t}^*$ that is integral (i.e. e^λ makes sense), you can obviously attach a holomorphic line bundle $\mathscr{L}_\lambda$. The normalization convention is that $\mathscr{L}_{-2\rho}$ is the canonical bundle of D.

(2) Consider $A^{0,i}(\mathscr{L}_\lambda)$, the space of compactly supported smooth $\mathscr{L}_\lambda$-valued $(0,i)$-forms on D. Then there is $\overline{\partial} : A^{0,i}(\mathscr{L}_\lambda) \to A^{0,i+1}(\mathscr{L}_\lambda)$, and from invariant hermitian metric there is adjoint $\overline{\partial}^* : A^{0,i}(\mathscr{L}_\lambda) \to A^{0,i-1}(\mathscr{L}_\lambda)$. The Laplacian $\square = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ is strongly elliptic. It can be extended to $L_2^{0,i}(\mathscr{L}_\lambda)$, namely square-integrable $(0,i)$-forms, and its kernel $\mathcal{H}^i(\mathscr{L}_\lambda)$ is the space of square-integrable harmonic $(0,i)$-forms. This is called L^2-cohomology; but it is not really a sheaf cohomology.

(3) On the other hand, because $A^{0,i}(\mathscr{L}_\lambda) = (C_c^\infty(G) \otimes \wedge^i \mathfrak{n}^*)_{-\lambda}$, $L_2^{0,i}(\mathscr{L}_\lambda) = (L^2(G) \otimes \wedge^i \mathfrak{n}^*)_{-\lambda}$. By Plancherel theorem,

$$L_2^{0,i}(\mathscr{L}_\lambda) = \int_{\widehat{G}} \pi_\iota \otimes (\pi_\iota \otimes \wedge^i \mathfrak{n}^*)_{-\lambda} d\iota.$$

(4) $\mathcal{H}^i(\mathscr{L}_\lambda)$ is expressed in terms of operators $\overline{\partial}$ and $\overline{\partial}^*$, and because we are picking a particular K-type, it follows that

$$\mathcal{H}^i(\mathscr{L}_\lambda) = \int_{\widehat{G}} \pi_\iota \otimes \mathcal{H}^i(\pi_\iota^*)_{-\lambda} d\iota,$$

where for any unitary irrep π of G, $\mathcal{H}^p(\pi)$ is the kernel of $\delta + \delta^*$ acting on $\pi \otimes \wedge^p \mathfrak{n}^*$, where $\delta : \pi_{(K)} \otimes \wedge \mathfrak{n}^* \to \pi_{(K)} \otimes \wedge \mathfrak{n}^*$ is the coboundary operator in Lie algebra cohomology and δ^* is the formal adjoint.

(5) Now modulo analytic difficulties ($\mathfrak{g}$-action vs G-action on Hilbert space) it is formality that, for any unitary G-irrep π, $\mathcal{H}^p(\pi) \cong H^p(\mathfrak{n}, \pi_{(K)})$. Casselman-Osborne says that $H^p(\mathfrak{n}, \pi_{(K)})_\mu \neq 0$ implies that π must have infinitesimal character $\chi_{-\mu-\rho}$, which means, by Harish-Chandra, that $\mathcal{H}^p(\mathscr{L}_\lambda)$ is a finite direct sum of discrete series.

(6) The rest becomes the calculation of $\mathfrak{n}$-cohomology of discrete series representations. This is pretty formal and very similar to Borel-Weil-Bott.

Theorem 4.1. *$H^p(\mathfrak{n}, (\pi_\lambda)_{(K)})_\mu$ vanishes in all dimensions unless $\mu - \rho \in W\lambda$. If $\mu - \rho = w\lambda$, then $H^p(\mathfrak{n}, (\pi_\lambda)_{(K)})_\mu$ is concentrated in one degree $p = k$, and is one-dimensional at that degree, where*

$$k = \#\{\alpha \in \Phi_c \cap \Psi \mid (\alpha, \mu - \rho) > 0\} + \#\{\alpha \in \Phi_n \cap \Psi \mid (\alpha, \mu - \rho) < 0\},$$

where Ψ is the root system using $\mathfrak{n}$ as negative roots.

