E_2 IS NOT OVERCONVERGENT

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We extract an argument from a paper of Coleman-Gouvea-Jochnowitz, *E_2, Θ, and Overconvergence*, to sketch a proof of the following

Theorem 1 (Coleman-Gouvea-Jochnowitz)**.** *The p-adic modular form $E_2(q) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n$ is not overconvergent.*

Proof. Let X be the j-line over $\mathbb{Z}_p$, C be the cusp, and $\widetilde{SS}$ be a lift of the set of supersingular points $SS \subset X \otimes \mathbb{F}_p$ (e.g. take the zeroes of E_{p-1}). Then as ordinary oldforms $S_2(\Gamma_0(p))$ embeds into $H^1_{\mathrm{dR}}(X - (C \cup \widetilde{SS}))$. Note that $\dim S_2(\Gamma_0(p)) = g(X_0(p)) = \#|SS| - 1$, but also note that $\dim H^1_{\mathrm{dR}}(X - \widetilde{SS}) = \#|SS| - 1$, and that $\dim H^1_{\mathrm{dR}}(X - (C \cup \widetilde{SS})) = \dim H_{\mathrm{dR}}(X - \widetilde{SS}) + 1$. Thus $S_2(\Gamma_0(p))$ embeds into $H^1_{\mathrm{dR}}(X - (C \cup \widetilde{SS}))$ as a codimension 1 subspace.

One then notices that $H^1_{\mathrm{dR}}(X - (C \cup \widetilde{SS})) \cong M_2/\theta M_0$ where M_i is the space of $p^{-p/(p+1)}$-overconvergent modular forms of weight i. This follows from that $M_2/\theta M_0 \cong \Omega^1_X(\log C)(X_{p^{-p/(p+1)}})$ by Kodaira-Spencer (here X_r is the r-overconvergent locus, e.g. $r = 1$ corresponds to the ordinary locus) and that $\Omega^1_X(\log C)(X_{p^{-p/(p+1)}}) \cong H^1_{\mathrm{dR}}(X - (C \cup \widetilde{SS}))$ as the rigid cohomology is the same as the de Rham cohomology and removing points gives the same de Rham cohomology as removing slightly larger "diskoid" (terminology of Coleman, see e.g. Coleman, *Reciprocity laws on curves*) regions around each point. Thus, if E_2 is overconvergent, then E_2 precisely gives the missing extra dimension, as E_2 is certainly not cuspidal (any image of theta is cuspidal in a p-adic sense). Thus $E_2 = \theta G + F$ for $G \in M_0$ and $F \in S_2(\Gamma_0(p))$. Note that eE_2, the ordinary projection of E_2, is a classical modular form of weight 2 level $\Gamma_0(p)$. Thus, as an image of theta is never ordinary, $F = eE_2$.

Now we know explicit q-expansions of E_2 and eE_2 as well as the effect of θ on q-expansions, so we see that G has to be, up to constant,

$$\frac{24p}{p-1}\sum_{n=1}^{\infty}\sigma^*_{-1}(n)q^n,$$

where the star means we miss multiples of p in the sum over divisors. This means

$$G(\mathrm{mod}\, p) =: \overline{G} = \sum_{n=1}^{\infty}\sigma_{p-2}(n)q^n,$$

is a mod p Katz modular form, which is a mod p reduction of a classical modular form of weight k for some k divisible by $p-1$. Let k be the filtration of $\overline{G}$, i.e. the minimal possible k. Then, by the congruence $\sigma_{p-2}(n) - \sigma_{p-2}(n/p) = n^{p-2}\sigma_1(n)(\mathrm{mod}\, p)$, we see that

$$\overline{G} - \overline{G}^p = \theta^{p-2}\Big(\sum_{n=1}^{\infty}\sigma_1(n)q^n\Big)(\mathrm{mod}\, p).$$

Obviously the LHS has the filtration pk. On the other hand, $\sum_{n=1}^{\infty}\sigma_1(n)q^n$ has filtration $p+1$, and by the theory of θ-cycles, as we do not face a multiple of p as a filtration in the course of applying θ operators multiple times, we see that the RHS has the filtration $(p+1)(p-2) + (p+1) = p^2 - 1$, which is not a multiple of p. ☐

In general, if f is r-overconvergent, then $\theta(f) - \frac{k}{12}E_2 f$ is r-overconvergent, so θ actually almost surely ruins overconvergence.