Some words about the proof:

(a) Let λ define positive roots $\widetilde{\Psi} = \{\alpha \in \Phi \mid (\alpha, \lambda) > 0\}$, and because only $W\lambda$ matters we assume that $\widetilde{\Psi} \cap \Phi^c = \Psi \cap \Phi^c$. Blattner's conjecture says that the minimal K-type of π_λ is of highest weight $\lambda + \widetilde{\rho}_n - \widetilde{\rho}_c$, where $\widetilde{\rho}$ means sum of positive roots wrt $\widetilde{\Psi}$ (and same for n, c), and all K-types are of htwt this minimal K-type + sum of positive noncompact roots in $\Phi^n \cap \widetilde{\Psi}$.

(b) Spectral sequence for Lie algebra cohomology, for the pair $(\mathfrak{n}, \mathfrak{k} \cap \mathfrak{n})$, says that there is a ss

$$E_1^{p,q} = H^q(\mathfrak{k} \cap \mathfrak{n}, (\pi_\lambda)_{(K)} \otimes \wedge^p(\mathfrak{n}/\mathfrak{k} \cap \mathfrak{n})^*)_\mu \Rightarrow H^{p+q}(\mathfrak{n}, (\pi_\lambda)_{(K)})_\mu.$$

Letting $\mathfrak{p}_- = \mathfrak{p} \cap \mathfrak{n}$, $(\mathfrak{n}/\mathfrak{k} \cap \mathfrak{n})^* \cong \mathfrak{p}/\mathfrak{p}_-$ the ss is rewritten as

$$E_1^{p,q} = H^q(\mathfrak{k} \cap \mathfrak{n}, (\pi_\lambda)_{(K)} \otimes \wedge^p(\mathfrak{p}/\mathfrak{p}_-))_\mu \Rightarrow H^{p+q}(\mathfrak{n}, (\pi_\lambda)_{(K)})_\mu.$$

(c) Now it's really a matter of calculating $H^q(\mathfrak{k} \cap \mathfrak{n}, V_\nu \otimes \wedge^p(\mathfrak{p}/\mathfrak{p}_-))_\mu$ for a K-type V_ν with htwt $\nu = \lambda + \widetilde{\rho}_n - \widetilde{\rho}_c + B$ for B sum of positive ($\widetilde{\Psi}$) noncompact roots. The weights appearing in $\wedge^p(\mathfrak{p}/\mathfrak{p}_-)$ are exactly sum of p-tuples of roots in $\Phi^n \cap \Psi$. So if $H^q(\mathfrak{k} \cap \mathfrak{n}, V_\nu \otimes \wedge^p(\mathfrak{p}/\mathfrak{p}_-))_\mu \neq 0$, then there are p distinct roots $\gamma_1, \cdots, \gamma_p \in \Psi^n \cap \Psi$ such that $H^q(\mathfrak{k} \cap \mathfrak{n}, V_\nu)_{\mu - \gamma_1 - \cdots - \gamma_p} \neq 0$. Then Lie algebra version of BWB (due to Kostant) says that there is $w \in W$ such that

$$\mu - \gamma_1 - \cdots - \gamma_p - \rho_c = w(\nu + \rho_c)$$
$$q = \#\{\alpha \in \Phi^c \cap \Psi \mid w\alpha \in \Phi^c \cap \Psi\}.$$

(d) On the other hand Dirac operator computes nilpotent Lie algebra cohomology, so it follows that $w^{-1}(\rho_n - \gamma_1 - \cdots - \gamma_p) = \widetilde{\rho}_n - \sum \beta_i$ for β_i distinct roots in $\Phi^n \cap \widetilde{\Psi}$. This implies that $\mu - \rho = w(\nu + \widetilde{\rho}_c - \widetilde{\rho}_n + \sum \beta_i)$. This implies that the minimal K-type only survives, etc.

To be continued:

- Milicic-Schmid-Vilonen-Wolf, ...
- Carayol, Griffiths-Green-Kerr, ...